PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 10, October 2009, Pages 3177–3185
S 0002-9939(09)09625-7
Article electronically published on May 15, 2009

# CONTINUED FRACTIONS AND HEAVY SEQUENCES

MICHAEL BOSHERNITZAN AND DAVID RALSTON

(Communicated by Michael T. Lacey)

Abstract. We initiate the study of the sets $\mathcal{H}(c)$, $0<c<1$, of real $x$ for which the sequence $(kx)_{k\geq 1}$ (viewed mod 1) consistently hits the interval $[0,c)$ at least as often as expected (i. e., with frequency $\geq c$). More formally,

$$\mathcal{H}(c) \stackrel{\text{def}}{=} \big\{\alpha \in \mathbb{R} \mid \mathbf{card}\big(\{1 \leq k \leq n \mid \langle k\alpha\rangle < c\}\big) \geq cn, \quad \text{for all } n \geq 1\big\},$$

where $\langle x\rangle = x - [x]$ stands for the fractional part of $x \in \mathbb{R}$.

We prove that, for rational $c$, the sets $\mathcal{H}(c)$ are of positive Hausdorff dimension and, in particular, are uncountable. For integers $m \geq 1$, we obtain a surprising characterization of the numbers $\alpha \in \mathcal{H}_m \stackrel{\text{def}}{=} \mathcal{H}(\frac{1}{m})$ in terms of their continued fraction expansions: The odd entries (partial quotients) of these expansions are divisible by $m$. The characterization implies that $x \in \mathcal{H}_m$ if and only if $\frac{1}{mx} \in \mathcal{H}_m$, for $x > 0$. We are unaware of a direct proof of this equivalence without making use of the mentioned characterization of the sets $\mathcal{H}_m$.

We also introduce the dual sets $\widehat{\mathcal{H}}_m$ of reals $y$ for which the sequence of integers $\big([ky]\big)_{k\geq 1}$ consistently hits the set $m\mathbb{Z}$ with the at least expected frequency $\frac{1}{m}$ and establish the connection with the sets $\mathcal{H}_m$:

$$\text{If } \ xy = m \ \text{ for } \ x, y > 0, \ \text{ then } x \in \mathcal{H}_m \iff y \in \widehat{\mathcal{H}}_m.$$

The motivation for the present study comes from Y. Peres's ergodic lemma.

## 1. Notation and results

We write $\mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$ for the sets of real numbers, rational numbers, integers and positive integers respectively.

In the paper we initiate the study of the sets $\mathcal{H}(c)$, $0<c<1$, of $x \in \mathbb{R}$ for which the sequence $(kx)_{k\geq 1}$ (viewed mod 1) consistently hits the interval $[0,c)$ at least as often as expected. More formally,

$$\mathcal{H}(c) = \big\{\alpha \in \mathbb{R} \ \big| \ \mathbf{card}\big(\{1 \leq k \leq n \mid \langle k\alpha\rangle < c\}\big) \geq cn, \quad \text{for all } \ n \in \mathbb{N}\big\}, \tag{1}$$

where $\langle x\rangle = x - \big[x\big]$ stands for the fractional part of $x \in \mathbb{R}$. Define

$$\mathcal{H}_m = \mathcal{H}(\tfrac{1}{m}), \quad \text{for } m \in \mathbb{N}. \tag{2}$$

Received by the editors October 24, 2007, and, in revised form, March 7, 2008.
2000 *Mathematics Subject Classification.* Primary 11K38, 11J71, 37A30.
The second author was supported in part by NSF VIGRE grant DMS–0240058.

The following notation will be used for CF (continued fraction) expansions of finite length $n+1$:

$$\big[a_0, a_1, a_2, \dots, a_n\big]_{\downarrow} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}, \quad n \geq 0,$$

or of infinite length

$$\big[a_0, a_1, a_2, \dots\big]_{\downarrow} = \lim_{n\to\infty} \big[a_0, a_1, a_2, \dots, a_n\big]_{\downarrow},$$

where $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$ for $k \geq 1$.

For some basic facts and standard notation from the theory of CFs we refer to [5] or [4]. (The first few pages in either book should suffice for our purposes.)

Every irrational number has a unique infinite CF expansion, and every rational number has exactly two finite CF expansions

$$\big[a_0, a_1, a_2, \dots, a_{n-1}+1\big]_{\downarrow} = \big[a_0, a_1, a_2, \dots, a_{n-1}, 1\big]_{\downarrow}, \quad \text{with } a_{n-1} \geq 1$$

(with the lengths being two consecutive integers, $n$ and $n+1$).

**Definition 1.** By the *odd* CF (odd continued fraction) expansion (of $\alpha \in \mathbb{R}$) we mean the CF expansion of length $L \in \{\infty, 1, 3, 5, \dots\}$. Similarly, in the *even* CF expansions one assumes $L \in \{\infty, 2, 4, 6, \dots\}$.

This way every number $\alpha \in \mathbb{R}$ has unique both odd and even CF expansions; the two coincide if and only if $\alpha$ is irrational. The sequence of (CF) convergents of $\alpha$,

$$\delta_k(\alpha) = \big[a_0, a_1, \dots, a_k\big]_{\downarrow}, \quad 0 \leq k < L,$$

can be alternatively defined as the sequence of rational numbers $\delta_k = \frac{p_k}{q_k}$ with numerators and denominators $p_k = p_k(\alpha)$, $q_k = q_k(\alpha)$ determined by the recurrence relations

$$\begin{cases} p_k = a_k p_{k-1} + p_{k-2}, \\ q_k = a_k q_{k-1} + q_{k-2}, \end{cases} \quad \text{for } 2 \leq k < L, \tag{3}$$

and the initial conditions $p_0 = a_0$; $q_0 = 1$; $p_1 = a_0 a_1 + 1$; $q_1 = a_1$.

The following theorem provides a criterion for the relation $\alpha \in \mathcal{H}_m$ to hold (see (2)).

**Theorem 1.** *Let $\alpha \in \mathbb{R}$ and assume that $\alpha = \big[a_0, a_1, a_2, \dots\big]_{\downarrow}$ is its odd CF expansion (i.e., of the length $L \in \{\infty, 1, 3, 5, \dots\}$). Let $m \in \mathbb{N}$ be given. Then the following three conditions are equivalent:*

- **(C1)** $\alpha \in \mathcal{H}_m$.
- **(C2)** $m \,\big|\, a_k$, *for all odd $k$, $1 \leq k < L$.*
- **(C3)** $m \,\big|\, q_k$, *for all odd $k$, $1 \leq k < L$, where $q_k = q_k(\alpha)$ are the denominators of the convergents for $\alpha$; see* (3).

*Remark* 1. For $m = 1$ the above theorem holds trivially because $\mathcal{H}_1 = \mathbb{R}$. It also holds trivially for $\alpha \in \mathbb{Z}$ (in this case $L = 1$).

**Examples.** (1) $\alpha = \frac{4}{3}$. The odd CF expansion is $\big[1, 2, 1\big]_{\downarrow}$, $L = 3$, $a_1 = 2$. Thus $\frac{4}{3} \in \mathcal{H}_m$ if and only if $m = 1$ or 2.

(2) $\alpha = \frac{\sqrt{5}}{2}$. The odd CF expansion is $\big[1, 8, 2, 8, 2, 8, \dots\big]_{\downarrow}$, $L = \infty$, $a_1 = a_3 = a_5 = \dots = 8$.
Thus $\frac{\sqrt{5}}{2} \in \mathcal{H}_m$ if and only if $m = 1, 2, 4$ or $8$.

**Corollary 1.** $\mathcal{H}_m \cap \mathcal{H}_n = \mathcal{H}_{\mathrm{LCM}(m,n)}$, *for all* $m, n \in \mathbb{N}$.

**Corollary 2.** *For real* $\alpha > 0$ *and* $m \in \mathbb{N}$, *we have* $\alpha \in \mathcal{H}_m$ *if and only if* $\frac{1}{m\alpha} \in \mathcal{H}_m$.

Both corollaries follow directly from the equivalence of **(C1)** and **(C2)** in Theorem 1; the proof of Corollary 2 also uses the identity

$$m\big[x_0, mx_1, x_2, mx_3, x_4, mx_5, \dots\big]_{\downarrow} = \big[mx_0, x_1, mx_2, x_3, mx_4, x_5, \dots\big]_{\downarrow}. \tag{4}$$

In the next three theorems we classify the numbers in the sets $\widehat{\mathcal{H}}_m$, $m \in \mathbb{N}$:

$$\widehat{\mathcal{H}}_m = \big\{\alpha \in \mathbb{R} \;\big|\; \mathbf{card}\big(\{1 \le k \le n \mid \big[k\alpha\big] \in m\mathbb{Z}\}\big) \ge \tfrac{n}{m}, \quad \text{for all } n \in \mathbb{N}\big\}. \tag{5}$$

**Theorem 2.** *For* $\alpha \in \mathbb{R}$ *and* $m \in \mathbb{N}$, *we have* $\alpha \in \mathcal{H}_m \iff m\alpha \in \widehat{\mathcal{H}}_m$.

The proof of Theorem 2 is derived from the comparison (1) and (5) and taking in account that, for $x \in \mathbb{R}$, $\langle x \rangle \in [0, 1/m) \iff \big[mx\big] \in m\mathbb{Z}$.

Note that we establish another, deeper connection (than the one indicated in Theorem 2) between the sets $\mathcal{H}_m$ and $\widehat{\mathcal{H}}_m$ in Theorem 4 below.

The following result provides an explicit description of the sets $\widehat{\mathcal{H}}_m$.

**Theorem 3.** *Let* $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$ *and assume that* $\alpha = \big[a_0, a_1, a_2, \dots\big]_{\downarrow}$ *is its even CF expansion (of the length* $L \in \{\infty, 2, 4, 6, \dots\}$*). Let* $m \in \mathbb{N}$ *be given. Then the following three conditions are equivalent:*

**(C1)** $\alpha \in \widehat{\mathcal{H}}_m$.

**(C2)** $m \,\big|\, a_k$, *for all even* $k$, $0 \le k < L$.

**(C3)** $m \,\big|\, p_k$, *for all even* $k$, $0 \le k < L$ *where* $p_k = p_k(\alpha)$ *are numerators of the convergents for* $\alpha$*; see* (3).

The proof of Theorem 3 easily follows from Theorems 1 and 2 using the identity (4).

Alternatively, Theorem 3 can be derived from the following.

**Theorem 4.** *For* $\alpha > 0$ *and* $m \in \mathbb{N}$, *we have* $\alpha \in \mathcal{H}_m \iff \frac{1}{\alpha} \in \widehat{\mathcal{H}}_m$.

Theorem 4 follows from Corollary 2 and identity (4).

The proof of Theorem 1 will be provided in the next section. We also prove (Theorems 5 and 6) that

$$\mathcal{H}(\tfrac{n}{m}) \supset \mathcal{H}(\tfrac{1}{m}) = \mathcal{H}_m, \quad \text{for arbitrary } n, m \in \mathbb{N},\ n < m,$$

and conclude that, for rational $c$, $0 < c < 1$, the sets $\mathcal{H}(c)$ have a positive Hausdorff dimension (Corollary 3).

Finally, in the last section we discuss briefly the motivation behind our study.

## 2. Proof of Theorem 1

The proof is subdivided into several lemmas, some of which are of independent interest. Let $\mathbb{I}_{0,1}$ stand for the open unit interval $(0, 1)$. For $n \in \mathbb{N}$, $\alpha > 0$ and $c \in \mathbb{I}_{0,1}$, consider the following finite subsets of $\mathbb{N}$:

$$S(n, \alpha) \stackrel{\text{def}}{=} \{k \in \mathbb{N} \,|\, k\alpha < n\} \tag{6}$$

and

$$S(n,\alpha,c) \stackrel{\text{def}}{=} \{k \in S(n,\alpha) \,|\, \langle k\alpha\rangle < c\} = \{k \in \mathbb{N} \,|\, k\alpha < n \;\&\; \langle k\alpha\rangle < c\}. \tag{7}$$

It is easy to see that

$$\mathbf{card}\big(S(n,\alpha)\big) = \big[\tfrac{n}{\alpha}\big]^- \tag{8}$$

and

$$\mathbf{card}\big(S(n,\alpha,c)\big) = \big[\tfrac{c}{\alpha}\big]^- + \sum_{k=1}^{n-1} \Big(\big[\tfrac{k+c}{\alpha}\big]^- - \big[\tfrac{k}{\alpha}\big]^-\Big), \tag{9}$$

where $\big[x\big]^-$ stands for the largest integer smaller than $x \in \mathbb{R}$:

$$\big[x\big]^- \stackrel{\text{def}}{=} \begin{cases} \big[x\big] & \text{if } x \notin \mathbb{Z}. \\ x-1 & \text{if } x \in \mathbb{Z}. \end{cases} \tag{10}$$

We observe the following.

**Lemma 1.** *Given* $\alpha > 0$ *and* $c \in \mathbb{I}_{0,1} = (0,1)$, *the following two conditions are equivalent:*

(1) $\alpha \in \mathcal{H}(c)$;
(2) $\mathbf{card}\big(S(n,\alpha,c)\big) \geq c\, \mathbf{card}\big(S(n,\alpha)\big)$, *for all* $n \in \mathbb{N}$.

*Proof.* The claim of Lemma 1 follows directly from the definitions of the sets $\mathcal{H}(c)$, $S(n,\alpha)$ and $S(n,\alpha,c)$ (see (1), (6), (7)). □

**Lemma 2.** *Let* $\alpha, \beta > 0$ *and* $c \in \mathbb{I}_{0,1}$. *Assume that the following two conditions are met:*

$$(1)\ \tfrac{1}{\alpha} - \tfrac{1}{\beta} \in \mathbb{Z} \qquad \textit{and} \qquad (2)\ \tfrac{c}{\alpha} - \tfrac{c}{\beta} \in \mathbb{Z}. \tag{11}$$

*Then:*

(A) $\mathbf{card}\big(S(n,\alpha)\big) - \mathbf{card}\big(S(n,\beta)\big) = \frac{n(\beta-\alpha)}{\alpha\beta}$;
(B) $\mathbf{card}\big(S(n,\alpha,c)\big) - \mathbf{card}\big(S(n,\beta,c)\big) = c\,\frac{n(\beta-\alpha)}{\alpha\beta}$;
(C) $\alpha \in \mathcal{H}(c) \iff \beta \in \mathcal{H}(c)$.

*Proof.* The claims (A) and (B) of the lemma follow from formulae (8), (9) and the obvious implications

$$x - y \in \mathbb{Z} \iff \langle x\rangle = \langle y\rangle \implies \big[x\big]^- - \big[y\big]^- = x - y.$$

Finally, (C) follows from (A), (B) and Lemma 1. □

The next lemma is just a specification of Lemma 2.

**Lemma 3.** *Let* $\alpha, \beta > 0$, $c \in \mathbb{I}_{0,1}$, $m \in \mathbb{N}$ *be given and assume that the following two conditions are met:*

$$\text{(a)}\ \tfrac{1}{\alpha} - \tfrac{1}{\beta} \in m\mathbb{Z}, \qquad \text{(b)}\ mc \in \mathbb{N}.$$

*Then* $\alpha \in \mathcal{H}(c)$ *if and only if* $\beta \in \mathcal{H}(c)$.

*Proof.* We need only validate condition (2) of Lemma 2. It follows from (a) that $\frac{1}{\alpha} - \frac{1}{\beta} = km$, for some $k \in \mathbb{Z}$. But then $\frac{c}{\alpha} - \frac{c}{\beta} = ckm = k(mc) \in \mathbb{Z}$, in view of (b). The proof is complete. □

For $\alpha \in \mathbb{R}$, denote by $L(\alpha)$ the length of the odd CF expansion of $\alpha = \big[a_0(\alpha), a_1(\alpha), \dots\big]_{\downarrow}$. Observe that $L(\alpha) = 1$ if and only if $\alpha \in \mathbb{Z}$; otherwise $L(\alpha) \geq 3$. Define the map $\phi : \mathbb{R} \to \mathbb{R}$ by the rule

$$\phi(\alpha) = \begin{cases} \frac{1}{\langle\alpha\rangle} = \big[a_1, a_2, \dots\big]_{\downarrow} & \text{if } \alpha \notin \mathbb{Z}, \\ 0 & \text{if } \alpha \in \mathbb{Z}. \end{cases}$$

One easily verifies that for $\alpha = \big[a_0, a_1, a_2, \dots\big]_{\downarrow} \notin \mathbb{Z}$, one has $\phi^2(\alpha) = \big[a_2, \dots\big]_{\downarrow}$; i. e., $\phi^2$ removes the first two entries (partial quotients) in the odd CF expansion of any noninteger.

Next we introduce the sets

$$\mathbb{R}(m) \stackrel{\text{def}}{=} \big\{\alpha \in \mathbb{R}\backslash\mathbb{Z} \;\big|\; a_1(\alpha) \in m\,\mathbb{N}\big\} \cup \mathbb{Z}, \quad m \in \mathbb{N}. \tag{12}$$

**Lemma 4.** *Let $m \in \mathbb{N}$, $c \in \mathbb{I}_{0,1}$ and assume that $mc \in \mathbb{N}$. Then, for every $\alpha \in \mathbb{R}(m)$,*

$$\alpha \in \mathcal{H}(c) \iff \phi^2(\alpha) \in \mathcal{H}(c).$$

*Proof.* Denote $\beta = \phi^2(\alpha)$ and $u = \frac{1}{\alpha} - \frac{1}{\langle\beta\rangle} = -a_1$. Since $\alpha \in \mathbb{R}(m)$, we conclude that $m\big|u$ and use Lemma 3 to complete the proof of Lemma 4: $\alpha \in \mathcal{H}(c) \iff \langle\beta\rangle \in \mathcal{H}(c) \iff \beta \in \mathcal{H}(c)$. □

It turns out that Lemma 4 can be used to explicitly exhibit uncountable subsets of $\mathcal{H}(c)$, for $c \in \mathbb{I}_{0,1} \cap \mathbb{Q}$. ($\mathbb{Q}$ stands for the set of rational numbers.) Those are the sets

$$\mathbb{R}_m \stackrel{\text{def}}{=} \{\alpha \in \mathbb{R} \,|\, \phi^{2k}(\alpha) \in \mathbb{R}(m), \text{ for all } k \geq 0\}, \quad m \in \mathbb{N} \tag{13}$$

(see Theorem 5 below).

The following lemma provides an alternative, more explicit description of the classes $\mathbb{R}_m$.

**Lemma 5.** *Let $m \in \mathbb{N}$ and assume that $\alpha \in \mathbb{R}$ is given in terms of its odd CF expansion $\alpha = \big[a_0, a_1, \dots\big]_{\downarrow}$ of length $L = L(\alpha) \in \{\infty, 1, 3, \dots\}$. Then $\alpha \in \mathbb{R}_m$ if and only if $a_k \in m\mathbb{Z}$,* for all odd $k < L$.

*Proof.* The proof follows directly from the nature of the map $\phi^2$ and the trivial fact that $\mathbb{Z} \subset \mathbb{R}(m)$. □

**Examples.** (1) $\alpha = \frac{4}{3}$. The odd CF expansion is $\big[1, 2, 1\big]_{\downarrow}$, $L = 3$, $a_1 = 2$. Thus $\frac{4}{3} \in \mathbb{R}_m \iff m = 1$ or $2$.

(2) $\alpha = \frac{\sqrt{5}}{2}$. The odd CF expansion is $\big[1, 8, 2, 8, 2, 8, \dots\big]_{\downarrow}$, $L = \infty$, $a_1 = a_3 = a_5 = \dots = 8$.
Thus $\frac{\sqrt{5}}{2} \in \mathbb{R}_m \iff m = 1, 2, 4$ or $8$.

**Theorem 5.** *Let $m \in \mathbb{N}$, $c \in \mathbb{I}_{0,1} = (0, 1)$ and assume that $cm \in \mathbb{N}$. Then $\mathbb{R}_m \subset \mathcal{H}(c)$.*

*Proof of Theorem* 5. Recall that $\mathbb{Q} \subset \mathbb{R}$ stands for the set of rational numbers. We prove that

$$\alpha \in \mathbb{R}_m \implies \alpha \in \mathcal{H}(c).$$

*Case* 1. $L < \infty$, i.e. $\alpha \in \mathbb{Q}$. The proof goes by induction in $L = L(\alpha) \in \{1, 3, 5, \ldots\}$.

If $L = 1$, then $\alpha \in \mathbb{Z}$ and one has both $\alpha \in \mathbb{R}_m$ and $\alpha \in \mathcal{H}(c)$. For the inductional step, we use Lemma 4 and the obvious fact that $\phi^2(\mathbb{R}_m) \subset \mathbb{R}_m$ (see (13) and Lemma 5).

*Case* 2. $L = \infty$, i.e. $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ ($\alpha$ is irrational). The proof uses an approximation argument. For a subset $S \subset \mathbb{R}$, denote by $\overline{S}$ the closure of $S$ in $\mathbb{R}$. Next we validate the following inclusion:

$$\overline{\mathcal{H}(c)} \cap (\mathbb{R}\backslash\mathbb{Q}) \subset \mathcal{H}(c) \qquad (\text{for } c \in \mathbb{Q} \cap (0,1)). \tag{14}$$

Indeed, it follows from (1) that $\mathcal{H}(c) = \bigcap_{n\in\mathbb{N}} \mathcal{H}(c,n)$, where

$$\mathcal{H}(c,n) = \big\{\alpha \in \mathbb{R} \mid \mathbf{card}(\{1 \le m \le n \mid \langle m\alpha\rangle < c\}) \ge cn\big\}$$

is a finite union of intervals of the form $[u, v)$ with rational endpoints $u, v$:

$$u, v \in \bigcup_{0\le r<s\le n} \big\{\tfrac{r}{s}, \tfrac{r+c}{s}\big\} \subset \mathbb{Q}$$

because $c \in \mathbb{Q}$. In particular, for all $n \ge 1$,

$$\overline{\mathcal{H}(c,n)} \cap (\mathbb{R}\backslash\mathbb{Q}) \subset \mathcal{H}(c,n).$$

The proof of (14) is completed as follows:

$$\begin{aligned}\overline{\mathcal{H}(c)} \cap (\mathbb{R}\backslash\mathbb{Q}) &= \overline{\bigcap_{n\in\mathbb{N}} \mathcal{H}(c,n)} \cap (\mathbb{R}\backslash\mathbb{Q}) \subset \Big(\bigcap_{n\in\mathbb{N}} \overline{\mathcal{H}(c,n)}\Big) \cap (\mathbb{R}\backslash\mathbb{Q}) \\ &= \bigcap_{n\in\mathbb{N}} \Big(\overline{\mathcal{H}(c,n)} \cap (\mathbb{R}\backslash\mathbb{Q})\Big) \subset \bigcap_{n\in\mathbb{N}} \mathcal{H}(c,n) = \mathcal{H}(c).\end{aligned}$$

Next one considers the sequence $\delta_{2k} = \delta_{2k}(\alpha)$, $k \ge 0$, of even CF convergents of $\alpha$ (with $L(\delta_{2k}) = 2k+1$, an odd number). By what has been proven in Case 1, $\delta_{2k} \in \mathbb{R}_m \cap \mathbb{Q} \subset \mathcal{H}(c)$, for all $k \ge 0$. Since $\lim_{k\to\infty} \delta_{2k} = \alpha \in \mathbb{R}\backslash\mathbb{Q}$, the proof is complete in view of (14). □

**Corollary 3.** *Let $C \subset \mathbb{Q} \cap \mathbb{I}_{0,1}$ be a finite subset of rational numbers. Then the intersection $\bigcap_{c\in C} \mathcal{H}(c)$ is an uncountable set of positive Hausdorff dimension.*

*Proof.* Let $m \in \mathbb{N}$ be a common denominator for all the numbers $c \in C$. Then $\mathbb{R}_m \subset \bigcap_{c\in C} \mathcal{H}(c)$, in view of Theorem 5.

The set $\mathbb{R}_m$ is clearly uncountable, and it is easily seen to have a positive Hausdorff dimension. (One way to see it is to observe that it contains the set $\mathbb{R}'_m$ of all numbers of the form $\big[0, m, a_1, m, a_2, m, \ldots\big]_\downarrow$, where $a_i \in \{1, 2\}$. This set is the disjoint union of its images under the two contractions:

$$f_i(x) = \frac{1}{m + \frac{1}{i+x}}, \quad i = 1, 2,$$

and therefore must be of positive Hausdorff dimension (see Chapter 9 in [2]).) □

**Corollary 4.** $\mathbb{R}_m \subset \mathcal{H}_m$, *for all* $m \in \mathbb{N}$.

*Proof of Corollary* 4. Take $c = \frac{1}{m}$ in Theorem 5. (Recall that $\mathcal{H}_m = \mathcal{H}(\frac{1}{m})$.) □

As is pointed out in Section 1, the inclusion in Corollary 4 can be reversed.

**Theorem 6.** $\mathbb{R}_m = \mathcal{H}_m$, *for all* $m \in \mathbb{N}$.

We first need to prove the following:

**Lemma 6.** *$\mathcal{H}_m \subset \mathbb{R}(m)$, for all $m \in \mathbb{N}$.*

*Proof of Lemma* 6. Since $\mathbb{Z} \subset \mathbb{R}(m)$ (see (12)), it suffices to prove that if $\alpha \in \big(\mathbb{R}\backslash\mathbb{Z}\big) \cap \mathcal{H}_m$, then $\alpha \in \mathbb{R}(m)$. We assume without loss of generality that $\big[\alpha\big] = 0$ (otherwise replacing $\alpha$ by $\langle\alpha\rangle$). Let $\big[0, a_1, a_2, \dots\big]_\downarrow$ be the odd CF expansion of $\alpha$. We have to show that $m \,\big|\, a_1$. If not, then $a_1 \equiv r \pmod m$, for some integer $r$, $1 \le r \le m-1$. Define $\beta \in \mathbb{R}$ by its CF expansion $\big[0, r, a_2, a_3, \dots\big]_\downarrow$, with all the entries $a_k$, for $k \ne 1$, being the same as in the CF expansion of $\alpha$.

Then, with $c = \frac{1}{m}$, we easily validate the conditions of Lemma 3. Indeed, $\alpha \in \mathcal{H}_m = \mathcal{H}(c)$, $mc = 1 \in \mathbb{N}$ and $\frac{1}{\alpha} - \frac{1}{\beta} = a_1 - r \in m\mathbb{Z}$ . By Lemma 3, $\beta \in \mathcal{H}(c) = \mathcal{H}_m$, which is impossible because $\beta = \langle\beta\rangle > \frac{1}{r+1} \ge \frac{1}{m} = c$, so that the relation $\beta \in \mathcal{H}(c)$ contradicts (1) for $n = 1$. □

*Proof of Theorem* 6. In view of Corollary 4, we only have to establish the inclusion $\mathcal{H}_m \subset \mathbb{R}_m$. Let $\alpha \in \mathcal{H}_m$ be given. We claim that then

$$\phi^{2k}(\alpha) \in \mathbb{R}(m) \cap \mathcal{H}_m, \quad \text{for all} \quad k \ge 0. \tag{15}$$

The proof goes by induction in $k$. For $k = 0$, (15) holds in view of Lemma 6. For the inductional step, we use Lemmas 4 and 6. This completes the proof of (15). In view of (13), $\alpha \in \mathbb{R}_m$, completing the proof. □

Now we are ready to complete the proof of Theorem 1 in the introduction. The main part of the work has already been done: Theorem 6 is a rephrasing of the equivalence **(C1)** $\Longleftrightarrow$ **(C2)**.

It remains to prove the equivalence of the following two conditions:

**(C2)** $m \,\big|\, a_k$, for all odd $k$, $1 \le k < L$.

**(C3)** $m \,\big|\, q_k$, for all odd $k$, $1 \le k < L$.

The proof goes by induction in $k$. For $k = 1$ the equivalence is immediate because $q_1 = a_1$.

Now assume that both **(C2)** and **(C3)** hold for some odd $k = n < L - 2$. It suffices to show that

$$m \,\big|\, a_{n+2} \iff m \,\big|\, q_{n+2}.$$

From the identity $q_{n+2} = a_{n+2}q_{n+1} + q_n$ we derive the congruence $q_{n+2} \equiv a_{n+2}q_{n+1} \pmod m$, so that the implication $\Longrightarrow$ is immediate. The opposite implication is also valid because $q_{n+2}, q_{n+1}$ are relatively prime.

## 3. Motivation: Heavy sequences

The following result by Y. Peres is closely related to the Maximal Ergodic Theorem:

**Lemma 7** (Peres)**.** *Let $T : X \to X$ be a continuous transformation of a compact space, and let $\mu$ be a probability measure preserved by $T$. For every continuous $g : X \to \mathbb{R}$ there exists some $x \in X$ such that*

$$\forall N \in \mathbb{N} \qquad \frac{1}{N}\sum_{n=0}^{N-1} g(T^n x) \ge \int_X g d\mu.$$

This lemma can be strengthened to upper semi-continuous functions (such as characteristic functions of closed sets), and has several interesting generalizations; see [7]. The application of most interest to us right now is to extend the ideas of equation (1) to the context of dynamical systems. Let $\{X, \mu\}$ be a compact space with probability Borel measure $\mu$, and let $T$ be a continuous map from $X$ to itself which preserves the measure $\mu$: $\mu(T^{-1}A) = \mu(A)$ for any Borel set $A$. Define the *heavy set of $f$*:

$$\mathcal{H}_T^f = \left\{x \in X\colon\ S_n(x) - n\int_X f d\mu \geq 0,\ \ \forall n \in \mathbb{N}\right\}. \tag{16}$$

Then Lemma 7 tells us that in this situation, for any $f \in L^1(X, \mu)$, $\mathcal{H}_T^f \neq \emptyset$. We also say that there is some point $x$ whose orbit is *heavy for $f$*. If $f$ is the characteristic function of a set $A$, we will generally simply refer to "the heavy set of $A$", or call a sequence "heavy for $A$". Restricting ourselves only to the reals modulo one, $\mathbb{R}/\mathbb{Z} = S^1$, we derive the following results:

**Example 1.** Fix $\alpha \in S^1$. Then for any closed subset $A \subset S^1$, there exists some point $x \in S^1$ such that the sequence $\{x + n\alpha\}_{n=0}^{\infty}$ is heavy for $A$.

The previous example can be viewed as the following: for any choice of a closed set $A \subset S^1$ and leading coefficient $\alpha$, there exists some choice of $\beta$ such that the polynomial $\alpha n + \beta$, considered modulo one, is heavy for $A$. This example may be generalized as follows:

**Example 2.** Fix $\alpha \in \mathbb{R}$, a closed set $A \subset S^1$, and a choice of $k \in \mathbb{N}$. Then there exists a choice of coefficients $a_0, a_1, \ldots, a_{k-1}$ such that the sequence

$$\{\alpha n^k + a_{k-1}n^{k-1} + \cdots + a_1 n + a_0\}_{n=0}^{\infty}$$

is heavy for $A$ (when taken modulo one). For details on how to derive this sequence as the orbit of a measure preserving system, we refer the reader to pp. 35–37 in [3] or, for a more detailed derivation of heaviness properties, to [7].

Finally, the reader may be tempted to try to generalize the results of Theorem 1 and Example 1 to claim that the set

$$\mathcal{H}[A] \stackrel{\text{def}}{=} \{x \in S^1 \mid \text{ the sequence } (kx)_{k\geq 1} \text{ is heavy for } A\}$$

is always nonempty. This cannot be done.

**Example 3.** There exists a closed set $A \subset S^1$, a finite union of closed intervals, whose measure is larger than $1/3$, such that $(x \in A) \Rightarrow (2x \notin A,\, 3x \notin A)$. In particular, for such an $A$ one has $\mathcal{H}[A] = \emptyset$.

The measure of the set $A$ in the above example can be made arbitrarily close to $\frac{1}{2}$. For details, see [7], where techniques from ergodic theory are used (in a nonconstructive way) to establish the existence of such a set $A$.

We have examples of closed subintervals $J \subset (0, 1) \subset S^1$ for which the set $\mathcal{H}[J]$ is countable or finite ($J = [\frac{1}{3}, \frac{2}{3}]$ and $J = [\frac{2}{7}, \frac{5}{7}]$, respectively). We don't know whether it can be made empty.

Note that the subject of our paper is somehow related to that in [1], where some sufficient conditions for the one-sided boundedness of the sequence

$$\mathbf{card}(\{1 \leq k \leq n \mid \langle k\alpha\rangle < c\}) - cn$$

have been established (cf. equation (1)). All our results are new and imply some of the results in [1].

## References

[1] Y. Dupain, T. Vera Sós, On the one-sided boundedness of discrepancy-function of the sequence $\{n\alpha\}$, *Acta Arith.* **37** (1980), 363–374. MR598889 (82c:10058)

[2] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons, Ltd., 1990. MR1102677 (92j:28008)

[3] H. Furstenberg, *Recurrence in ergodic theorem and combinatorial number theory*, Princeton University Press, 1981. MR603625 (82j:28010)

[4] A. Y. Khinchin, *Continued Fractions*, The University of Chicago Press, 1964. MR0161833 (28:5037)

[5] S. Lang, *Introduction to Diophantine Approximations*, Springer-Verlag, 1995. MR1348400 (96h:11067)

[6] Y. Peres, A combinatorial application of the maximal ergodic theorem, *Bull. London Math. Soc.* **20** (1988), 248–252. MR931186 (89e:28033)

[7] D. Ralston, Heaviness—An Extension of a Lemma of Y. Peres, *Houston Journal of Mathematics*. To appear.

Department of Mathematics, Rice University, Houston, Texas 77005
*E-mail address*: michael@rice.edu

Department of Mathematics, Rice University, Houston, Texas 77005
*Current address*: Department of Mathematics, Ohio State University, 231 W. 18th Avenue, Columbus, Ohio 43210