CONTINUED FRACTIONS AND HEAVY SEQUENCES

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Abstract. We initiate the study of the sets \( \mathcal{H}(c) \), \( 0 < c < 1 \), of real \( x \) for which the sequence \( (kx)_{k \geq 1} \) (viewed mod 1) consistently hits the interval \([0, c)\) at least as often as expected (i.e., with frequency \( \geq c \)). More formally,

\[ \mathcal{H}(c) \overset{\text{def}}{=} \{ \alpha \in \mathbb{R} \mid \text{card}\{1 \leq k \leq n \mid \langle k\alpha \rangle < c\} \geq cn, \text{ for all } n \geq 1\}, \]

where \( \langle x \rangle = x - \lfloor x \rfloor \) stands for the fractional part of \( x \in \mathbb{R} \).

We prove that, for rational \( c \), these sets \( \mathcal{H}(c) \) are of positive Hausdorff dimension and, in particular, are uncountable. For integers \( m \geq 1 \), we obtain a surprising characterization of the numbers \( \alpha \in \mathcal{H}(1/m) \) in terms of their continued fraction expansions: The odd entries (partial quotients) of these expansions are divisible by \( m \). The characterization implies that \( x \in \mathcal{H}(m) \) if and only if \( \frac{1}{m}x \in \mathcal{H}(1) \), for \( x > 0 \). We are unaware of a direct proof of this equivalence without making use of the mentioned characterization of the sets \( \mathcal{H}(m) \).

We also introduce the dual sets \( \hat{\mathcal{H}}_m \) of reals \( y \) for which the sequence of integers \( ([ky])_{k \geq 1} \) consistently hits the set \( m\mathbb{Z} \) with the at least expected frequency \( \frac{1}{m} \) and establish the connection with the sets \( \mathcal{H}_m \):

If \( xy = m \) for \( x, y > 0 \), then \( x \in \mathcal{H}_m \iff y \in \hat{\mathcal{H}}_m \).

The motivation for the present study comes from Y. Peres’s ergodic lemma.

1. Notation and results

We write \( \mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N} \) for the sets of real numbers, rational numbers, integers and positive integers respectively.

In the paper we initiate the study of the sets \( \mathcal{H}(c) \), \( 0 < c < 1 \), of \( x \in \mathbb{R} \) for which the sequence \( (kx)_{k \geq 1} \) (viewed mod 1) consistently hits the interval \([0, c)\) at least as often as expected. More formally,

\[ \mathcal{H}(c) = \{ \alpha \in \mathbb{R} \mid \text{card}\{1 \leq k \leq n \mid \langle k\alpha \rangle < c\} \geq cn, \text{ for all } n \in \mathbb{N}\}, \]

where \( \langle x \rangle = x - \lfloor x \rfloor \) stands for the fractional part of \( x \in \mathbb{R} \). Define

\[ \mathcal{H}_m = \mathcal{H}(\frac{1}{m}), \text{ for } m \in \mathbb{N}. \]
The following notation will be used for CF (continued fraction) expansions of finite length \( n + 1 \):

\[
[a_0, a_1, a_2, \ldots, a_n] \downarrow = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}, \quad n \geq 0,
\]

or of infinite length

\[
[a_0, a_1, a_2, \ldots] \downarrow = \lim_{n \to \infty} [a_0, a_1, a_2, \ldots, a_n] \downarrow,
\]

where \( a_0 \in \mathbb{Z} \) and \( a_k \in \mathbb{N} \) for \( k \geq 1 \).

For some basic facts and standard notation from the theory of CFs we refer to [5] or [4]. (The first few pages in either book should suffice for our purposes.)

Every irrational number has a unique infinite CF expansion, and every rational number has exactly two finite CF expansions

\[
[a_0, a_1, a_2, \ldots, a_{n-1} + 1] \downarrow = [a_0, a_1, a_2, \ldots, a_{n-1}, 1] \downarrow, \quad \text{with } a_{n-1} \geq 1
\]

(with the lengths being two consecutive integers, \( n \) and \( n + 1 \)).

**Definition 1.** By the odd CF (odd continued fraction) expansion (of \( \alpha \in \mathbb{R} \)) we mean the CF expansion of length \( L \in \{\infty, 1, 3, 5, \ldots \} \). Similarly, in the even CF expansions one assumes \( L \in \{\infty, 2, 4, 6, \ldots \} \).

This way every number \( \alpha \in \mathbb{R} \) has unique both odd and even CF expansions; the two coincide if and only if \( \alpha \) is irrational. The sequence of (CF) convergents of \( \alpha \),

\[
\delta_k(\alpha) = [a_0, a_1, \ldots, a_k] \downarrow, \quad 0 \leq k < L,
\]

can be alternatively defined as the sequence of rational numbers \( \delta_k = \frac{p_k}{q_k} \) with numerators and denominators \( p_k = p_k(\alpha), q_k = q_k(\alpha) \) determined by the recurrence relations

\[
\begin{align*}
p_k &= a_k p_{k-1} + p_{k-2}, \quad 2 \leq k < L, \\
q_k &= a_k q_{k-1} + q_{k-2}, \quad \text{for } 2 \leq k < L,
\end{align*}
\]

and the initial conditions \( p_0 = a_0; q_0 = 1; \quad p_1 = a_0 a_1 + 1; q_1 = a_1 \).

The following theorem provides a criterion for the relation \( \alpha \in \mathcal{H}_m \) to hold (see [2]).

**Theorem 1.** Let \( \alpha \in \mathbb{R} \) and assume that \( \alpha = [a_0, a_1, a_2, \ldots] \downarrow \) is its odd CF expansion (i.e., of the length \( L \in \{\infty, 1, 3, 5, \ldots \} \)). Let \( m \in \mathbb{N} \) be given. Then the following three conditions are equivalent:

(C1) \( \alpha \in \mathcal{H}_m \).

(C2) \( m \mid a_k \), for all odd \( k \), \( 1 \leq k < L \).

(C3) \( m \mid q_k \), for all odd \( k \), \( 1 \leq k < L \), where \( q_k = q_k(\alpha) \) are the denominators of the convergents for \( \alpha \); see [2].

**Remark 1.** For \( m = 1 \) the above theorem holds trivially because \( \mathcal{H}_1 = \mathbb{R} \). It also holds trivially for \( \alpha \in \mathbb{Z} \) (in this case \( L = 1 \)).

**Examples.**

1. \( \alpha = \frac{4}{3} \). The odd CF expansion is \([1, 2, 1] \downarrow \), \( L = 3 \), \( a_1 = 2 \).

Thus \( \frac{4}{3} \in \mathcal{H}_m \) if and only if \( m = 1 \) or \( 2 \).
Theorem 2) between the sets in account that, for rational

Corollary 2. For real $\alpha > 0$ and $m \in \mathbb{N}$, we have $\alpha \in \mathcal{H}_m$ if and only if $\frac{\alpha}{m} \in \mathcal{H}_m$.

Both corollaries follow directly from the equivalence of (C1) and (C2) in Theorem 1; the proof of Corollary 2 also uses the identity

Theorem 2. For $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$, we have $\alpha \in \mathcal{H}_m \iff m\alpha \in \tilde{\mathcal{H}}_m$.

The proof of Theorem 2 is derived from the comparison (1) and (5) and taking into account that, for $x \in \mathbb{R}$, $\langle x \rangle \in [0, 1/m) \iff \lfloor mx \rfloor \in m\mathbb{Z}$.

Note that we establish another, deeper connection (than the one indicated in Theorem 2) between the sets $\mathcal{H}_m$ and $\tilde{\mathcal{H}}_m$ in Theorem 4 below.

The following result provides an explicit description of the sets $\tilde{\mathcal{H}}_m$.

Theorem 3. Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and assume that $\alpha = [a_0, a_1, a_2, \ldots]$ is its even CF expansion (of the length $L \in \{\infty, 2, 4, 6, \ldots\}$). Let $m \in \mathbb{N}$ be given. Then the following three conditions are equivalent:

1. $\alpha \in \mathcal{H}_m$.
2. $m \mid a_k$, for all even $k$, $0 \leq k < L$.
3. $m \mid p_k$, for all even $k$, $0 \leq k < L$ where $p_k = p_k(\alpha)$ are numerators of the convergents for $\alpha$; see (3).

The proof of Theorem 3 easily follows from Theorems 1 and 2 using the identity (4).

Alternatively, Theorem 3 can be derived from the following.

Theorem 4. For $\alpha > 0$ and $m \in \mathbb{N}$, we have $\alpha \in \mathcal{H}_m \iff \frac{1}{\alpha} \in \tilde{\mathcal{H}}_m$.

Theorem 4 follows from Corollary 2 and identity (4).

The proof of Theorem 4 will be provided in the next section. We also prove (Theorems 5 and 6) that

$$\mathcal{H}(\frac{n}{m}) \supset \mathcal{H}(\frac{1}{n}) = \mathcal{H}_m,$$

for arbitrary $n, m \in \mathbb{N}$, $n < m$.

and conclude that, for rational $c$, $0 < c < 1$, the sets $\mathcal{H}(c)$ have a positive Hausdorff dimension (Corollary 3).

Finally, in the last section we discuss briefly the motivation behind our study.

2. PROOF OF THEOREM 1

The proof is subdivided into several lemmas, some of which are of independent interest. Let $\mathbb{I}_{0,1}$ stand for the open unit interval $(0, 1)$. For $n \in \mathbb{N}$, $\alpha > 0$ and $c \in \mathbb{I}_{0,1}$, consider the following finite subsets of $\mathbb{N}$:

$$S(n, \alpha) \seteq \{k \in \mathbb{N} \mid k\alpha < n\}$$
and
\begin{equation}
S(n, \alpha, c) \overset{\text{def}}{=} \{ k \in S(n, \alpha) \mid \langle k \alpha \rangle < c \} = \{ k \in \mathbb{N} \mid k \alpha < n \& \langle k \alpha \rangle < c \}.
\end{equation}

It is easy to see that
\begin{equation}
\text{card}(S(n, \alpha)) = \left\lceil \frac{n}{\alpha} \right\rceil^{-}
\end{equation}
and
\begin{equation}
\text{card}(S(n, \alpha, c)) = \left\lceil \frac{c}{\alpha} \right\rceil^{-} + \sum_{k=1}^{n-1} \left( \left\lceil \frac{k+c}{\alpha} \right\rceil^{-} - \left\lfloor \frac{k}{\alpha} \right\rfloor^{-} \right),
\end{equation}

where $[x]^{-}$ stands for the largest integer smaller than $x \in \mathbb{R}$:
\begin{equation}
[x]^{-} \overset{\text{def}}{=} \begin{cases} [x] & \text{if } x \notin \mathbb{Z}, \\ x - 1 & \text{if } x \in \mathbb{Z}. \end{cases}
\end{equation}

We observe the following.

**Lemma 1.** Given $\alpha > 0$ and $c \in \mathbb{I}_{0,1} = (0,1)$, the following two conditions are equivalent:
\begin{enumerate}
\item $\alpha \in \mathcal{H}(c)$;
\item $\text{card}(S(n, \alpha, c)) \geq c \text{ card}(S(n, \alpha))$, for all $n \in \mathbb{N}$.
\end{enumerate}

**Proof.** The claim of Lemma follows directly from the definitions of the sets $\mathcal{H}(c)$, $S(n, \alpha)$ and $S(n, \alpha, c)$ (see (1), (6), (7)). \hfill $\Box$

**Lemma 2.** Let $\alpha, \beta > 0$ and $c \in \mathbb{I}_{0,1}$. Assume that the following two conditions are met:
\begin{enumerate}
\item $\frac{1}{\alpha} - \frac{1}{\beta} \in \mathbb{Z}$ and $\frac{c}{\alpha} - \frac{c}{\beta} \in \mathbb{Z}$.
\end{enumerate}

Then:
\begin{enumerate}
\item (A) $\text{card}(S(n, \alpha)) - \text{card}(S(n, \beta)) = \frac{n(\beta - \alpha)}{\alpha \beta}$;
\item (B) $\text{card}(S(n, \alpha, c)) - \text{card}(S(n, \beta, c)) = \frac{c(n(\beta - \alpha))}{\alpha \beta}$;
\item (C) $\alpha \in \mathcal{H}(c) \iff \beta \in \mathcal{H}(c)$.
\end{enumerate}

**Proof.** The claims (A) and (B) of the lemma follow from formulae (8), (9) and the obvious implications
\[ x - y \in \mathbb{Z} \iff \langle x \rangle = \langle y \rangle \implies [x]^{-} - [y]^{-} = x - y. \]

Finally, (C) follows from (A), (B) and Lemma \hfill $\Box$

The next lemma is just a specification of Lemma 2

**Lemma 3.** Let $\alpha, \beta > 0$, $c \in \mathbb{I}_{0,1}$, and $m \in \mathbb{N}$ be given and assume that the following two conditions are met:
\begin{enumerate}
\item (a) $\frac{1}{\alpha} - \frac{1}{\beta} \in m \mathbb{Z}$;
\item (b) $mc \in \mathbb{N}$.
\end{enumerate}

Then $\alpha \in \mathcal{H}(c)$ if and only if $\beta \in \mathcal{H}(c)$.

**Proof.** We need only validate condition (2) of Lemma 2. It follows from (a) that
\[ \frac{1}{\alpha} - \frac{1}{\beta} = km, \text{ for some } k \in \mathbb{Z}. \]
But then $\frac{c}{\alpha} - \frac{c}{\beta} = ckm = k(mc) \in \mathbb{Z}$, in view of (b).

The proof is complete. \hfill $\Box$
For \( \alpha \in \mathbb{R} \), denote by \( L(\alpha) \) the length of the odd CF expansion of \( \alpha = [a_0(\alpha), a_1(\alpha), \ldots] \). Observe that \( L(\alpha) = 1 \) if and only if \( \alpha \in \mathbb{Z} \); otherwise \( L(\alpha) \geq 3 \). Define the map \( \phi : \mathbb{R} \to \mathbb{R} \) by the rule
\[
\phi(\alpha) = \begin{cases} 
\frac{1}{(\alpha)} = [a_1, a_2, \ldots] & \text{if } \alpha \notin \mathbb{Z}, \\
0 & \text{if } \alpha \in \mathbb{Z}.
\end{cases}
\]

One easily verifies that for \( \alpha = [a_0, a_1, a_2, \ldots] \notin \mathbb{Z} \), one has \( \phi^2(\alpha) = [a_2, \ldots] \); i.e., \( \phi^2 \) removes the first two entries (partial quotients) in the odd CF expansion of any noninteger.

Next we introduce the sets
\[
\mathbb{R}(m) \overset{\text{def}}{=} \{ \alpha \in \mathbb{R} \setminus \mathbb{Z} \mid a_1(\alpha) \in m \mathbb{N} \} \cup \mathbb{Z}, \ m \in \mathbb{N}.
\]

**Lemma 4.** Let \( m \in \mathbb{N}, \ c \in \mathbb{I}_{n,1} \) and assume that \( mc \in \mathbb{N} \). Then, for every \( \alpha \in \mathbb{R}(m) \),
\[
\alpha \in \mathcal{H}(c) \iff \phi^2(\alpha) \in \mathcal{H}(c).
\]

**Proof.** Denote \( \beta = \phi^2(\alpha) \) and \( u = \frac{1}{\alpha} - \frac{1}{(\beta)} = -a_1. \) Since \( \alpha \in \mathbb{R}(m) \), we conclude that \( m|u \) and use Lemma 3 to complete the proof of Lemma 4: \( \alpha \in \mathcal{H}(c) \iff \langle \beta \rangle \in \mathcal{H}(c) \iff \beta \in \mathcal{H}(c). \)

It turns out that Lemma 4 can be used to explicitly exhibit uncountable subsets of \( \mathcal{H}(c) \), for \( c \in \mathbb{I}_{n,1} \cap \mathbb{Q} \). (\( \mathbb{Q} \) stands for the set of rational numbers.) Those are the sets
\[
\mathbb{R}_m \overset{\text{def}}{=} \{ \alpha \in \mathbb{R} \mid \phi^{2k}(\alpha) \in \mathbb{R}(m), \text{ for all } k \geq 0 \}, \ m \in \mathbb{N}
\]
(see Theorem 5 below).

The following lemma provides an alternative, more explicit description of the classes \( \mathbb{R}_m \).

**Lemma 5.** Let \( m \in \mathbb{N} \) and assume that \( \alpha \in \mathbb{R} \) is given in terms of its odd CF expansion \( \alpha = [a_0, a_1, \ldots] \) of length \( L = L(\alpha) \in \{ \infty, 1, 3, \ldots \} \). Then \( \alpha \in \mathbb{R}_m \) if and only if \( a_k \in m\mathbb{Z} \), for all odd \( k < L \).

**Proof.** The proof follows directly from the nature of the map \( \phi^2 \) and the trivial fact that \( \mathbb{Z} \subset \mathbb{R}(m) \).

**Examples.**

1. \( \alpha = \frac{4}{7} \). The odd CF expansion is \( [1, 2, 1] \) \( \downarrow \), \( L = 3 \), \( a_1 = 2 \).

   Thus \( \frac{4}{7} \in \mathbb{R}_m \iff m = 1 \) or 2.

2. \( \alpha = \frac{\sqrt{2}}{2} \). The odd CF expansion is \( [1, 8, 2, 8, 2, 8, \ldots] \) \( \downarrow \), \( L = \infty \), \( a_1 = a_3 = a_5 = \cdots = 8 \).

   Thus \( \frac{\sqrt{2}}{2} \in \mathbb{R}_m \iff m = 1, 2, 4 \) or 8.

**Theorem 5.** Let \( m \in \mathbb{N}, \ c \in \mathbb{I}_{n,1} = (0,1) \) and assume that \( cm \in \mathbb{N} \). Then \( \mathbb{R}_m \subset \mathcal{H}(c) \).

**Proof of Theorem 5.** Recall that \( \mathbb{Q} \subset \mathbb{R} \) stands for the set of rational numbers. We prove that
\[
\alpha \in \mathbb{R}_m \implies \alpha \in \mathcal{H}(c).
\]
Indeed, it follows from (1) that $\delta_1 < \frac{3}{182}$. M. Bo Sheridan and D. Ralston

Case 1. $L < \infty$, i.e. $\alpha \in \mathbb{Q}$. The proof goes by induction in $L = L(\alpha) \in \{1, 3, 5, \ldots\}$. If $L = 1$, then $\alpha \in \mathbb{Z}$ and one has both $\alpha \in \mathbb{R}_m$ and $\alpha \in \mathcal{H}(c)$. For the induction step, we use Lemma 4 and the obvious fact that $\phi^2(\mathbb{R}_m) \subset \mathbb{R}_m$ (see (13) and Lemma 5).

Case 2. $L = \infty$, i.e. $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ ($\alpha$ is irrational). The proof uses an approximation argument. For a subset $S \subset \mathbb{R}$, denote by $\overline{S}$ the closure of $S$ in $\mathbb{R}$. Next we validate the following inclusion:

\begin{equation}
\mathcal{H}(c) \cap (\mathbb{R} \setminus \mathbb{Q}) \subset \mathcal{H}(c) \quad \text{(for } c \in \mathbb{Q} \cap (0, 1)).
\end{equation}

Indeed, it follows from (14) that $\mathcal{H}(c) = \bigcap_{n \in \mathbb{N}} \mathcal{H}(c, n)$, where

\[ \mathcal{H}(c, n) = \{ \alpha \in \mathbb{R} \mid \text{card}([1 \leq m \leq n \mid \langle m \alpha \rangle < c] \geq cn \} \]

is a finite union of intervals of the form $[u, v)$ with rational endpoints $u, v$:

\[ u, v \in \bigcup_{0 \leq r < s \leq n} \left\{ \frac{r}{s}, \frac{r + c}{s} \right\} \subset \mathbb{Q} \]

because $c \in \mathbb{Q}$. In particular, for all $n \geq 1$,

\[ \mathcal{H}(c, n) \cap (\mathbb{R} \setminus \mathbb{Q}) \subset \mathcal{H}(c, n). \]

The proof of (14) is completed as follows:

\[ \mathcal{H}(c) \cap (\mathbb{R} \setminus \mathbb{Q}) = \bigcap_{n \in \mathbb{N}} \mathcal{H}(c, n) \cap (\mathbb{R} \setminus \mathbb{Q}) \subset \left( \bigcap_{n \in \mathbb{N}} \mathcal{H}(c, n) \right) \cap (\mathbb{R} \setminus \mathbb{Q}) \]

\[ = \bigcap_{n \in \mathbb{N}} \left( \mathcal{H}(c, n) \cap (\mathbb{R} \setminus \mathbb{Q}) \right) \subset \bigcap_{n \in \mathbb{N}} \mathcal{H}(c, n) = \mathcal{H}(c). \]

Next one considers the sequence $\delta_{2k} = \delta_{2k}(\alpha)$, $k \geq 0$, of even CF convergents of $\alpha$ (with $L(\delta_{2k}) = 2k + 1$, an odd number). By what has been proven in Case 1, $\delta_{2k} \in \mathbb{R}_m \cap \mathbb{Q} \subset \mathcal{H}(c)$, for all $k \geq 0$. Since $\lim_{k \to \infty} \delta_{2k} = \alpha \in \mathbb{R} \setminus \mathbb{Q}$, the proof is complete in view of (14).

\[ \square \]

Corollary 3. Let $C \subset \mathbb{Q} \cap \mathbb{N}$ be a finite subset of rational numbers. Then the intersection $\bigcap_{c \in C} \mathcal{H}(c)$ is an uncountable set of positive Hausdorff dimension.

Proof. Let $m \in \mathbb{N}$ be a common denominator for all the numbers $c \in C$. Then $\mathbb{R}_m \subset \bigcap_{c \in C} \mathcal{H}(c)$, in view of Theorem 5.

The set $\mathbb{R}_m$ is clearly uncountable, and it is easily seen to have a positive Hausdorff dimension. (One way to see it is to observe that it contains the set $\mathbb{R}_m$ of all numbers of the form $\frac{[0, m, a_1, m, a_2, m, \ldots]}{m}$, where $a_i \in \{1, 2\}$. This set is the disjoint union of its images under the two contractions:

\[ f_i(x) = \frac{1}{m + 1 + x}, \quad i = 1, 2, \]

and therefore must be of positive Hausdorff dimension (see Chapter 9 in 2).)

\[ \square \]

Corollary 4. $\mathbb{R}_m \subset \mathcal{H}_m$, for all $m \in \mathbb{N}$.

Proof of Corollary 4. Take $c = \frac{1}{m}$ in Theorem 5 (Recall that $\mathcal{H}_m = \mathcal{H}(\frac{1}{m})$.)

As is pointed out in Section 1, the inclusion in Corollary 4 can be reversed.

Theorem 6. $\mathbb{R}_m = \mathcal{H}_m$, for all $m \in \mathbb{N}$.
We first need to prove the following:

**Lemma 6.** $\mathcal{H}_m \subset \mathbb{R}(m)$, for all $m \in \mathbb{N}$.

*Proof of Lemma 6.* Since $\mathbb{Z} \subset \mathbb{R}(m)$ (see (12)), it suffices to prove that if $\alpha \in (\mathbb{R} \setminus \mathbb{Z}) \cap \mathcal{H}_m$, then $\alpha \in \mathbb{R}(m)$. We assume without loss of generality that $[\alpha] = 0$ (otherwise replacing $\alpha$ by $\langle \alpha \rangle$). Let $[0, a_1, a_2, \ldots]_4$ be the odd CF expansion of $\alpha$.

We have to show that $m \mid a_1$. If not, then $a_1 \equiv r \pmod{m}$, for some integer $r$, $1 \leq r \leq m - 1$. Define $\beta \in \mathbb{R}$ by its CF expansion $[0, r, a_2, a_3, \ldots]_4$, with all the entries $a_k$, for $k \neq 1$, being the same as in the CF expansion of $\alpha$.

Then, with $c = \frac{1}{m}$, we easily validate the conditions of Lemma 3. Indeed, $\alpha \in \mathcal{H}_m = \mathcal{H}(c)$, $mc = 1 \in \mathbb{N}$ and $\frac{1}{m} - \frac{1}{c} = a_1 - r \in m\mathbb{Z}$. By Lemma 3, $\beta \in \mathcal{H}(c) = \mathcal{H}_m$, which is impossible because $\beta = \langle \beta \rangle > \frac{1}{r+1} \geq \frac{1}{m} = c$, so that the relation $\beta \in \mathcal{H}(c)$ contradicts (11) for $n = 1$. □

*Proof of Theorem 6.* In view of Corollary 4, we only have to establish the inclusion $\mathcal{H}_m \subset \mathbb{R}_m$. Let $\alpha \in \mathcal{H}_m$ be given. We claim that then

$$\phi^{2k}(\alpha) \in \mathbb{R}(m) \cap \mathcal{H}_m, \text{ for all } k \geq 0.$$  (15)

The proof goes by induction in $k$. For $k = 0$, (15) holds in view of Lemma 3. For the inductive step, we use Lemmas 4 and 5. This completes the proof of (15). In view of (13), $\alpha \in \mathbb{R}_m$, completing the proof. □

Now we are ready to complete the proof of Theorem 6 in the introduction. The main part of the work has already been done: Theorem 6 is a rephrasing of the equivalence $(C1) \iff (C2)$.

It remains to prove the equivalence of the following two conditions:

$(C2)$ $m \mid a_k$, for all odd $k$, $1 \leq k < L$.

$(C3)$ $m \mid q_k$, for all odd $k$, $1 \leq k < L$.

The proof goes by induction in $k$. For $k = 1$ the equivalence is immediate because $q_1 = a_1$.

Now assume that both $(C2)$ and $(C3)$ hold for some odd $k = n < L - 2$. It suffices to show that

$$m \mid a_{n+2} \iff m \mid q_{n+2}.$$  

From the identity $q_{n+2} = a_{n+2}q_{n+1} + q_n$ we derive the congruence $q_{n+2} \equiv a_{n+2}q_{n+1} \pmod{m}$, so that the implication $\implies$ is immediate. The opposite implication is also valid because $q_{n+2}, q_{n+1}$ are relatively prime.

3. Motivation: Heavy sequences

The following result by Y. Peres is closely related to the Maximal Ergodic Theorem:

**Lemma 7** (Peres). Let $T : X \to X$ be a continuous transformation of a compact space, and let $\mu$ be a probability measure preserved by $T$. For every continuous $g : X \to \mathbb{R}$ there exists some $x \in X$ such that

$$\forall N \in \mathbb{N} \quad \frac{1}{N} \sum_{n=0}^{N-1} g(T^nx) \geq \int_X g d\mu.$$
This lemma can be strengthened to upper semi-continuous functions (such as characteristic functions of closed sets), and has several interesting generalizations; see [7]. The application of most interest to us right now is to extend the ideas of equation (1) to the context of dynamical systems. Let \( \{X, \mu\} \) be a compact metric space with probability Borel measure \( \mu \), and let \( T \) be a continuous map from \( X \) to itself which preserves the measure \( \mu \): \( \mu(T^{-1}A) = \mu(A) \) for any Borel set \( A \). Define the heavy set of \( f \):

\[
\mathcal{H}_T^f = \{ x \in X : S_n(x) - n \int_X f \, d\mu \geq 0, \quad \forall n \in \mathbb{N} \}. 
\]

Then Lemma [7] tells us that in this situation, for any \( f \in L^1(X, \mu) \), \( \mathcal{H}_T^f \neq \emptyset \). We also say that there is some point \( x \) whose orbit is heavy for \( f \). If \( f \) is the characteristic function of a closed set \( S \), we will generally simply refer to “the heavy set of \( S \)”, or call a sequence “heavy for \( S \)”. Restricting ourselves only to the reals modulo one, \( \mathbb{R}/\mathbb{Z} = S^1 \), we derive the following results:

**Example 1.** Fix \( \alpha \in S^1 \). Then for any closed subset \( A \subset S^1 \), there exists some point \( x \in S^1 \) such that the sequence \( \{x + n\alpha\}_{n=0}^{\infty} \) is heavy for \( A \).

The previous example can be viewed as the following: for any choice of a closed set \( A \subset S^1 \) and leading coefficient \( \alpha \), there exists some choice of \( \beta \) such that the polynomial \( \alpha n + \beta \), considered modulo one, is heavy for \( A \). This example may be generalized as follows:

**Example 2.** Fix \( \alpha \in \mathbb{R} \), a closed set \( A \subset S^1 \), and a choice of \( k \in \mathbb{N} \). Then there exists a choice of coefficients \( a_0, a_1, \ldots, a_{k-1} \) such that the sequence

\[
\{\alpha n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0\}_{n=0}^{\infty}
\]

is heavy for \( A \) (when taken modulo one). For details on how to derive this sequence as the orbit of a measure preserving system, we refer the reader to pp. 35–37 in [3] or, for a more detailed derivation of heaviness properties, to [7].

Finally, the reader may be tempted to try to generalize the results of Theorem [1] and Example 1 to claim that the set

\[
\mathcal{H}[A] \overset{\text{def}}{=} \{ x \in S^1 \mid \text{the sequence } (kx)_{k \geq 1} \text{ is heavy for } A \}
\]

is always nonempty. This cannot be done.

**Example 3.** There exists a closed set \( A \subset S^1 \), a finite union of closed intervals, whose measure is larger than \( 1/3 \), such that \( (x \in A) \Rightarrow (2x \notin A, 3x \notin A) \). In particular, for such an \( A \) one has \( \mathcal{H}[A] = \emptyset \).

The measure of the set \( A \) in the above example can be made arbitrarily close to \( \frac{1}{2} \). For details, see [7], where techniques from ergodic theory are used (in a nonconstructive way) to establish the existence of such a set \( A \).

We have examples of closed subintervals \( J \subset (0, 1) \subset S^1 \) for which the set \( \mathcal{H}[J] \) is countable or finite (\( J = \left[ \frac{1}{3}, \frac{2}{3} \right] \) and \( J = \left[ \frac{2}{5}, \frac{3}{5} \right] \), respectively). We don’t know whether it can be made empty.

Note that the subject of our paper is somehow related to that in [11], where some sufficient conditions for the one-sided boundedness of the sequence

\[
\text{card}(\{1 \leq k \leq n \mid \langle k\alpha \rangle < c \}) - cn
\]
have been established (cf. equation (1)). All our results are new and imply some of the results in [1].

References


