CONTINUED FRACTIONS AND HEAVY SEQUENCES

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Abstract. We initiate the study of the sets $\mathcal{H}(c)$, $0 < c < 1$, of real $x$ for which the sequence $(kx)_{k \geq 1}$ (viewed mod 1) consistently hits the interval $[0, c)$ at least as often as expected (i.e., with frequency $\geq c$). More formally,

$$\mathcal{H}(c) \doteq \{ \alpha \in \mathbb{R} \mid \text{card}(\{1 \leq k \leq n \mid \langle k\alpha \rangle < c\}) \geq cn, \text{ for all } n \geq 1\},$$

where $\langle x \rangle = x - \lfloor x \rfloor$ stands for the fractional part of $x \in \mathbb{R}$.

We prove that, for rational $c$, the sets $\mathcal{H}(c)$ are of positive Hausdorff dimension and, in particular, are uncountable. For integers $m \geq 1$, we obtain a surprising characterization of the numbers $\alpha \in \mathcal{H}_m \doteq \mathcal{H}(\frac{1}{m})$ in terms of their continued fraction expansions: The odd entries (partial quotients) of these expansions are divisible by $m$. The characterization implies that $x \in \mathcal{H}_m$ if and only if $\frac{1}{mx} \in \mathcal{H}_m$, for $x > 0$. We are unaware of a direct proof of this equivalence without making use of the mentioned characterization of the sets $\mathcal{H}_m$.

We also introduce the dual sets $\hat{\mathcal{H}}_m$ of reals $y$ for which the sequence of integers $([ky])_{k \geq 1}$ consistently hits the set $m\mathbb{Z}$ with the at least expected frequency $\frac{1}{m}$ and establish the connection with the sets $\mathcal{H}_m$:

If $xy = m$ for $x, y > 0$, then $x \in \mathcal{H}_m \iff y \in \hat{\mathcal{H}}_m$.

The motivation for the present study comes from Y. Peres’s ergodic lemma.

1. Notation and results

We write $\mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$ for the sets of real numbers, rational numbers, integers and positive integers respectively.

In the paper we initiate the study of the sets $\mathcal{H}(c)$, $0 < c < 1$, of $x \in \mathbb{R}$ for which the sequence $(kx)_{k \geq 1}$ (viewed mod 1) consistently hits the interval $[0, c)$ at least as often as expected. More formally,

$$\mathcal{H}(c) = \{ \alpha \in \mathbb{R} \mid \text{card}(\{1 \leq k \leq n \mid \langle k\alpha \rangle < c\}) \geq cn, \text{ for all } n \in \mathbb{N}\},$$

where $\langle x \rangle = x - \lfloor x \rfloor$ stands for the fractional part of $x \in \mathbb{R}$. Define

$$\mathcal{H}_m = \mathcal{H}(\frac{1}{m}), \text{ for } m \in \mathbb{N}.$$
The following notation will be used for CF (continued fraction) expansions of finite length \( n + 1 \):

\[
[a_0, a_1, a_2, \ldots, a_n]_n = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}, \quad n \geq 0,
\]

or of infinite length

\[
[a_0, a_1, a_2, \ldots] = \lim_{n \to \infty} [a_0, a_1, a_2, \ldots, a_n]_n,
\]

where \( a_0 \in \mathbb{Z} \) and \( a_k \in \mathbb{N} \) for \( k \geq 1 \).

For some basic facts and standard notation from the theory of CFs we refer to [5] or [4]. (The first few pages in either book should suffice for our purposes.)

Every irrational number has a unique infinite CF expansion, and every rational number has exactly two finite CF expansions

\[
[a_0, a_1, a_2, \ldots, a_{n-1} + 1]_n = [a_0, a_1, a_2, \ldots, a_{n-1}, 1]_n, \quad \text{with } a_{n-1} \geq 1
\]

(with the lengths being two consecutive integers, \( n \) and \( n + 1 \)).

**Definition 1.** By the **odd CF** (odd continued fraction) expansion (of \( \alpha \in \mathbb{R} \)) we mean the CF expansion of length \( L \in \{\infty, 1, 3, 5, \ldots\} \). Similarly, in the **even CF** expansions one assumes \( L \in \{\infty, 2, 4, 6, \ldots\} \).

This way every number \( \alpha \in \mathbb{R} \) has unique both odd and even CF expansions; the two coincide if and only if \( \alpha \) is irrational. The sequence of (CF) convergents for \( \alpha \),

\[
\delta_k(\alpha) = [a_0, a_1, \ldots, a_k], \quad 0 \leq k < L,
\]

can be alternatively defined as the sequence of rational numbers \( \delta_k = \frac{p_k}{q_k} \) with numerators and denominators \( p_k = p_k(\alpha), q_k = q_k(\alpha) \) determined by the recurrence relations

\[
\begin{cases}
p_k = a_k p_{k-1} + p_{k-2}, & \text{for } 2 \leq k < L, \\
q_k = a_k q_{k-1} + q_{k-2}, & \text{for } 2 \leq k < L,
\end{cases}
\]

and the initial conditions \( p_0 = a_0; \ q_0 = 1; \ p_1 = a_0 a_1 + 1; \ q_1 = a_1 \).

The following theorem provides a criterion for the relation \( \alpha \in \mathcal{H}_m \) to hold (see [2]).

**Theorem 1.** Let \( \alpha \in \mathbb{R} \) and assume that \( \alpha = [a_0, a_1, a_2, \ldots]_n \) is its odd CF expansion (i.e., of the length \( L \in \{\infty, 1, 3, 5, \ldots\} \)). Let \( m \in \mathbb{N} \) be given. Then the following three conditions are equivalent:

\[(C1) \ \alpha \in \mathcal{H}_m. \]

\[(C2) \ m \mid a_k, \text{ for all odd } k, \ 1 \leq k < L. \]

\[(C3) \ m \mid q_k, \text{ for all odd } k, \ 1 \leq k < L, \text{ where } q_k = q_k(\alpha) \text{ are the denominators of the convergents for } \alpha; \text{ see } [2]. \]

**Remark 1.** For \( m = 1 \) the above theorem holds trivially because \( \mathcal{H}_1 = \mathbb{R} \). It also holds trivially for \( \alpha \in \mathbb{Z} \) (in this case \( L = 1 \)).

**Examples.**

1. \( \alpha = \frac{4}{3} \). The odd CF expansion is \([1, 2, 1]_3 \), \( L = 3, a_1 = 2 \). Thus \( \frac{4}{3} \in \mathcal{H}_m \) if and only if \( m = 1 \) or \( 2 \).
(2) \( \alpha = \frac{\sqrt{5} - 1}{2} \). The odd CF expansion is \([1, 8, 2, 8, 2, 8, \ldots] \)\(^{\ast} \), \( L = \infty \), \( a_1 = a_3 = a_5 = \cdots = 8 \).
Thus \( \frac{\sqrt{5}}{2} \in \mathcal{H}_m \) if and only if \( m = 1, 2, 4 \) or \( 8 \).

**Corollary 1.** \( \mathcal{H}_m \cap \mathcal{H}_n = \mathcal{H}_{\text{LCM}(m,n)} \), for all \( m, n \in \mathbb{N} \).

**Corollary 2.** For real \( \alpha > 0 \) and \( m \in \mathbb{N} \), we have \( \alpha \in \mathcal{H}_m \) if and only if \( \frac{1}{\alpha m} \in \mathcal{H}_m \).

Both corollaries follow directly from the equivalence of (C1) and (C2) in Theorem 1, the proof of Corollary 2 also uses the identity
\[
(4) \quad m \lfloor x, mx, mx_2, mx_3, mx_4, \ldots \rfloor = [mx, x_1, mx_2, x_3, mx_4, x_5, \ldots].
\]

In the next three theorems we classify the numbers in the sets \( \mathcal{H}_m \), \( m \in \mathbb{N} \):
\[
(5) \quad \mathcal{H}_m = \{ \alpha \in \mathbb{R} \mid \text{card}(\{1 \leq k \leq n \mid [k\alpha] \in m\mathbb{Z}\}) \geq \frac{n}{m}, \text{ for all } n \in \mathbb{N} \}.
\]

**Theorem 2.** For \( \alpha \in \mathbb{R} \) and \( m \in \mathbb{N} \), we have \( \alpha \in \mathcal{H}_m \iff \alpha m \in \mathcal{H}_m \).

The proof of Theorem 2 is derived from the comparison (11) and (5) and taking account that, for \( x \in \mathbb{R} \), \( \langle x \rangle \in [0, 1/m) \iff [mx] \in m\mathbb{Z} \).

Note that we establish another, deeper connection (than the one indicated in Theorem 1) between the sets \( \mathcal{H}_m \) and \( \mathcal{H}_n \) in Theorem 1 below.

The following result provides an explicit description of the sets \( \mathcal{H}_m \).

**Theorem 3.** Let \( m \in \mathbb{N} \), \( \alpha \in \mathbb{R} \) and assume that \( \alpha = [a_0, a_1, a_2, \ldots] \) is its even CF expansion (of the length \( L \in \{\infty, 2, 4, 6, \ldots\} \)). Let \( m \in \mathbb{N} \) be given. Then the following three conditions are equivalent:

(C1) \( \alpha \in \mathcal{H}_m \).

(C2) \( m \mid a_k \), for all even \( k \), \( 0 \leq k < L \).

(C3) \( m \mid p_k \), for all even \( k \), \( 0 \leq k < L \) where \( p_k = p_k(\alpha) \) are numerators of the convergents for \( \alpha \); see (5).

The proof of Theorem 3 easily follows from Theorems 1 and 2 using the identity (4).

Alternatively, Theorem 3 can be derived from the following.

**Theorem 4.** For \( \alpha > 0 \) and \( m \in \mathbb{N} \), we have \( \alpha \in \mathcal{H}_m \iff \frac{1}{\alpha m} \in \mathcal{H}_m \).

Theorem 4 follows from Corollary 2 and identity (4).

The proof of Theorem 4 will be provided in the next section. We also prove (Theorems 5 and 6) that
\( \mathcal{H}(\frac{n}{m}) \supset \mathcal{H}(\frac{1}{m}) = \mathcal{H}_m \), for arbitrary \( n, m \in \mathbb{N} \), \( n < m \),
and conclude that, for rational \( c \), \( 0 < c < 1 \), the sets \( \mathcal{H}(c) \) have a positive Hausdorff dimension (Corollary 3).

Finally, in the last section we discuss briefly the motivation behind our study.

2. Proof of Theorem 1

The proof is subdivided into several lemmas, some of which are of independent interest. Let \( \mathbb{I}_{0,1} \) stand for the open unit interval \((0, 1)\). For \( n \in \mathbb{N} \), \( \alpha > 0 \) and \( c \in \mathbb{I}_{0,1} \), consider the following finite subsets of \( \mathbb{N} \):
\[
(6) \quad S(n, \alpha) \overset{\text{def}}{=} \{ k \in \mathbb{N} \mid k\alpha < n \}
\]
\[ S(n, \alpha, c) \stackrel{\text{def}}{=} \{ k \in S(n, \alpha) \mid \langle k\alpha \rangle < c \} = \{ k \in \mathbb{N} \mid k\alpha < n \& \langle k\alpha \rangle < c \}. \]

It is easy to see that
\[ \text{card} \{ S(n, \alpha) \} = \left\lceil \frac{n}{\alpha} \right\rceil - \]
\[ \text{and} 
\[ \text{card} \{ S(n, \alpha, c) \} = \left\lceil \frac{c}{\alpha} \right\rceil + \sum_{k=1}^{n-1} \left( \left\lceil \frac{k+c}{\alpha} \right\rceil - \left\lfloor \frac{k}{\alpha} \right\rfloor \right), \]

where \( \lfloor x \rfloor \) stands for the largest integer smaller than \( x \in \mathbb{R} \):
\[ \lfloor x \rfloor = \begin{cases} [x] & \text{if } x \notin \mathbb{Z} \\ x - 1 & \text{if } x \in \mathbb{Z}. \end{cases} \]

We observe the following.

**Lemma 1.** Given \( \alpha > 0 \) and \( c \in \mathbb{I}_{0,1} = (0, 1) \), the following two conditions are equivalent:

1. \( \alpha \in \mathcal{H}(c) \);
2. \( \text{card} \{ S(n, \alpha, c) \} \geq c \text{ card} \{ S(n, \alpha) \}, \) for all \( n \in \mathbb{N} \).

**Proof.** The claim of Lemma 1 follows directly from the definitions of the sets \( \mathcal{H}(c) \), \( S(n, \alpha) \) and \( S(n, \alpha, c) \) (see (1), (6), (7)). \( \square \)

**Lemma 2.** Let \( \alpha, \beta > 0 \) and \( c \in \mathbb{I}_{0,1} \). Assume that the following two conditions are met:
\[ (11) \quad \left( \begin{array}{c} 1 \frac{1}{\alpha} - 1 \frac{1}{\beta} \in \mathbb{Z} \\ m \in \mathbb{N} \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} n \frac{1}{\alpha} - n \frac{1}{\beta} \in \mathbb{Z} \end{array} \right). \]

Then:

- (A) \( \text{card} \{ S(n, \alpha) \} - \text{card} \{ S(n, \beta) \} = \frac{n(\beta - \alpha)}{\alpha \beta}; \)
- (B) \( \text{card} \{ S(n, \alpha, c) \} - \text{card} \{ S(n, \beta, c) \} = c \frac{n(\beta - \alpha)}{\alpha \beta}; \)
- (C) \( \alpha \in \mathcal{H}(c) \iff \beta \in \mathcal{H}(c). \)

**Proof.** The claims (A) and (B) of the lemma follow from formulae (8), (9) and the obvious implications
\[ x - y \in \mathbb{Z} \iff \langle x \rangle = \langle y \rangle \implies [x] - [y] = x - y. \]

Finally, (C) follows from (A), (B) and Lemma 1. \( \square \)

The next lemma is just a specification of Lemma 2.

**Lemma 3.** Let \( \alpha, \beta > 0 \), \( c \in \mathbb{I}_{0,1} \), \( m \in \mathbb{N} \) be given and assume that the following two conditions are met:
\[ (a) \quad \frac{1}{\alpha} - \frac{1}{\beta} \in m\mathbb{Z}, \quad \text{and} \quad (b) \quad mc \in \mathbb{N}. \]

Then \( \alpha \in \mathcal{H}(c) \) if and only if \( \beta \in \mathcal{H}(c). \)

**Proof.** We need only validate condition (2) of Lemma 2. It follows from (a) that \( \frac{1}{\alpha} - \frac{1}{\beta} = km \), for some \( k \in \mathbb{Z} \). But then \( \frac{c}{\alpha} - \frac{c}{\beta} = ckm = k(mc) \in \mathbb{Z} \), in view of (b). The proof is complete. \( \square \)
For \( \alpha \in \mathbb{R} \), denote by \( L(\alpha) \) the length of the odd CF expansion of \( \alpha = [a_0, a_1, a_2, \ldots] \). Observe that \( L(\alpha) = 1 \) if and only if \( \alpha \in \mathbb{Z} \); otherwise \( L(\alpha) \geq 3 \). Define the map \( \phi : \mathbb{R} \to \mathbb{R} \) by the rule
\[
\phi(\alpha) = \begin{cases} 
\frac{1}{(\alpha)} = [a_1, a_2, \ldots] & \text{if } \alpha \notin \mathbb{Z}, \\
0 & \text{if } \alpha \in \mathbb{Z}.
\end{cases}
\]

One easily verifies that for \( \alpha = [a_0, a_1, a_2, \ldots] \notin \mathbb{Z} \), one has \( \phi^2(\alpha) = [a_2, \ldots] \); i.e., \( \phi^2 \) removes the first two entries (partial quotients) in the odd CF expansion of any noninteger.

Next we introduce the sets
\[
\mathbb{R}(m) \overset{\text{def}}{=} \{ \alpha \in \mathbb{R} \setminus \mathbb{Z} \mid a_1(\alpha) \in m\mathbb{N} \} \cup \mathbb{Z}, \quad m \in \mathbb{N}.
\]

**Lemma 4.** Let \( m \in \mathbb{N}, \ c \in \mathbb{I}_{b,1} \) and assume that \( mc \in \mathbb{N} \). Then, for every \( \alpha \in \mathbb{R}(m) \),
\[
\alpha \in \mathcal{H}(c) \iff \phi^2(\alpha) \in \mathcal{H}(c).
\]

**Proof.** Denote \( \beta = \phi^2(\alpha) \) and \( u = \frac{1}{a} \frac{1}{(\beta)} = -a_1. \) Since \( \alpha \in \mathbb{R}(m) \), we conclude that \( m\mid u \) and use Lemma 3 to complete the proof of Lemma 4: \( \alpha \in \mathcal{H}(c) \iff \langle \beta \rangle \in \mathcal{H}(c) \iff \beta \in \mathcal{H}(c) \). \( \square \)

It turns out that Lemma 4 can be used to explicitly exhibit uncountable subsets of \( \mathcal{H}(c) \), for \( c \in \mathbb{I}_{b,1} \cap \mathbb{Q} \). (\( \mathbb{Q} \) stands for the set of rational numbers.) Those are the sets
\[
\mathbb{R}_m \overset{\text{def}}{=} \{ \alpha \in \mathbb{R} \mid \phi^{2k}(\alpha) \in \mathbb{R}(m), \text{ for all } k \geq 0 \}, \quad m \in \mathbb{N}
\]
(see Theorem 5 below).

The following lemma provides an alternative, more explicit description of the classes \( \mathbb{R}_m \).

**Lemma 5.** Let \( m \in \mathbb{N} \) and assume that \( \alpha \in \mathbb{R} \) is given in terms of its odd CF expansion \( \alpha = [a_0, a_1, a_2, \ldots] \) of length \( L = L(\alpha) \in \{\infty, 1, 3, \ldots\} \). Then \( \alpha \in \mathbb{R}_m \) if and only if \( a_k \equiv m \mathbb{Z} \), for all odd \( k < L \).

**Proof.** The proof follows directly from the nature of the map \( \phi^2 \) and the trivial fact that \( \mathbb{Z} \subset \mathbb{R}(m) \). \( \square \)

**Examples.**

1. \( \alpha = \frac{4}{5} \). The odd CF expansion is \( [1, 2, 1] \), \( L = 3 \), \( a_1 = 2 \). Thus \( \frac{4}{5} \in \mathbb{R}_m \iff m = 1 \) or 2.

2. \( \alpha = \frac{3\sqrt{2}}{2} \). The odd CF expansion is \( [1, 8, 2, 8, 2, 8, \ldots] \), \( L = \infty \), \( a_1 = a_3 = a_5 = \cdots = 8 \). Thus \( \frac{3\sqrt{2}}{2} \in \mathbb{R}_m \iff m = 1, 2, 4 \) or 8.

**Theorem 5.** Let \( m \in \mathbb{N}, \ c \in \mathbb{I}_{b,1} = (0, 1) \) and assume that \( cm \in \mathbb{N} \). Then \( \mathbb{R}_m \subset \mathcal{H}(c) \).

**Proof of Theorem 5.** Recall that \( \mathbb{Q} \subset \mathbb{R} \) stands for the set of rational numbers. We prove that
\[
\alpha \in \mathbb{R}_m \implies \alpha \in \mathcal{H}(c).
\]
Case 1. \( L < \infty \), i.e. \( \alpha \in \mathbb{Q} \). The proof goes by induction in \( L = L(\alpha) \in \{1, 3, 5, \ldots \} \).

If \( L = 1 \), then \( \alpha \in \mathbb{Z} \) and one has both \( \alpha \in \mathbb{R}_m \) and \( \alpha \in \mathcal{H}(c) \). For the inductive step, we use Lemma 4 and the obvious fact that \( \phi^2(\mathbb{R}_m) \subset \mathbb{R}_m \) (see (13) and Lemma 5).

Case 2. \( L = \infty \), i.e. \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) (\( \alpha \) is irrational). The proof uses an approximation argument. For a subset \( S \subset \mathbb{R} \), denote by \( \overline{S} \) the closure of \( S \) in \( \mathbb{R} \). Next we validate the following inclusion:

\[
(14) \quad \overline{\mathcal{H}(c)} \cap (\mathbb{R} \setminus \mathbb{Q}) \subset \mathcal{H}(c) \quad \text{for } c \in \mathbb{Q} \cap (0, 1)).
\]

Indeed, it follows from (11) that \( \mathcal{H}(c) = \bigcap_{n \in \mathbb{N}} \mathcal{H}(c, n) \), where

\[
\mathcal{H}(c, n) = \{ \alpha \in \mathbb{R} \mid \text{card}(\{1 \leq m \leq n \mid \langle ma \rangle < c\}) \geq cn \}
\]

is a finite union of intervals of the form \([u, v)\) with rational endpoints \( u, v \):

\[
u, v \in \bigcup_{0 \leq r < s \leq n} \left\{ \frac{r}{s}, \frac{r + c}{s} \right\} \subset \mathbb{Q}
\]

because \( c \in \mathbb{Q} \). In particular, for all \( n \geq 1 \),

\[
\overline{\mathcal{H}(c, n)} \cap (\mathbb{R} \setminus \mathbb{Q}) \subset \mathcal{H}(c, n).
\]

The proof of (14) is completed as follows:

\[
\overline{\mathcal{H}(c)} \cap (\mathbb{R} \setminus \mathbb{Q}) = \bigcap_{n \in \mathbb{N}} \overline{\mathcal{H}(c, n)} \cap (\mathbb{R} \setminus \mathbb{Q}) \subset \left( \bigcap_{n \in \mathbb{N}} \overline{\mathcal{H}(c, n)} \right) \cap (\mathbb{R} \setminus \mathbb{Q})
\]

\[
= \bigcap_{n \in \mathbb{N}} \left( \overline{\mathcal{H}(c, n)} \cap (\mathbb{R} \setminus \mathbb{Q}) \right) \subset \bigcap_{n \in \mathbb{N}} \mathcal{H}(c, n) = \mathcal{H}(c).
\]

Next one considers the sequence \( \delta_{2k} = \delta_{2k}(\alpha), k \geq 0 \), of even CF convergents of \( \alpha \) (with \( L(\delta_{2k}) = 2k + 1 \), an odd number). By what has been proven in Case 1, \( \delta_{2k} \in \mathbb{R}_m \cap \mathbb{Q} \subset \mathcal{H}(c) \), for all \( k \geq 0 \). Since \( \lim_{k \to \infty} \delta_{2k} = \alpha \in \mathbb{R} \setminus \mathbb{Q} \), the proof is complete in view of (14).

\[\square\]

Corollary 3. Let \( C \subset \mathbb{Q} \cap \mathbb{I}_{1,1} \) be a finite subset of rational numbers. Then the intersection \( \bigcap_{c \in C} \mathcal{H}(c) \) is an uncountable set of positive Hausdorff dimension.

Proof. Let \( m \in \mathbb{N} \) be a common denominator for all the numbers \( c \in C \). Then \( \mathbb{R}_m \subset \bigcap_{c \in C} \mathcal{H}(c) \), in view of Theorem 5.

The set \( \mathbb{R}_m \) is clearly uncountable, and it is easily seen to have a positive Hausdorff dimension. (One way to see it is to observe that it contains the set \( \mathbb{R}_m \) of all numbers of the form \([0, m, a_1, m, a_2, m, \ldots]_1 \) where \( a_i \in \{1, 2\} \). This set is the disjoint union of its images under the two contractions:

\[
f_i(x) = \frac{1}{m + 1}, \quad i = 1, 2,
\]

and therefore must be of positive Hausdorff dimension (see Chapter 9 in [2]).) \(\square\)

Corollary 4. \( \mathbb{R}_m \subset \mathcal{H}_m \), for all \( m \in \mathbb{N} \).

Proof of Corollary 4. Take \( c = \frac{1}{m} \) in Theorem 5 (Recall that \( \mathcal{H}_m = \mathcal{H}(\frac{1}{m}) \)). \(\square\)

As is pointed out in Section 1, the inclusion in Corollary 4 can be reversed.

Theorem 6. \( \mathbb{R}_m = \mathcal{H}_m \), for all \( m \in \mathbb{N} \).
We first need to prove the following:

**Lemma 6.** $\mathcal{H}_m \subset \mathbb{R}(m)$, for all $m \in \mathbb{N}$.

**Proof of Lemma 6.** Since $\mathbb{Z} \subset \mathbb{R}(m)$ (see (12)), it suffices to prove that if $\alpha \in (\mathbb{R} \setminus \mathbb{Z}) \cap \mathcal{H}_m$, then $\alpha \in \mathbb{R}(m)$. We assume without loss of generality that $[\alpha] = 0$ (otherwise replacing $\alpha$ by $\langle \alpha \rangle$). Let $[0, a_1, a_2, \ldots]_\downarrow$ be the odd CF expansion of $\alpha$.

We have to show that $m \mid a_1$. If not, then $a_1 \equiv r \,(\text{mod } m)$, for some integer $r$, $1 \leq r \leq m - 1$. Define $\beta \in \mathbb{R}$ by its CF expansion $[0, r, a_2, a_3, \ldots]_\downarrow$, with all the entries $a_k$, for $k \neq 1$, being the same as in the CF expansion of $\alpha$.

Then, with $c = \frac{1}{m}$, we easily validate the conditions of Lemma 3. Indeed, $\alpha \in \mathcal{H}_m = \mathcal{H}(c)$, $mc = 1 \in \mathbb{N}$ and $\frac{1}{n} - \frac{1}{c} = a_1 - r \in m\mathbb{Z}$ . By Lemma 3, $\beta \in \mathcal{H}(c) = \mathcal{H}_m$, which is impossible because $\beta = \langle \beta \rangle > \frac{1}{r+1} \geq \frac{1}{m} = c$, so that the relation $\beta \in \mathcal{H}(c)$ contradicts (1) for $n = 1$. \hfill \Box

**Proof of Theorem 6.** In view of Corollary 4, we only have to establish the inclusion $\mathcal{H}_m \subset \mathbb{R}_m$. Let $\alpha \in \mathcal{H}_m$ be given. We claim without loss of generality that $\alpha \in \mathcal{H}_m$.

The proof goes by induction in $k$. For $k = 0$, (13) holds in view of Lemma 6. For the inductive step, we use Lemmas 4 and 6. This completes the proof of (15). In view of (13), $\alpha \in \mathbb{R}_m$, completing the proof. \hfill \Box

Now we are ready to complete the proof of Theorem 6 in the introduction. The main part of the work has already been done: Theorem 6 is a rephrasing of the equivalence (C1) $\iff$ (C2).

It remains to prove the equivalence of the following two conditions:

(C2) $m \mid a_k$, for all odd $k$, $1 \leq k < L$.

(C3) $m \mid q_k$, for all odd $k$, $1 \leq k < L$.

The proof goes by induction in $k$. For $k = 1$ the equivalence is immediate because $q_1 = a_1$.

Now assume that both (C2) and (C3) hold for some odd $k = n < L - 2$. It suffices to show that $m \mid a_{n+2} \iff m \mid q_{n+2}$.

From the identity $q_{n+2} = a_{n+2}q_{n+1} + q_n$ we derive the congruence $q_{n+2} \equiv a_{n+2}q_{n+1} \,(\text{mod } m)$, so that the implication $\implies$ is immediate. The opposite implication is also valid because $q_{n+2}, q_{n+1}$ are relatively prime.

**3. Motivation: Heavy sequences**

The following result by Y. Peres is closely related to the Maximal Ergodic Theorem:

**Lemma 7 (Peres).** Let $T : X \to X$ be a continuous transformation of a compact space, and let $\mu$ be a probability measure preserved by $T$. For every continuous $g : X \to \mathbb{R}$ there exists some $x \in X$ such that

$$\forall N \in \mathbb{N} \quad \frac{1}{N} \sum_{n=0}^{N-1} g(T^n x) \geq \int_X g d\mu.$$
This lemma can be strengthened to upper semi-continuous functions (such as characteristic functions of closed sets), and has several interesting generalizations; see [7]. The application of most interest to us right now is to extend the ideas of equation (11) to the context of dynamical systems. Let \( \{X, \mu\} \) be a compact space with probability Borel measure \( \mu \), and let \( T \) be a continuous map from \( X \) to itself which preserves the measure \( \mu: \mu(T^{-1}A) = \mu(A) \) for any Borel set \( A \). Define the heavy set of \( f \):

\[
\mathcal{H}_T^f = \{ x \in X: S_n(x) - n \int_X f d\mu \geq 0, \quad \forall n \in \mathbb{N} \}.
\]

Then Lemma 7 tells us that in this situation, for any \( f \in L^1(X, \mu), \mathcal{H}_T^f \neq \emptyset \). We also say that there is some point \( x \) whose orbit is heavy for \( f \). If \( f \) is the characteristic function of a set \( A \), we will generally simply refer to “the heavy set of \( A \)”, or call a sequence “heavy for \( A \)”. Restricting ourselves only to the reals modulo one, \( \mathbb{R}/\mathbb{Z} = S^1 \), we derive the following results:

**Example 1.** Fix \( \alpha \in S^1 \). Then for any closed subset \( A \subset S^1 \), there exists some point \( x \) whose orbit is heavy for \( A \).

The previous example can be viewed as the following: for any choice of a closed set \( A \subset S^1 \) and leading coefficient \( \alpha \), there exists some choice of \( \beta \) such that the polynomial \( \alpha n + \beta \), considered modulo one, is heavy for \( A \). This example may be generalized as follows:

**Example 2.** Fix \( \alpha \in \mathbb{R} \), a closed set \( A \subset S^1 \), and a choice of \( k \in \mathbb{N} \). Then there exists a choice of coefficients \( a_0, a_1, \ldots, a_{k-1} \) such that the sequence

\[
\{\alpha n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0\}_{n=0}^\infty
\]

is heavy for \( A \) (when taken modulo one). For details on how to derive this sequence as the orbit of a measure preserving system, we refer the reader to pp. 35–37 in [3] or, for a more detailed derivation of heaviness properties, to [7].

Finally, the reader may be tempted to try to generalize the results of Theorem 3 and Example 1 to claim that the set

\[
\mathcal{H}[A] \overset{\text{def}}{=} \{ x \in S^1 \mid \text{the sequence } (kx)_{k \geq 1} \text{ is heavy for } A \}
\]

is always nonempty. This cannot be done.

**Example 3.** There exists a closed set \( A \subset S^1 \), a finite union of closed intervals, whose measure is larger than 1/3, such that \( (x \in A) \Rightarrow (2x \notin A, 3x \notin A) \). In particular, for such an \( A \) one has \( \mathcal{H}[A] = \emptyset \).

The measure of the set \( A \) in the above example can be made arbitrarily close to \( \frac{1}{3} \). For details, see [7], where techniques from ergodic theory are used (in a nonconstructive way) to establish the existence of such a set \( A \).

We have examples of closed subintervals \( J \subset (0, 1) \subset S^1 \) for which the set \( \mathcal{H}[J] \) is countable or finite (\( J = [\frac{1}{3}, \frac{2}{3}] \) and \( J = [\frac{2}{3}, \frac{5}{3}] \), respectively). We don’t know whether it can be made empty.

Note that the subject of our paper is somehow related to that in [4], where some sufficient conditions for the one-sided boundedness of the sequence

\[
\text{card}(\{1 \leq k \leq n \mid \langle k\alpha \rangle < c \}) - cn
\]
have been established (cf. equation (1)). All our results are new and imply some of the results in [1].

References


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