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# MENGER SUBSETS OF THE SORGENFREY LINE

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Abstract. A space $X$ is said to have the Menger property if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of $X$, there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$ such that $\bigcup_{n\in\omega} \mathcal{V}_n$ is a cover of $X$. Let $i : \mathbb{S} \to \mathbb{R}$ be the identity map from the Sorgenfrey line onto the real line and let $X_{\mathbb{S}} = i^{-1}(X)$ for $X \subset \mathbb{R}$. Lelek noted in 1964 that for every Lusin set $L$ in $\mathbb{R}$, $L_{\mathbb{S}}$ has the Menger property. In this paper we further investigate Menger subsets of the Sorgenfrey line. Among other things, we show: (1) If $X_{\mathbb{S}}$ has the Menger property, then $X$ has Marczewski's property $(s^0)$. (2) Let $X$ be a zero-dimensional separable metric space. If $X$ has a countable subset $Q$ satisfying that $X \setminus A$ has the Menger property for every countable set $A \subset X \setminus Q$, then there is an embedding $e : X \to \mathbb{R}$ such that $e(X)_{\mathbb{S}}$ has the Menger property. (3) For a Lindelöf subspace of a real GO-space (for instance the Sorgenfrey line), total paracompactness, total metacompactness and the Menger property are equivalent.

## 1. Introduction and preliminaries

In this paper all spaces are assumed to be regular $T_1$. The symbols $\mathbb{R}, \mathbb{P}$ and $\mathbb{Q}$ are respectively the space of real numbers, the space of irrational numbers and the space of rational numbers. The symbol $\mathbb{C}$ is Cantor's "middle thirds" set in the closed unit interval [0,1]. In other words, $\mathbb{C} = \{\sum_{n=0}^{\infty} \frac{2k_n}{3^{n+1}} : k_n \in \{0,1\},\ n \in \omega\}$. We let $\mathbb{S}$ stand for the Sorgenfrey line and let $i : \mathbb{S} \to \mathbb{R}$ be the identity map. For $X \subset \mathbb{R}$, we put $X_{\mathbb{S}} = i^{-1}(X)$. The Sorgenfrey line has the topology generated by all half-open intervals $[p, q)$. The Sorgenfrey line is zero-dimensional, hereditarily Lindelöf and hereditarily separable [7]. For a set $X$, $[X]^{\leq\omega}, [X]^{\omega}$ and $[X]^{<\omega}$ are respectively the set of countable subsets of $X$, the set of countably infinite subsets of $X$ and the set of finite subsets of $X$. Unexplained notions and terminology are the same as in [7], [15] and [18].

**Definition 1.1.** According to [21], a space $X$ has *the Menger property* (or simply we say $X$ is *Menger*) if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of $X$, there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$ such that $\bigcup_{n\in\omega} \mathcal{V}_n$ is a cover of $X$.

Hurewicz [11] introduced this covering property and showed that for a metric space this covering property is equivalent to property $E$ introduced by Menger [17]. In the classic literature a space with the Menger property is called a *Hurewicz*

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*space*. Every $\sigma$-compact space has the Menger property and every space with the Menger property is Lindelöf. Every analytic set (i.e. the continuous image of $\mathbb{P}$) with the Menger property is $\sigma$-compact (Hurewicz, 1925). For a space $X$, $X^\omega$ has the Menger property iff $X$ is compact [1, Proposition 1].

Since Scheepers' paper [21], classical and new covering properties including the Menger property have been extensively studied by many researchers; for instance, see Tsaban [25] for a survey. Tsaban's paper [26] contains interesting open problems in this field.

Lelek showed in [16, Example] that for every Lusin set $L$ in $\mathbb{R}$, $L_{\mathbb{S}}$ is Menger, and if $L$ satisfies $L \times L \cap \{(x, y) : x + y = 0\}$ is uncountable, then the square of $L_{\mathbb{S}}$ is not normal (hence not Menger). In this paper we give some further results on Menger subsets of the Sorgenfrey line.

**Definition 1.2** ([6])**.** A quasi-ordering $\leq^*$ $(<^*)$ is defined on the Baire space $\omega^\omega$ as follows: $f \leq^* g$ $(f <^* g)$ if $f(n) \leq g(n)$ $(f(n) < g(n))$ for all but finitely many $n \in \omega$. We say that a subset $Y$ of $\omega^\omega$ is *bounded* if there is a $g \in \omega^\omega$ such that for each $f \in Y, f \leq^* g$. Otherwise, we say that $Y$ is *unbounded*. The symbol $\mathfrak{b}$ denotes the minimal cardinality of an unbounded subset in $\omega^\omega$. We say that a subset $Y$ of $\omega^\omega$ is *dominating* if for every $f \in \omega^\omega$ there is $g \in Y$ with $f \leq^* g$. The minimal cardinality of a dominating subset in $\omega^\omega$ is denoted by the symbol $\mathfrak{d}$.

The following proposition is essentially due to Hurewicz [12]. For instance, the proof can be found in Wingers' paper [27, Theorem 3.12].

**Proposition 1.3.** *Let $X$ be a zero-dimensional Lindelöf space. Then $X$ is Menger iff every continuous image of $X$ in $\omega^\omega$ is not dominating.*

Thus $\mathbb{P}$ is not Menger. We summarize some results on the Menger property.

**Lemma 1.4.** (1) *The Menger property is preserved under continuous images.*
(2) *The Menger property is hereditary for closed subsets.*
(3) $\mathfrak{d} = \min\{|X| : X \subset \mathbb{R}$ *and $X$ is not Menger*$\}$.
(4) *If a subset $X$ of $\mathbb{R}$ is $\mathfrak{d}$-concentrated on a countable subset $Q$ of $X$ (i.e., for every open subset $U$ in $X$ containing $Q$, $|X \setminus U| < \mathfrak{d}$), then $X$ is Menger.*
(5) *If $\kappa < \mathfrak{b}$ and $\{X_\alpha : \alpha < \kappa\}$ is a cover consisting of Menger subsets of a Lindelöf space $X$, then $X$ is Menger.*

*Proof.* (1) and (2) are obvious. (3) immediately follows from Proposition 1.3. (4). Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of $X$. Let $Q = \{q_n : n \in \omega\}$. For each $n \in \omega$, take $U_n \in \mathcal{U}_n$ with $q_n \in U_n$ and let $U = \bigcup_{n\in\omega} U_n$. Since $|X \setminus U| < \mathfrak{d}$, $X \setminus U$ is Menger by (3) in this lemma. Take finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$ such that $\bigcup_{n\in\omega} \mathcal{V}_n$ covers $X \setminus U$. The sequence $\{\mathcal{V}_n \cup \{U_n\} : n \in \omega\}$ is what we need. (5) is due to Wingers [27, Corollary 3.9]. □

By Proposition 1.3 and Lemma 1.4 (1) (3), we obtain $\mathfrak{d} = \min\{|X| : X \subset \mathbb{S}$ and $X$ is not Menger$\}$.

## 2. A necessary condition

**Lemma 2.1.** *The following two statements hold.*

(1) *Every non-empty dense-in-itself completely metrizable space contains a subset which is homeomorphic to the Cantor cube $\{0, 1\}^\omega$* [7, 4.5.5].

(2) *Every $G_\delta$-set which is both dense and co-dense in a separable zero-dimensional completely metrizable space is homeomorphic to the Baire space $\omega^\omega$* [7, 6.2.A.(a)].

**Definition 2.2.** Let $X$ be a subset of $\mathbb{R}$. $X$ is *totally imperfect* [15] if it has no subset which is homeomorphic to the Cantor cube $\{0,1\}^\omega$. $X$ has *property ($s^0$)* [23] if for every perfect set $P$ in $\mathbb{R}$ (i.e. $P$ is dense-in-itself and closed) there is a perfect set $P'$ in $\mathbb{R}$ such that $P' \subset P \setminus X$.

Obviously every set with property ($s^0$) is totally imperfect. A Bernstein set is totally imperfect, but it does not have property ($s^0$). Property ($s^0$) is a common generalization of a universal measure-zero set and a perfectly meager set [18].

**Lemma 2.3.** *Let $X$ be a subset of $\mathbb{R}$. If $X$ is Menger and totally imperfect, then it has property ($s^0$).*

*Proof.* Let $P$ be a non-empty perfect set in $\mathbb{R}$. By Lemma 2.1 (1), we may suppose that $P$ is homeomorphic to the Cantor cube $\{0,1\}^\omega$. Since $X$ is totally imperfect, $P \setminus X$ is dense in $P$. Let $D$ be a countable dense subset of $P \setminus X$. By Lemma 2.1 (2), $P \setminus D$ is homeomorphic to $\omega^\omega$. Let $\varphi$ be a homeomorphism from $P \setminus D$ onto $\omega^\omega$. Since $X \cap P$ is Menger, $\varphi(X \cap P)$ is not dominating in $\omega^\omega$ by Proposition 1.3. Hence there is some $h \in \omega^\omega$ such that for every $f \in \varphi(X \cap P)$ the set $\{n \in \omega : f(n) < h(n)\}$ is infinite. Let $h(n) < k_n < l_n$ $(k_n, l_n \in \omega)$ for every $n \in \omega$. Let

$$K = \{g \in \omega^\omega : g(n) \in \{k_n, l_n\} \text{ for every } n \in \omega\}.$$

Obviously $K$ is homeomorphic to $\{0,1\}^\omega$ and $K \cap \varphi(X \cap P) = \emptyset$. The perfect set $\varphi^{-1}(K)$ satisfies $\varphi^{-1}(K) \subset P \setminus X$. □

**Theorem 2.4.** *Let $X$ be a subset of $\mathbb{R}$. If $X_{\mathbb{S}}$ is Menger, then $X$ is Menger and has property ($s^0$); in particular, $X$ is zero-dimensional.*

*Proof.* Obviously $X$ is Menger by Lemma 1.4 (1). In view of Lemma 2.3, we have only to show that $X$ is totally imperfect. Suppose that $X$ has a subset $C$ which is homeomorphic to $\{0,1\}^\omega$. Let

$$I = \{r \in C : r \text{ is isolated in } C_{\mathbb{S}}\}.$$

Since $\mathbb{S}$ is hereditarily Lindelöf, $I$ is countable. For every non-empty clopen subset $U$ of $C$, the greatest element of $U$ is in $I$. Hence $I$ is dense in $C$. By Lemma 2.1 (2), $C \setminus I$ is homeomorphic to $\omega^\omega$, so it is not Menger. On the other hand, since $(C \setminus I)_{\mathbb{S}}$ is closed in $X_{\mathbb{S}}$, it is Menger by Lemma 1.4 (2). Hence $C \setminus I$ must be Menger. This is a contradiction. □

## 3. Sufficient conditions

**Definition 3.1.** Let $X$ be a subset of $\mathbb{R}$ and let $Y$ be a space. A map $\eta : X \to Y$ is *right-continuous at $x \in X$* if for every neighborhood $U$ of $\eta(x)$ in $Y$ there is an $\varepsilon > 0$ such that $\eta([x, x+\varepsilon) \cap X) \subset U$. A map $\eta : X \to Y$ is *right-continuous* if it is right-continuous at every point in $X$.

Continuous functions on $\mathbb{S}$ are exactly right-continuous functions on $\mathbb{R}$; thus by Proposition 1.3 we obtain the following.

**Lemma 3.2.** *Let $X$ be a subset of $\mathbb{R}$. Then $X_{\mathbb{S}}$ is Menger iff every right-continuous image of $X$ in $\omega^\omega$ is not dominating.*

The following is folklore; for instance, see [14, p. 173].

**Lemma 3.3.** *The set of all discontinuous points of a right-continuous map into a metric space is countable.*

**Lemma 3.4.** *Let $X$ be a subset of $\mathbb{R}$. If $X \setminus A$ is Menger for every $A \in [X]^{\leq\omega}$, then $X_{\mathbb{S}}$ is Menger.*

*Proof.* Let $\eta : X \to \omega^\omega$ be right-continuous. Let $A$ be the set of all discontinuous points of $\eta$. By Lemma 3.3, $A$ is countable. Since $X \setminus A$ is Menger and $\eta|_{X\setminus A}$ is continuous, $\eta(X \setminus A)$ is not dominating by Proposition 1.3. Since $\eta(A)$ is countable, $\eta(X)$ is not dominating. By Lemma 3.2, $X_{\mathbb{S}}$ is Menger. □

**Proposition 3.5.** *Let $X$ be a subset of $\mathbb{R}$. The following two statements hold.*

(1) *$X \setminus A$ is Menger for every $A \in [X]^{\leq\omega}$ iff $(X \setminus A)_{\mathbb{S}}$ is Menger for every $A \in [X]^{\leq\omega}$.*
(2) *$X$ is hereditarily Menger iff $X_{\mathbb{S}}$ is hereditarily Menger.*

*Proof.* (1) Suppose that $X \setminus A$ is Menger for every $A \in [X]^{\leq\omega}$. Let $A \in [X]^{\leq\omega}$. Then for every $B \in [X \setminus A]^{\leq\omega}$, $(X \setminus A) \setminus B = X \setminus (A \cup B)$ is Menger. Applying Lemma 3.4 to $X \setminus A$, $(X \setminus A)_{\mathbb{S}}$ is Menger. The converse is trivial.

(2) Suppose that $X$ is hereditarily Menger and let $Y \subset X$. Obviously $Y$ satisfies that $Y \setminus A$ is Menger for every $A \in [Y]^{\leq\omega}$. Hence $Y_{\mathbb{S}}$ is Menger. The converse is trivial. □

A subset of $\mathbb{R}$ is called a *Lusin set* if it is uncountable and has countable intersection with every meager set in $\mathbb{R}$. A subset of $\mathbb{R}$ is called a *Sierpiński set* if it is uncountable and has countable intersection with every Lebesque measure-zero set in $\mathbb{R}$. Such sets exist under the continuum hypothesis; see [18]. A Lusin set is not meager and a Sierpiński set is not Lebesque measurable. It is known that both a Lusin set and a Sierpiński set are hereditarily Menger; see [19, pp. 20, 30].

**Corollary 3.6.** *If $X$ is a Lusin set or a Sierpiński set, then $X_{\mathbb{S}}$ is hereditarily Menger.*

Boaz Tsaban pointed out that the preceding corollary follows also from the results in [22, Theorems 3, 13].

For each $n \in \omega$ we denote by $2^n$ the set of all sequences of 0's and 1's with length $n$. For each $s \in 2^n$ let $[s] = \{f \in \{0,1\}^\omega : f|n = s\}$. Let $C$ be a subset of $\mathbb{R}$ which is homeomorphic to $\{0,1\}^\omega$ and let $\varphi_C : \{0,1\}^\omega \to C$ be a homeomorphism. Since $\varphi_C([s])$ is compact, it has the minimum number $p_s$ and the maximum number $q_s$. We put

$$Q_0(C) = \{p_s : s \in \bigcup_{n\in\omega} 2^n\};\ Q_1(C) = \{q_s : s \in \bigcup_{n\in\omega} 2^n\}.$$

Note that both $Q_0(C)$ and $Q_1(C)$ are countable and dense in $C$.

**Lemma 3.7.** *Let $C$ be a subset of $\mathbb{R}$ which is homeomorphic to $\{0,1\}^\omega$. Let $X$ be a zero-dimensional separable metrizable space and $A$ be a countable subset of $X$. Then there are embeddings $e_0, e_1 : X \to C$ such that $e_0(A) \subset Q_0(C)$ and $e_1(A) \subset Q_1(C)$.*

*Proof.* Take any embedding $e : X \to C$. Let $Q$ be a countable dense subset of $C$ containing $e(A)$. There is a homeomorphism $e' : C \to C$ such that $e'(Q) = Q_0(C)$; see [7, 4.3.H.(e)]. The composition $e_0 = e' \cdot e$ is an embedding of $X$ with $e_0(A) \subset Q_0(C)$. An embedding $e_1$ can be obtained similarly. □

**Theorem 3.8.** *Let $X$ be a zero-dimensional separable metrizable space. The following two statements hold.*

(1) *$X \setminus A$ is Menger for every $A \in [X]^{\leq\omega}$ iff for every embedding $e : X \to \mathbb{R}$, $e(X)_{\mathbb{S}}$ is Menger.*
(2) *Let $C$ be a subset of $\mathbb{R}$ which is homeomorphic to $\{0,1\}^\omega$. If there is a countable subset $Q$ of $X$ such that $X \setminus A$ is Menger for every $A \in [X \setminus Q]^{\leq\omega}$, then there is an embedding $e : X \to C$ such that $e(X)_{\mathbb{S}}$ is Menger.*

*Proof.* (1) ($\Rightarrow$) This is obvious by Lemma 3.4. ($\Leftarrow$) Let $A \in [X]^{\leq\omega}$. By Lemma 3.7, there is an embedding $e_1 : X \to \mathbb{C}$ such that $e_1(A) \subset Q_1(\mathbb{C})$. Note that each point of $Q_1(\mathbb{C})_{\mathbb{S}}$ is isolated in $\mathbb{C}_{\mathbb{S}}$. Since $e_1(X)_{\mathbb{S}}$ is Menger, the closed set $(e_1(X) \setminus Q_1(\mathbb{C}))_{\mathbb{S}}$ is Menger. Since the union of a Menger set and a countable set is Menger, $(e_1(X) \setminus e_1(A))_{\mathbb{S}}$ is Menger. Hence $X \setminus A$ is Menger.

(2) Let $Q$ be a countable subset of $X$ such that $X \setminus A$ is Menger for every $A \in [X \setminus Q]^{\leq\omega}$. By Lemma 3.7, there is an embedding $e_0 : X \to C$ such that $e_0(Q) \subset Q_0(C)$. We show that $e_0(X)_{\mathbb{S}}$ is Menger. Let $\eta : e_0(X) \to \omega^\omega$ be a right-continuous map and let $A$ be the set of all discontinuous points of $\eta$. By Lemma 3.3 $A$ is countable. Note that $\eta$ is continuous at any point in $e_0(X) \cap Q_0(C)$, because each point $r$ of $Q_0(C)$ has a neighborhood base consisting of half-open intervals $[r, r+\varepsilon)$. Hence $e_0(Q) \cap A = \emptyset$; therefore $e_0(X) \setminus A$ is Menger. Since $\eta|_{e_0(X)\setminus A} : e_0(X) \setminus A \to \omega^\omega$ is continuous, $\eta(e_0(X) \setminus A)$ is not dominating by Proposition 1.3. Since $A$ is countable, $\eta(e_0(X))$ is also not dominating. Thus by Lemma 3.2, $e_0(X)_{\mathbb{S}}$ is Menger. □

**Corollary 3.9.** *Let $C$ be a subset of $\mathbb{R}$ which is homeomorphic to $\{0,1\}^\omega$. If $X$ is a zero-dimensional separable metric space which is $\mathfrak{d}$-concentrated on a countable subset of $X$, then there is an embedding $e : X \to C$ such that $e(X)_{\mathbb{S}}$ is Menger.*

*Proof.* If $X$ is $\mathfrak{d}$-concentrated on a countable subset $Q$ of $X$, then $X \setminus A$ is also $\mathfrak{d}$-concentrated on $Q$ for every $A \in [X \setminus Q]^{\leq\omega}$; hence $X \setminus A$ is Menger by Lemma 1.4 (4). □

We observe that there are a zero-dimensional separable metrizable space $X$ and two embeddings $e, e'$ of $X$ into $\mathbb{R}$ such that $e(X)_{\mathbb{S}}$ is not Menger, but $e'(X)_{\mathbb{S}}$ is Menger.

**Example 3.10.** Our construction is essentially due to [4, Theorem 16]. Let $D = \{f_\alpha : \alpha < \mathfrak{d}\}$ be a dominating subset of $\omega^\omega$ which satisfies the condition (*): for every $f \in \omega^\omega$ there is $\gamma < \mathfrak{d}$ such that $f_\alpha \not\leq^* f$ for every $\alpha > \gamma$. Let $Q$ be a countable dense subset of $\mathbb{C}$. By Lemma 2.1 (2), $\mathbb{C} \setminus Q$ is homeomorphic to $\omega^\omega$. Let $\varphi : \omega^\omega \to \mathbb{C} \setminus Q$ be a homeomorphism. Let

$$X = \varphi(D) \cup Q \subset \mathbb{C}.$$

By Proposition 1.3, $X \setminus Q = \varphi(D)$ is not Menger. Hence by Theorem 3.8 (1), there is an embedding $e : X \to \mathbb{R}$ such that $e(X)_{\mathbb{S}}$ is not Menger. On the other hand, the condition (*) on $D$ implies that $X$ is $\mathfrak{d}$-concentrated on $Q$. By Corollary 3.9, there is an embedding $e' : X \to \mathbb{C}$ such that $e'(X)_{\mathbb{S}}$ is Menger.

The converse of Theorem 3.8 (2) does not hold.

**Example 3.11.** Under $\mathfrak{b} = \mathfrak{d}$, we give a subset $Y$ of $\mathbb{R}$ such that $Y_{\mathbb{S}}$ is Menger, but for every $A \in [Y]^{\leq\omega}$ there is $B \in [Y \setminus A]^{\leq\omega}$ such that $Y \setminus B$ is not Menger.

First we observe that there is a pairwise disjoint family $\{C_\alpha : \alpha < \mathfrak{d}\}$ consisting of subsets of $\mathbb{C} \setminus Q_0(\mathbb{C})$ such that each $C_\alpha$ is homeomorphic to $\{0,1\}^\omega$ and moreover for every open subset $U$ of $\mathbb{C}$ containing $Q_0(\mathbb{C})$ there is $\gamma < \mathfrak{d}$ with $\bigcup_{\alpha>\gamma} C_\alpha \subset U$. Indeed let $E = \{f_\alpha : \alpha < \mathfrak{d}\}$ be a dominating subset of $\omega^\omega$ such that $f_\alpha <^* f_\beta$ for $\alpha < \beta$. We may assume that $E$ satisfies $\lim_{n\to\infty} |f_\alpha(n) - f_\beta(n)| = \infty$ for $\alpha < \beta$. Let

$$K_\alpha = \{f \in \omega^\omega : f(n) \in \{f_\alpha(n), f_\alpha(n)+1\} \text{ for } n \in \omega\}.$$

Obviously $K_\alpha$ is homeomorphic to $\{0,1\}^\omega$ and $K_\alpha \cap K_\beta = \emptyset$ for $\alpha < \beta$. Let $\psi : \omega^\omega \to \mathbb{C} \setminus Q_0(\mathbb{C})$ be a homeomorphism and let $C_\alpha = \psi(K_\alpha)$. The family $\{C_\alpha : \alpha < \mathfrak{d}\}$ is what we need.

Recall the space $X = \varphi(D) \cup Q \subset \mathbb{C}$ constructed in Example 3.10 which is $\mathfrak{d}$-concentrated on $Q$. Applying Corollary 3.9, we take an embedding $e_\alpha : X \to C_\alpha$ such that $e_\alpha(X)_\mathbb{S}$ is Menger. Let

$$Y = Q_0(\mathbb{C}) \cup (\textstyle\bigcup_{\alpha<\mathfrak{d}} e_\alpha(X)).$$

To show that $Y_\mathbb{S}$ is Menger, we need the following.

*Claim* 1. For every $\gamma < \mathfrak{d}$, (a) $Y_\gamma = Q_0(\mathbb{C}) \cup (\bigcup_{\alpha>\gamma} e_\alpha(X))$ is Menger, and (b) $(\bigcup_{\alpha\le\gamma} e_\alpha(X))_\mathbb{S}$ is also Menger.

*Proof of Claim.* Fix $\gamma < \mathfrak{d}$. The statement (a) can be proved by the same arguments as in Lemma 1.4 (4). Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of $Y_\gamma$. Let $Q_0(\mathbb{C}) = \{q_n : n \in \omega\}$. For each $n \in \omega$, take $U_n \in \mathcal{U}_n$ with $q_n \in U_n$. Since the open set $U = \bigcup_{n\in\omega} U_n$ contains $Q_0(\mathbb{C})$, there is $\gamma' < \mathfrak{d}$ with $\bigcup_{\alpha>\gamma'} e_\alpha(X) \subset U$. Hence $Y_\gamma \setminus U \subset \bigcup\{e_\alpha(X) : \gamma < \alpha \le \gamma'\}$. Since $\bigcup\{e_\alpha(X) : \gamma < \alpha \le \gamma'\}$ is Menger by Lemma 1.4 (5), we can take finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$ such that $\bigcup_{n\in\omega} \mathcal{V}_n$ covers $\bigcup\{e_\alpha(X) : \gamma < \alpha \le \gamma'\}$. The sequence $\{\mathcal{V}_n \cup \{U_n\} : n \in \omega\}$ is what we need. The statement (b) follows from Lemma 1.4 (5) and the fact that $e_\alpha(X)_\mathbb{S}$ is Menger. The proof of the claim is complete. □

Now we show that $Y_\mathbb{S}$ is Menger. Let $\eta : Y \to \omega^\omega$ be a right-continuous map and let $A$ be the set of all discontinuous points of $\eta$. Since $\eta$ is continuous at each point of $Q_0(\mathbb{C})$, $A \cap Q_0(\mathbb{C}) = \emptyset$. Hence $A \subset \bigcup_{\alpha\le\gamma} e_\alpha(X)$ for some $\gamma < \mathfrak{d}$. Since $\eta|_{Y_\gamma} : Y_\gamma \to \omega^\omega$ is continuous and $Y_\gamma$ is Menger by Claim (a), $\eta(Y_\gamma)$ is non-dominating by Proposition 1.3. Since $(\bigcup_{\alpha\le\gamma} e_\alpha(X))_\mathbb{S}$ is Menger by Claim (b), $\eta(\bigcup_{\alpha\le\gamma} e_\alpha(X))$ is also non-dominating by Lemma 3.2. Consequently $\eta(Y)$ is non-dominating; therefore $Y_\mathbb{S}$ is Menger.

On the other hand, let $A \in [Y]^{\le\omega}$. There is $\gamma < \mathfrak{d}$ with $A \cap e_\gamma(X) = \emptyset$ (hence $A \cap e_\gamma(Q) = \emptyset$). Since $e_\gamma(X) \setminus e_\gamma(Q)$ is closed in $Y \setminus e_\gamma(Q)$ and $e_\gamma(X) \setminus e_\gamma(Q)$ is not Menger, $Y \setminus e_\gamma(Q)$ is not Menger.

**Example 3.12.** The notion of property $(\gamma)$ was introduced by Gerlits and Nagy. A space with property $(\gamma)$ (i.e. a $\gamma$-set) is Menger and totally imperfect [10]. Property $(\gamma)$ is much stronger than the Menger property, but there is a $\gamma$-set $Y$ in $\mathbb{R}$ such that $Y_\mathbb{S}$ is not Menger. Indeed, under $\mathfrak{p} = \mathfrak{c}$, Galvin and Miller constructed in [9] a $\gamma$-set $X \subset 2^\omega$ such that $X \setminus A$ is not Menger for some countable set $A \subset X$. By Theorem 3.8 (1) there is an embedding $e : X \to \mathbb{R}$ such that $e(X)_\mathbb{S}$ is not Menger.

The following question is open.

**Question 3.13.** Let $X$ be a zero-dimensional separable metric space. If $X$ is totally imperfect and Menger, then is there an embedding $e : X \to \mathbb{R}$ such that $e(X)_\mathbb{S}$ is Menger?

## 4. Hurewicz subsets of the Sorgenfrey line

In this section we observe analogous results on the Hurewicz property.

**Definition 4.1.** According to [21], a space $X$ has *the Hurewicz property* (or simply we say $X$ is *Hurewicz*) if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of $X$, there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$ such that every point of $X$ is contained in $\bigcup \mathcal{V}_n$ for all but finitely many $n \in \omega$.

This covering property was introduced by Hurewicz [12]. Obviously the Hurewicz property implies the Menger property. The following proposition can be proved in a similar way to Proposition 1.3, and it is necessary to show Theorem 4.4 below.

**Proposition 4.2.** *Let $X$ be a zero-dimensional Lindelöf space. Then $X$ is Hurewicz iff every continuous image of $X$ in $\omega^\omega$ is bounded.*

**Definition 4.3.** A subset $X$ of $\mathbb{R}$ is *universally meager* [28] if every Borel isomorphic image of $X$ is meager in $\mathbb{R}$. A subset $X$ of $\mathbb{R}$ is a *$\lambda$-set* [18] if every countable subset of $X$ is a $G_\delta$-set in $X$.

**Theorem 4.4.** *The following three statements hold.*

(1) *Let $X$ be a subset of $\mathbb{R}$. If $X_{\mathbb{S}}$ is Hurewicz, then $X$ is universally meager.*
(2) *Let $X$ be a zero-dimensional separable metrizable space. Then $X$ is a $\lambda$-set with the Hurewicz property iff for every embedding $e : X \to \mathbb{R}$, $e(X)_{\mathbb{S}}$ is Hurewicz.*
(3) *Let $X$ be a subset of $\mathbb{R}$. Then $X$ is hereditarily Hurewicz iff $X_{\mathbb{S}}$ is hereditarily Hurewicz.*

*Proof.* We sketch the proofs. (1) If $X_{\mathbb{S}}$ is Hurewicz, then $X$ is totally imperfect by Theorem 2.4. Since every totally imperfect Hurewicz subset of $\mathbb{R}$ is universally meager [28, Proposition 2.3], $X$ is universally meager.

(2) Using the same arguments as in Theorem 3.8 (1), we obtain that $X \setminus A$ is Hurewicz for every $A \in [X]^{\leq\omega}$ iff for every embedding $e : X \to \mathbb{R}$, $e(X)_{\mathbb{S}}$ is Hurewicz. On the other hand, $X \setminus A$ is Hurewicz for every $A \in [X]^{\leq\omega}$ iff $X$ is a $\lambda$-set with the Hurewicz property [13, Theorem 3].

(3) can be proved by the same method as in Proposition 3.5 (2). □

## 5. Totally paracompact subsets of the Sorgenfrey line

**Definition 5.1.** A space $X$ is *totally paracompact* [8] (*totally metacompact*) if every open base of $X$ contains a locally finite (point-finite) cover of $X$.

Neither the Michael line nor the Sorgenfrey line is totally metacompact [24], [20]. Every totally paracompact space is trivially totally metacompact. But the converse is not true; see [2, p. 753]. Curtis proved in [5, Theorem 3.1] that every space with the Menger property is totally paracompact. On the other hand, Lelek proved in [16] that for separable metrizable spaces, total metacompactness, total paracompactness and the Menger property are equivalent. Total paracompactness of real GO-spaces was studied by Balogh and Bennett [2], where a real GO-space is a generalized ordered space constructed on the real line $\mathbb{R}$ with the usual order. They gave a characterization of total paracompactness of real GO-spaces and showed that for real GO-spaces total metacompactness and total paracompactness are equivalent. Note that both the Michael line and the Sorgenfrey line are real GO-spaces.

Balogh and Bennett asked in [2, Problem 3.1] a necessary and sufficient condition for subspaces of a real GO-space to be totally paracompact (or totally metacompact). Concerning this problem, we show that for subspaces of the Sorgenfrey line total metacompactness, total paracompactness and the Menger property are equivalent.

We denote by $\mathbb{N}$ the set of positive integers.

**Theorem 5.2.** *For a Lindelöf subspace $X$ of a real GO-space, total paracompactness, total metacompactness and the Menger property are equivalent.*

*Proof.* Since $X$ is also a GO-space, it is the pairwise disjoint union of subsets $I, R, L$ and $E$, where

$I = \{x \in X : \{x\}$ is open in $X\}$,
$R = \{x \in X \setminus I : [x, x+r) \cap X$ is open in $X\}$,
$L = \{x \in X \setminus I : (x-r, x] \cap X$ is open in $X\}$ and
$E = X \setminus (I \cup R \cup L)$.

For convenience, we put $[x, x+r) \cap X = [x, x+r)_X, (x-r, x] \cap X = (x-r, x]_X$ and $(x-r, x+r) \cap X = (x-r, x+r)_X$.

We have only to show that total metacompactness of $X$ implies the Menger property of $X$. Let $X$ be totally metacompact. Since $X = \bigcup\{X \cap [n, n+1] : n = 0, \pm 1, \cdots\}$ and each $X \cap [n, n+1]$ is totally metacompact, we may suppose $X \subset [0, 1]$. Recall that the usual metric $d(x, y) = |x - y|$ on $[0, 1]$ is totally bounded (in particular, totally bounded on $X$). Hence for each $n \in \mathbb{N}$ there is a finite set $F_n \subset X$ such that for each $x \in X$, $d(x, F_n) < 1/2^n$.

Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of $X$. Our goal is to find finite subfamilies $\mathcal{V}_n$ $(n \in \mathbb{N})$ such that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ covers $X$. For each $x \in X \setminus I$ and $n \in \mathbb{N}$, take $U_n(x) \in \mathcal{U}_n$ containing $x$. Moreover, we define a positive real number $r_n(x)$ which satisfies (1) $r_n(x) < 1/2^n$, (2) $V_n(x) = [x, x+r_n(x))_X \subset U_n(x)$ if $x \in R$, $V_n(x) = (x - r_n(x), x]_X \subset U_n(x)$ if $x \in L$ and $V_n(x) = (x - r_n(x), x + r_n(x))_X \subset U_n(x)$ if $x \in E$. If $x \in I$, then we put $V_n(x) = \{x\}$. Let

$$\mathcal{B}_n = \{V_n(x) \cup V_n(z) : (x, z) \in (R \cup L \cup E) \times F_n\}.$$

We show that $\mathcal{B} = \{\{x\} : x \in I\} \cup \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$ is a base of $X$. Let $x \in X \setminus I$ and let $U$ be an open neighborhood of $x$. We examine only the case $x \in R$. Other cases can be proved similarly. Take $m \in \mathbb{N}$ with $[x, x + 1/2^m)_X \subset U$. Since $x \in R$, there is $y \in X$ with $x < y < x + 1/2^m$. Take $n \in \mathbb{N}$ which satisfies

(a) $n > m$,
(b) $y + 1/2^{n-1} < x + 1/2^m$ and
(c) $x < y - 1/2^{n-1}$.

Moreover for this $n \in \mathbb{N}$ take $z \in F_n$ with $d(y, z) = |y - z| < 1/2^n$. Then $V_n(x) \cup V_n(z) = [x, x + r_n(x))_X \cup V_n(z)$ is a member of $\mathcal{B}_n$. Obviously the condition (a) implies $V_n(x) = [x, x + r_n(x))_X \subset [x, x + 1/2^m)_X \subset U$. Note that the conditions (a), (b) and (c) above imply the order

$$x < y - 1/2^{n-1} < y - 1/2^n < z < y + 1/2^n < y + 1/2^{n-1} < x + 1/2^m.$$

If $z \in I$, then $V_n(z) = \{z\} \subset [x, x + 1/2^m)_X \subset U$. If $z \in R$, then $V_n(z) = [z, z + r_n(z))_X$. Observe the order

$$z + r_n(z) < y + 1/2^n + 1/2^n = y + 1/2^{n-1} < x + 1/2^m;$$

hence $V_n(z) = [z, z + r_n(z))_X \subset [x, x + 1/2^m)_X \subset U$. If $z \in L$, then $V_n(z) = (z - r_n(z), z]_X$. Observe the order

$$z - r_n(z) > y - 1/2^n - 1/2^n = y - 1/2^{n-1} > x;$$

hence $V_n(z) = (z - r_n(z), z]_X \subset [x, x + 1/2^m)_X \subset U$. The case $z \in E$ follows from the cases $z \in R$ and $z \in L$.

By total metacompactness of $X$, there are subfamilies $\mathcal{B}_n^{'} \subset \mathcal{B}_n (n \in \mathbb{N})$ such that $\bigcup\{\mathcal{B}_n^{'} : n \in \mathbb{N}\}$ is point-finite and covers $X \setminus I$. Each member of $\mathcal{B}_n$ contains a point of $F_n$. Therefore each $\mathcal{B}_n^{'}$ must be finite. Since each set of the form $V_n(x)$ is contained in a member of $\mathcal{U}_n$, there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \mathbb{N})$ such that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ covers $X \setminus I$. Since $X$ is Lindelöf, $X \setminus \bigcup\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is countable. Taking a suitable member $W_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$, we can have a cover $\bigcup\{\mathcal{V}_n \cup \{W_n\} : n \in \mathbb{N}\}$ of $X$. The proof is complete. □

**Corollary 5.3.** *For subspaces of the Sorgenfrey line, total metacompactness, total paracompactness and the Menger property are equivalent.*

**Corollary 5.4** (O'Farrell [20])**.** *The Sorgenfrey line is not totally metacompact.*

We would like to revisit some problems on total paracompactness. Curtis asked in [5, Problem 3.2] whether every Lindelöf totally paracompact space has the Menger property. Bandy asked in [3, Question 2] whether for a paracompact space (or a Lindelöf space) total paracompactness and total metacompactness are equivalent. So far as the author knows, these problems are still open.

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## References

1. A. V. Arhangel'skii, *Hurewicz spaces, analytic sets and fan tightness of function spaces*, Soviet Math. Dokl. **33**(1986), 396–399.
2. Z. Balogh and H. Bennett, *Total paracompactness of real GO-spaces*, Proc. Amer. Math. Soc. **101**(1987), 753–760. MR911046 (89a:54042)
3. C. Bandy, *A characterization of Hurewicz space*, Fund. Math. **84**(1974), 169–171. MR0339075 (49:3838)
4. T. Bartoszyński and B. Tsaban, *Hereditary topological diagonalizations and the Menger-Hurewicz conjectures*, Proc. Amer. Math. Soc. **134**(2006), 605–615. MR2176030 (2006f:54038)
5. D. W. Curtis, *Total and absolute paracompactness*, Fund. Math. **77**(1973), 277–283. MR0321005 (47:9538)
6. E. K. van Douwen, *The integers and topology*, in: Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984, 111–167. MR776619 (85k:54001)
7. R. Engelking, *General Topology*, Helderman Verlag, Berlin, 1989. MR1039321 (91c:54001)
8. R. M. Ford, *Basis properties in dimension theory*, Doctoral Dissertation, Auburn University, Auburn, AL, 1963.
9. F. Galvin and A. W. Miller, *$\gamma$-sets and other singular sets of real numbers*, Topology Appl. **17**(1984), 145–155. MR738943 (85f:54011)
10. J. Gerlits and Zs. Nagy, *Some properties of $C(X)$. I*, Topology Appl. **14**(1982), 151–161. MR667661 (84f:54021)

11. W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Mathematische Zeitschrift **24**(1926), 401–421. MR1544773
12. W. Hurewicz, *Über Folgen stetiger Funktionen*, Fund. Math. **9**(1927), 193–204.
13. F. Jordan, *There are no hereditary productive $\gamma$-spaces*, Topology Appl. **155**(2008), 1786–1791. MR2445301
14. P. Komjáth and V. Totik, *Problems and Theorems in Classical Set Theory*, Springer-Verlag, New York, 2006. MR2220838 (2007g:03057)
15. C. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966. MR0217751 (36:840)
16. A. Lelek, *Some cover properties of spaces*, Fund. Math. **64**(1964), 209–218. MR0242108 (39:3442)
17. M. K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Sitzungsberichte der Wiener Akademie, **133**(1924), 421–444.
18. A. W. Miller, *Special subsets of the real line*, in: Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984, 201–233. MR776624 (86i:54037)
19. A. W. Miller and D. H. Fremlin, *On some properties of Hurewicz, Menger and Rothberger*, Fund. Math. **129**(1988), 17–33. MR954892 (89g:54061)
20. J. M. O'Farrell, *The Sorgenfrey line is not totally metacompact*, Houston J. Math. **9**(1983), 271–273. MR703275 (84i:54039)
21. M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology Appl. **69**(1996), 31–62. MR1378387 (97h:90123)
22. M. Scheepers and B. Tsaban, *The combinatorics of Borel covers*, Topology Appl. **121**(2002), 357–382. MR1908999 (2003e:03091)
23. E. Szpilrajn (Marczewski), *Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles*, Fund. Math. **24**(1935), 17–34.
24. R. Telgársky and H. Kok, *The space of rationals is not absolutely paracompact*, Fund. Math. **73**(1971/72), 75–78. MR0293585 (45:2662)
25. B. Tsaban, *Some new directions in infinite-combinatorial topology*, in: Set Theory (J. Bagaria and S. Todorčević, eds.), Birkhäuser, Boston, 2006, 225–256. MR2267150 (2007f:03064)
26. B. Tsaban, *Selection principles and special sets of reals*, in: Open Problems in Topology II, edited by Elliott Pearl, Elsevier, 2007, 91–108.
27. L. Wingers, *Box products and Hurewicz spaces*, Topology Appl. **64**(1995), 9–21. MR1339755 (96f:54010)
28. P. Zakrzewski, *Universally meager sets*, Proc. Amer. Math. Soc. **129**(2000), 1793–1798. MR1814112 (2001m:03097)

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