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COMMUTATORS OF WEIGHTED HARDY OPERATORS ON $\mathbb{R}^n$

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Abstract. The purpose of this paper is to establish a sufficient and necessary condition on weight functions which ensures the boundedness of the commutators of weighted Hardy operators (with symbols in $\mathrm{BMO}(\mathbb{R}^n)$) on $L^p(\mathbb{R}^n)$, where $1 < p < \infty$.

1. Introduction

Let f be in $L^p(\mathbb{R}^1)$, and let the Hardy operator U be defined by

$$Uf(x) = \frac{1}{x}\int_0^x f(t)\,dt,\ x \neq 0.$$

It is obvious that $|Uf| \leq Mf$, where M is the Hardy-Littlewood maximal operator. A celebrated Hardy integral inequality [8] can be formulated as

$$\|Uf\|_{L^p(\mathbb{R}^1)} \leq \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^1)}, \tag{1.1}$$

where $1 < p < \infty$ and the constant $\frac{p}{p-1}$ is the best possible.

The Hardy integral inequality has received considerable attention. A number of papers involved its alternative proofs, various generalizations, variants and applications. For the earlier development of these kinds of inequalities and many important applications in analysis, we refer to the book [8]. Among numerous papers dealing with such inequalities, we choose to refer to the papers [1], [11], [14] and [15].

Definition 1.1. Let $b \in L_{loc}(\mathbb{R}^n)$. We say that $b \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if

$$\sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b - b_Q| < \infty,$$

where $b_Q = \frac{1}{|Q|}\int_Q b$. The BMO norm of b is defined by

$$\|b\|_* = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b - b_Q|.$$

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In 1984, Carton-Lebrun and Fosset [3] defined the weighted Hardy operator U_ψ. Let us now recall its definition. Let $\psi : [0,1] \to [0,\infty)$ be a function, and let f be a measurable complex-valued function on $\mathbb{R}^n$. The weighted Hardy operator U_ψ is defined by

$$U_\psi f(x) = \int_0^1 f(tx)\psi(t)\,dt, \quad x \in \mathbb{R}^n.$$

Under certain conditions on ψ, Carton-Lebrun and Fosset [3] found that U_ψ is bounded from $\mathrm{BMO}(\mathbb{R}^n)$ into itself, moreover, commutes with the Hilbert transform in the case $n = 1$ and with a certain Calderón-Zygmund singular integral operator (and thus with the Riesz transform) in the case $n \geq 2$.

Recently, Xiao [19] obtained that U_ψ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\mathcal{A} := \int_0^1 t^{-n/p}\psi(t)dt < \infty. \tag{1.2}$$

Meanwhile, the corresponding operator norm was worked out. The result seems to be of interest as it is related closely to the Hardy integral inequality. For example, if $\psi \equiv 1$ and $n = 1$, then U_ψ may be reduced to the Hardy operator U mentioned above, and (1.1) can be deduced immediately. In [19], Xiao also obtained the $\mathrm{BMO}(\mathbb{R}^n)$ bounds of U_ψ, which sharpened and extended the main result of Carton-Lebrun and Fosset in [3].

More recently, great attention was paid to the study on commutators of operators. A well-known result of Coifman, Rochberg and Weiss [4] states that the commutator

$$T_b f = bTf - T(bf)$$

(where T is a Calderón-Zygmund singular integral operator) is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Many results have been generalized to commutators of other operators, not only Calderón-Zygmund singular integral operators (see [2], [5], [6], [9], [10], [12] and [17]).

Let us first give the definition of commutators of weighted Hardy operators as follows.

Definition 1.2. Let b be a measurable, locally integrable function and $\psi : [0,1] \to [0,\infty)$ be a function. We define the commutator of the weighted Hardy operator, U_ψ^b, as

$$U_\psi^b f := bU_\psi f - U_\psi(bf).$$

The weighted Cesàro operator V_ψ is defined by

$$V_\psi f(x) = \int_0^1 f(x/t)t^{-n}\psi(t)\,dt, \quad x \in \mathbb{R}^n.$$

When $\psi \equiv 1$ and $n = 1$, V_ψ is the classical Cesàro operator:

$$Vf(x) = \begin{cases} \displaystyle\int_x^\infty \frac{f(y)}{y}\,dy, & x > 0, \\ \displaystyle -\int_{-\infty}^x \frac{f(y)}{y}\,dy, & x < 0. \end{cases}$$

The weighted Hardy operator U_ψ and the weighted Cesàro operator V_ψ are mutually adjoint,

$$\int_{\mathbb{R}^n} g(x)U_\psi f(x)\,dx = \int_{\mathbb{R}^n} f(x)V_\psi g(x)\,dx, \tag{1.3}$$

where $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p < \infty, 1/p + 1/q = 1$. We can see [19] for more details.

Definition 1.3. Let b be a measurable, locally integrable function and $\psi : [0,1] \to [0,\infty)$ be a function. We define the commutator of the weighted Cesàro operator, V_ψ^b, as

$$V_\psi^b f := V_\psi(bf) - bV_\psi f.$$

According to (1.3), U_ψ^b and V_ψ^b satisfy

$$\int_{\mathbb{R}^n} g(x) U_\psi^b f(x)\, dx = \int_{\mathbb{R}^n} f(x) V_\psi^b g(x)\, dx, \tag{1.4}$$

where $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p < \infty, 1/p + 1/q = 1$. Thus, U_ψ^b and V_ψ^b are also adjoint.

In general, when symbols are in $\mathrm{BMO}(\mathbb{R}^n)$, the properties of commutators are worse than those of the operators themselves (for example, the singularity, [16]). Therefore, we imagine that the condition (1.2) on weight functions cannot ensure the boundedness of U_ψ^b (with symbol $b \in \mathrm{BMO}(\mathbb{R}^n)$) on $L^p(\mathbb{R}^n)$. In this paper, we try to find a sufficient and necessary condition on weight functions ψ which ensures that U_ψ^b is bounded on $L^p(\mathbb{R}^n)$, where $1 < p < \infty$.

The crux of our idea is to control the commutators of weighted Hardy operators by the Hardy-Littlewood maximal operators instead of common methods in the study of commutators of singular integrals by using sharp maximal functions to control the commutators. Then we prove that the condition on weight functions is also necessary.

In this paper, C will often be used to denote a constant, but C may not be the same constant from one occurrence to the next.

2. Main results

For simplicity, we define

$$\mathcal{B} := \int_0^1 t^{-n/p} \psi(t) \log \frac{1}{t}\, dt$$

and

$$\mathcal{C} := \int_0^1 t^{-n/p} \psi(t) \log \frac{2}{t}\, dt.$$

Theorem 2.1. *Let $\psi : [0,1] \to [0,\infty)$ be a function and $1 < p < \infty$. Then the following statements are equivalent:*

(i) *U_ψ^b is bounded on $L^p(\mathbb{R}^n)$ for all $b \in \mathrm{BMO}(\mathbb{R}^n)$;*

(ii) *$\mathcal{C} < \infty$.*

Remark. For $\mathcal{C} = \mathcal{A} \log 2 + \mathcal{B}$, it is easy to see that the condition $\mathcal{C} < \infty$ implies $\mathcal{A} < \infty$. But $\mathcal{A} < \infty$ does not imply $\mathcal{C} < \infty$. For example, let $0 < \alpha < 1$. Take

$$e^{s(n/p-1)} \tilde{\psi}(s) = \begin{cases} s^{-1+\alpha}, & 0 < s \le 1, \\ s^{-1-\alpha}, & 1 < s < \infty, \\ 0, & s = 0, \infty \end{cases}$$

and $\psi(t) = \tilde{\psi}(\log \frac{1}{t})$, $0 \le t \le 1$. Then $\mathcal{A} < \infty$ and $\mathcal{C} = \infty$.

By (1.4), we can deduce the following result.

Theorem 2.2. *Let $\psi : [0,1] \to [0,\infty)$ be a function, $1 < q < \infty$ and $\mathcal{D} := \int_0^1 t^{-n(1-1/q)}\psi(t)\log\frac{2}{t}\,dt$. Then the following statements are equivalent:*

(i) *V_ψ^b is bounded on $L^q(\mathbb{R}^n)$ for all $b \in \mathrm{BMO}(\mathbb{R}^n)$;*

(ii) *$\mathcal{D} < \infty$.*

We start with a technical lemma that is a result of Torre and Torrea (Lemma 1.10 in [18]), with slight modifications.

Lemma 2.3. *Let C be a positive constant, Q_1 and Q_2 be two balls such that if Q is the smallest ball containing both Q_1 and Q_2, then $|Q| \le C|Q_i|, i = 1, 2$. Then for each function $b \in \mathrm{BMO}(\mathbb{R}^n)$, we have*

$$|b_{Q_1} - b_{Q_2}| \le 2C\|b\|_*.$$

Proof. It is clear that

$$|b_{Q_1} - b_{Q_2}| \le |b_{Q_1} - b_Q| + |b_Q - b_{Q_2}|.$$

Now

$$|b_{Q_1} - b_Q| \le \frac{C}{|Q|}\int_Q |b - b_Q| \le C\|b\|_*.$$

The other term can be handled in the same way. □

Lemma 2.4. *Let $1 \le p < \infty$. The John-Nirenberg Lemma inequality implies that*

$$\sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b - b_Q|^p \sim \|b\|_*^p.$$

Define the Hardy-Littlewood maximal operator M as

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|dy.$$

It is well known that M is a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Let us now give the proof of Theorem 2.1.

Proof. (i)⇒(ii) We assume that $\|U_\psi^b\|_{p\to p} \le C\|b\|_* < \infty$. For $\mathcal{C} = \mathcal{A}\log 2 + \mathcal{B}$, we will show that $\mathcal{A} < \infty$ and $\mathcal{B} < \infty$, respectively.

Set $b = \log|x| \in \mathrm{BMO}(\mathbb{R}^n)$. Then $\|U_\psi^b\|_{p\to p} < \infty$. For any $0 < \epsilon < 1$, take

$$f_\epsilon(x) = \begin{cases} 0, & |x| \le 1, \\ |x|^{-n/p-\epsilon}, & |x| > 1. \end{cases}$$

Then

$$\|f_\epsilon\|_p^p = \frac{c_n}{p\epsilon}, \quad c_n = \frac{n\pi^{n/2}}{\Gamma(1+n/2)}$$

and

$$(U_\psi^b f_\epsilon)(x) = \begin{cases} 0, & |x| \le 1, \\ |x|^{-n/p-\epsilon}\int_{1/|x|}^1 t^{-n/p-\epsilon}\psi(t)\log\frac{1}{t}\,dt, & |x| > 1. \end{cases}$$

Putting $\delta = \epsilon^{-1} > 1$ and making some simple computations, we have

$$\begin{aligned}\|U_\psi^b f_\epsilon\|_p^p &= \int_{|x|>1} \left(|x|^{-n/p-\epsilon} \int_{1/|x|}^1 t^{-n/p-\epsilon}\psi(t) \log \frac{1}{t}\, dt \right)^p dx \\ &\geq \int_{|x|>\delta} \left(|x|^{-n/p-\epsilon} \int_{1/\delta}^1 t^{-n/p-\epsilon}\psi(t) \log \frac{1}{t}\, dt \right)^p dx \\ &= \epsilon^{\epsilon p} \|f_\epsilon\|_p^p \left(\int_\epsilon^1 t^{-n/p-\epsilon}\psi(t) \log \frac{1}{t}\, dt \right)^p.\end{aligned}$$

So,

$$\|U_\psi^b f_\epsilon\|_p \geq \epsilon^\epsilon \|f_\epsilon\|_p \int_\epsilon^1 t^{-n/p-\epsilon}\psi(t) \log \frac{1}{t}\, dt.$$

Letting $\epsilon \to 0^+$, we have

$$\mathcal{B} \leq \|U_\psi^b\|_{p\to p} < \infty. \tag{2.1}$$

On the other hand, consider the family of functions b given by

$$b(x) = 1\chi_{Q(0,1)}(x)\mathrm{sgn}(\sin \pi d|x|), \quad d \in \mathbb{N}.$$

Then $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\|U_\psi^b\|_{p\to p} < \infty$.

Take $f = 1\chi_{Q(0,1)}$. Then,

$$U_\psi^b f(x) = b(x) \int_0^1 \psi(t)\, dt - \int_0^{\min\{1,1/|x|\}} \psi(t)\mathrm{sgn}(\sin \pi dt|x|)\, dt.$$

By the Riemann-Lebesgue Lemma, we can take d very large, depending on ψ, so large that

$$|U_\psi^b f(x)| \geq \frac{1}{2} \int_0^1 \psi(t)\, dt, \quad \frac{1}{2} < |x| < 1.$$

Hence, we have $\int_0^1 \psi(t)\, dt \leq C\|U_\psi^b\|_{p\to p} < \infty$. Moreover, $\int_{\frac{1}{2}}^1 \psi(t)\, dt \leq \int_0^1 \psi(t)\, dt < \infty$. For $t^{-n/p}$ locally integrable at $t = 1$, we have

$$\int_{1/2}^1 t^{-n/p}\psi(t)\, dt < \infty.$$

By (2.1), we have

$$\int_0^{1/2} t^{-n/p}\psi(t)\, dt \leq C \int_0^{1/2} t^{-n/p}\psi(t) \log \frac{1}{t}\, dt \leq C\mathcal{B} < \infty.$$

Hence,

$$\mathcal{A} < \infty. \tag{2.2}$$

Combining (2.1) and (2.2), we get the desired result.

(ii)⇒(i) We assume $\mathcal{C} < \infty$. Fix a ball $Q \subset \mathbb{R}^n$ and $x \in Q$. By Fubini's theorem, we get

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |U_\psi^b f(y)|dy &\le \frac{1}{|Q|}\int_Q\int_0^1 |(b(y)-b(ty))f(ty)|\,\psi(t)dtdy \\
&= \frac{1}{|Q|}\int_0^1\int_Q |(b(y)-b(ty))f(ty)|\,dy\psi(t)dt \\
&\le \frac{1}{|Q|}\int_0^1\int_Q |(b(y)-b_Q)f(ty)|\,dy\psi(t)dt \\
&\quad +\frac{1}{|Q|}\int_0^1\int_Q |(b_Q-b_{tQ})f(ty)|\,dy\psi(t)dt \\
&\quad +\frac{1}{|Q|}\int_0^1\int_Q |(b(ty)-b_{tQ})f(ty)|\,dy\psi(t)dt \\
&:= I_1+I_2+I_3.
\end{aligned}$$

Suppose $1 < r < p$. By Hölder's inequality $(1/r + 1/r' = 1)$, we have

$$\begin{aligned}
I_1 &\le \int_0^1 \left(\frac{1}{|Q|}\int_Q |f(ty)|^r\,dy\right)^{1/r}\left(\frac{1}{|Q|}\int_Q |b(y)-b_Q|^{r'}\,dy\right)^{1/r'}\psi(t)dt \\
&\le C\|b\|_*\int_0^1 \left(\frac{1}{|tQ|}\int_{tQ} |f(y)|^r\,dy\right)^{1/r}\psi(t)dt \\
&\le C\|b\|_*\int_0^1 (Mf^r(tx))^{1/r}\psi(t)dt.
\end{aligned}$$

Here, set $Q = Q(x_Q, r_Q)$. tQ denotes $Q(tx_Q, tr_Q)$.

Similarly, we have

$$\begin{aligned}
I_3 &\le \int_0^1 \left(\frac{1}{|Q|}\int_Q |f(ty)|^r\,dy\right)^{1/r}\left(\frac{1}{|Q|}\int_Q |b(ty)-b_{tQ}|^{r'}\,dy\right)^{1/r'}\psi(t)dt \\
&= \int_0^1 \left(\frac{1}{|tQ|}\int_{tQ} |f(y)|^r\,dy\right)^{1/r}\left(\frac{1}{|tQ|}\int_{tQ} |b(y)-b_{tQ}|^{r'}\,dy\right)^{1/r'}\psi(t)dt \\
&\le C\|b\|_*\int_0^1 (Mf^r(tx))^{1/r}\psi(t)dt.
\end{aligned}$$

Now let us estimate I_2. Select an appropriate real number $a \in (1, 2]$ such that $a^{-i}Q$ and $a^{-i-1}Q$, $i \in \mathbb{N}\cup\{0\}$, are two balls whose intersection is not empty. For $1 < r < p$, we obtain

$$\begin{aligned}
I_2 &= \int_0^1 \left(\frac{1}{|Q|}\int_Q |f(ty)|\,dy\right)|b_Q - b_{tQ}|\psi(t)dt \\
&\le \int_0^1 (Mf^r(tx))^{1/r}|b_Q - b_{tQ}|\psi(t)dt \\
&= \sum_{k=0}^{\infty}\int_{a^{-k-1}}^{a^{-k}} (Mf^r(tx))^{1/r}|b_Q - b_{tQ}|\psi(t)dt \\
&\le \sum_{k=0}^{\infty}\int_{a^{-k-1}}^{a^{-k}} (Mf^r(tx))^{1/r}\left\{\left(\sum_{i=0}^{k}|b_{a^{-i}Q} - b_{a^{-i-1}Q}|\right)+|b_{a^{-k-1}Q} - b_{tQ}|\right\}\psi(t)dt.
\end{aligned}$$

By Lemma 2.3, we have $|b_{a^{-i}Q}-b_{a^{-i-1}Q}| \le C\|b\|_*$, $i \in \mathbb{N}\cup\{0\}$. For $t \in [a^{-k-1}, a^{-k}]$, we have $|b_{a^{-k-1}Q} - b_{tQ}| \le C\|b\|_*$. Thus

$$\begin{aligned} I_2 &\le C\|b\|_* \sum_{k=0}^{\infty} \int_{a^{-k-1}}^{a^{-k}} (k+1)(Mf^r(tx))^{1/r}\psi(t)dt \\ &\le C\|b\|_* \left\{ \left(\sum_{k=0}^{\infty} \int_{a^{-k-1}}^{a^{-k}} (Mf^r(tx))^{1/r}\psi(t)\log a^k dt \right) + \int_0^1 (Mf^r(tx))^{1/r}\psi(t)dt \right\} \\ &\le C\|b\|_* \int_0^1 (Mf^r(tx))^{1/r}\psi(t)(1+\log\frac{1}{t})dt. \end{aligned}$$

Combining the estimates of I_1, I_2 and I_3, we have

$$\frac{1}{|Q|}\int_Q |U_\psi^b f(y)|dy \le C\|b\|_* \int_0^1 (Mf^r(tx))^{1/r}(1+\log\frac{1}{t})\psi(t)dt.$$

Take the supremum over all Q such that $x \in Q$, and take the L^p norm of both sides of the inequality above. We get

$$\left\|M\left(U_\psi^b f\right)(\cdot)\right\|_p \le C\|b\|_* \left\| \int_0^1 (Mf^r(t\cdot))^{1/r}(1+\log\frac{1}{t})\psi(t)dt \right\|_p .$$

The Minkowski inequality yields

$$\begin{aligned} \left\|M\left(U_\psi^b f\right)(\cdot)\right\|_p &\le C\|b\|_* \left(\int_{\mathbb{R}^n} \left(\int_0^1 (Mf^r(tx))^{1/r}\psi(t)(1+\log\frac{1}{t})dt \right)^p dx \right)^{1/p} \\ &\le C\|b\|_* \int_0^1 \left(\int_{\mathbb{R}^n} (Mf^r(tx))^{p/r}dx \right)^{1/p} \psi(t)(1+\log\frac{1}{t})dt \\ &\le C\|b\|_* \int_0^1 \left(\int_{\mathbb{R}^n} (Mf^r(x))^{p/r}dx \right)^{1/p} t^{-n/p}\psi(t)(1+\log\frac{1}{t})dt \\ &\le C\|b\|_*\|f\|_p \int_0^1 t^{-n/p}\psi(t)(1+\log\frac{1}{t})dt. \end{aligned}$$

It is well known that $|U_\psi^b f(x)| \le M\left(U_\psi^b f\right)(x)$ *a.e.* We arrive at

$$\|U_\psi^b\|_{p\to p} \le C\|b\|_* \int_0^1 t^{-n/p}\psi(t)(1+\log\frac{1}{t})dt.$$

On the other hand, from the Remark following Theorem 2.1, we can get

$$\int_0^1 t^{-n/p}\psi(t)(1+\log\frac{1}{t})dt \le C\int_0^1 t^{-n/p}\psi(t)\log\frac{2}{t}dt.$$

This finishes the proof of Theorem 2.1. □

3. Higher order commutators

Given $k \ge 0$ and a vector $\vec{b} = (b_1, ..., b_k)$, we define the higher order commutator of the weighted Hardy operator as

$$U_\psi^{\vec{b}} f(x) = \int_0^1 (\prod_{j=1}^k (b_j(x) - b_j(tx)))f(tx)\psi(t)\,dt, \quad x \in \mathbb{R}^n.$$

When $k = 0$, we understand that $U_\psi^{\vec{b}} = U_\psi$. Notice that if $k = 1$, then $U_\psi^{\vec{b}} = U_\psi^b$.

Using the method in the proof of Theorem 2.1 as well as induction, we can get a similar result about the higher order commutator of the weighted Hardy operator.

The notation $\vec{b} \in \mathrm{BMO}(\mathbb{R}^n)$ below will mean that all $b_i \in \mathrm{BMO}(\mathbb{R}^n)$ for $1 \le i \le k$.

Theorem 3.1. *Let $\psi : [0,1] \to [0,\infty)$ be a function, $1 < p < \infty$ and $\mathcal{C}' := \int_0^1 t^{-n/p}\psi(t)\left(\log \frac{2}{t}\right)^k dt$. Then the following statements are equivalent:*

(i) *$U_\psi^{\vec{b}}$ is bounded on $L^p(\mathbb{R}^n)$ for all $\vec{b} \in \mathrm{BMO}(\mathbb{R}^n)$;*

(ii) *$\mathcal{C}' < \infty$.*

Obviously, if $k = 0$ in Theorem 3.1, $\mathcal{C}' < \infty$ is equivalent to (1.2), and Theorem 3.1 becomes Theorem 2.1 if $k = 1$.

Similarly, we can also define the higher order commutator of the weighted Cesàro operator as

$$V_\psi^{\vec{b}} f(x) = \int_0^1 \left(\prod_{j=1}^k (b_j(x/t) - b_j(x))\right) f(x/t)\psi(t)\,dt, \quad x \in \mathbb{R}^n.$$

We have

Theorem 3.2. *Let $\psi : [0,1] \to [0,\infty)$ be a function, $1 < q < \infty$ and $\mathcal{D}' := \int_0^1 t^{-n(1-1/q)}\psi(t)\left(\log \frac{2}{t}\right)^k dt$. Then the following statements are equivalent:*

(i) *$V_\psi^{\vec{b}}$ is bounded on $L^q(\mathbb{R}^n)$ for all $\vec{b} \in \mathrm{BMO}(\mathbb{R}^n)$;*

(ii) *$\mathcal{D}' < \infty$.*

Some comments are further put forward in the following part. In [13], Lu and Yang introduced the definition of central bounded mean oscillation space.

Definition 3.3. Let $1 < q < \infty$. A function $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the space $\dot{\mathrm{CMO}}^q(\mathbb{R}^n)$ if

$$\sup_{r>0}\left(|Q(0,r)|^{-1}\int_{Q(0,r)} |f(x) - f_Q|^q dx\right)^{1/q} := \|b\|_{\dot{\mathrm{CMO}}^q(\mathbb{R}^n)} < \infty,$$

where

$$f_Q := |Q(0,r)|^{-1}\int_{Q(0,r)} f(x)dx.$$

Remark. $\mathrm{BMO}(\mathbb{R}^n) \subsetneq \dot{\mathrm{CMO}}^q(\mathbb{R}^n)$, where $1 \le q < \infty$.

The space $\dot{\mathrm{CMO}}^q(\mathbb{R}^n)$ can be regarded as a local version of BMO at the origin, but they have quite different properties. By Lemma 2.4, we have $\|b\|_* \sim \sup_{Q\subset\mathbb{R}^n}\left\{\frac{1}{|Q|}\int_Q |b - b_Q|^p\right\}^{1/p}$. However, the spaces $\dot{\mathrm{CMO}}^q(\mathbb{R}^n)$ depend on q. If $1 < q_1 < q_2 < \infty$, then $\dot{\mathrm{CMO}}^{q_2}(\mathbb{R}^n) \subsetneq \dot{\mathrm{CMO}}^{q_1}(\mathbb{R}^n)$. Therefore, there is no analogy of the celebrated John-Nirenberg inequality of BMO for the space $\dot{\mathrm{CMO}}^q(\mathbb{R}^n)$. One can imagine that the behavior of $\dot{\mathrm{CMO}}^q(\mathbb{R}^n)$ may be quite different from that of $\mathrm{BMO}(\mathbb{R}^n)$.

Up until now, we have found that there are two versions of generalization for the classical Hardy operator in the higher-dimensional case. One of them is the n-dimensional Hardy operator. The problem of whether the n-dimensional Hardy operator can characterize the central mean oscillation space has been proved by Fu, Liu, Lu and Wang [7]. The result in [7] can be regarded as a generalization of Long

and Wang's result [12] in the higher-dimensional case. Another is the weighted Hardy operator which is the object of study in this paper. Since $\mathrm{BMO}(\mathbb{R}^n) \subsetneq \dot{\mathrm{CMO}}^q(\mathbb{R}^n)$ when $b \in \dot{\mathrm{CMO}}^q(\mathbb{R}^n)$, then what kinds of conditions on the weight functions might ensure the boundedness of U_ψ^b on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ is still an interesting question.

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