PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 10, October 2009, Pages 3259–3269
S 0002-9939(09)09899-2
Article electronically published on March 24, 2009

DISCRIMINANTS OF CHEBYSHEV-LIKE POLYNOMIALS AND THEIR GENERATING FUNCTIONS

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(Communicated by Ken Ono)

Abstract. In his paper of 2000, Kenneth B. Stolarsky made various observations and conjectures about discriminants and generating functions of certain types of Chebyshev-like polynomials. We prove several of these conjectures. One of our proofs involves Wilf-Zeilberger pairs and a contiguous relation for hypergeometric series.

1. Introduction

We begin by recalling the notion of a discriminant. Suppose that $P_m(x)$ is a polynomial of degree m whose roots are x_1, x_2, …, x_m. Then the disciminant of $P_m(x)$, which we will denote by $\Delta_x P_m(x)$, is

$$\Delta_x P_m(x) = a_m^{2m-2} \prod_{1\le i<j\le m} (x_i - x_j)^2,$$

where a_m is the leading coefficient. If $P_m(x)$ is monic, it is easy to prove the useful fact that

$$\Delta_x P_m(x) = (-1)^{m(m-1)/2} P'(x_1)\cdots P'(x_m). \tag{1}$$

The study of discriminants and resultants of specific types of polynomials has a long history and includes contributions from both Stieltjes and Hilbert (see [1] and [4] respectively). Formulas for various specific types of polynomials are given, e.g., in [1], [2], [3], [4], and [5]. The importance of this subject is discussed in the introduction to [5]. Here we contribute to this area by establishing some of the conjectures in [9].

The discriminant of the product of the two polynomials

$$K_n(x,q) = (1+x)^{2n} + qx^n$$

and

$$f_m(x) = (x^{2m+1} - 1)/(x-1)$$

has some remarkable properties in terms of roots, generating functions and divisibility. The following observation of Stolarsky limits the possible range of the roots of the resulting polynomial [9].

Received by the editors November 13, 2008, and, in revised form, January 27, 2009.
2000 *Mathematics Subject Classification.* Primary 11C08.

TABLE 1. The polynomials $H_m^{(n)}(q)$.

- $n = 1$
  $H_0^{(1)}(q) = 1$
  $H_1^{(1)}(q) = 1 + q$
  $H_2^{(1)}(q) = 1 + 3q + q^2$
  $H_3^{(1)}(q) = 1 + 6q + 5q^2 + q^3$
  $H_4^{(1)}(q) = 1 + 10q + 15q^2 + 7q^3 + q^4$

- $n = 2$
  $H_0^{(2)}(q) = 1$
  $H_1^{(2)}(q) = 1 + q$
  $H_2^{(2)}(q) = 1 + 7q + q^2$
  $H_3^{(2)}(q) = 1 + 26q + 13q^2 + q^3$
  $H_4^{(2)}(q) = 1 + 70q + 87q^2 + 19q^3 + q^4$

- $n = 3$
  $H_0^{(3)}(q) = 1$
  $H_1^{(3)}(q) = 1 + q$
  $H_2^{(3)}(q) = 1 + 18q + q^2$
  $H_3^{(3)}(q) = 1 + 129q + 38q^2 + q^3$
  $H_4^{(3)}(q) = 1 + 571q + 627q^2 + 58q^3 + q^4$

Proposition (K. Stolarsky, 2002)**.** *Let $a < b$ be real. Let $K(x, q)$ be any polynomial in x and q such that*
(i) $K(x, q)$ is never zero for $|x| = 1$ unless $a \le q \le b$;
(ii) $K(x, q)$ has no multiple zeros unless $a \le q \le b$.

Say $f_m(x)$ is a sequence of polynomials with no multiple zeros, and the zeros of $f_m(x)$ all lie on $|x| = 1$. Then

$$g_m(q) := \Delta_x(K(x, q)f_m(x))$$

is a sequence of polynomials in q whose roots are all in $[a, b]$.

An application of this theorem shows that the discriminant of the product of the two polynomials $K_n(x, q)$ and $f_m(x)$ defined above has all real roots in $[-2^{2n}, 0]$. Calculation of this discriminant gives rise to the following polynomial:

$$H_m^{(n)}(q) := \prod_{k=1}^{m} (q + (2\cos k\theta_m + 2)^n),$$

where

$$\theta_m = 2\pi/(2m + 1).$$

See Table 1 for a short tabulation of these polynomials for $1 \le n \le 3$. It immediately suggests that for $n = 1$ we have a familiar type of Fibonacci polynomial (for $q = 1$ the values are 1, 2, 5, 13, 34, etc.). However, for $n \ge 2$ the nature of these polynomials is much less obvious.

This polynomial has several interesting properties conjectured by Stolarsky. Some of these properties are clear from Proposition 2.3. The previously conjectured formulas for the generating functions of $H_m^{(1)}(q)$ and $H_m^{(2)}(q)$ are obtained in Sections 3 and 4. It is still unknown to the author what the generating function of $H_m^{(n)}(q)$ is for $n \geq 3$. Stolarsky [9] conjectured the following generating function for $H_m^{(3)}(q)$:

$$\frac{(1-t)^7 - qt^2(1-t)(t+3)(1+3t)}{(1-t)^8 - qt(1-t)^2(1+14t+34t^2+14t^3+t^4) + x^2t^4}.$$

To find these generating functions, we use some knowledge about hypergeometric series and the Wilf-Zeilberger algorithm. A hypergeometric series is defined as

$${}_pF_q\left(\begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array}; x\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{x^n}{n!}.$$

Euler obtained the following contiguous relation [1, equation (2.5.3)] for a hypergeometric series ${}_2F_1$:

$$\begin{aligned} {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array}; u\right) &= \frac{c + (a-b+1)u}{c} {}_2F_1\left(\begin{array}{c} a+1, b \\ c+1 \end{array}; u\right) \\ &\quad - \frac{(a+1)(c-b+1)u}{c(c+1)} {}_2F_1\left(\begin{array}{c} a+2, b \\ c+2 \end{array}; u\right). \end{aligned}$$

A Wilf-Zeilberger pair (F, G) [7] satisfies the equation

$$F(m+1, i) - F(m, i) = G(m, i+1) - G(m, i).$$

By telescoping summation, if $G(m, a) = G(m, b+1) = 0$, then $\sum_{i=a}^{b} F(m, i)$ does not depend on m. We will use this information to prove the identity

$$\sum_{i=0}^{2k} (-1)^i \binom{m+k+i}{m+k-i} \binom{m+3k-i}{m-k+i} = (-1)^k \sum_{i=0}^{m} \binom{2k}{m-i} \binom{4k+i}{i}$$

in Section 5. This identity is crucial for finding the generating function of $H_m^{(2)}(q)$.

2. A general form of the discriminant

In this section we will compute the discriminant of $K_n(x, q)$. For further computations of the resultants and discriminants of different kinds of polynomials, see [2], [3], [4], and [8]. We first note that in the cases $q = 0$ and $q = -2^{2n}$ the polynomial $K_n(x, q)$ has multiple roots. These roots arise solely from the $K_n(x, q)$ polynomial. So one expects that $\Delta_x(K_n f_m)$ has factors q and $(q + 2^{2n})$. Other factors of $\Delta_x(K_n f_m)$ appear when $K_n(x, q)$ and $f_m(x)$ have common roots.

Proposition 2.1. *The discriminant of $K_n(x, q)$ is given by*

$$\Delta_x(K_n(x, q)) = n^{2n} q^{2n-1} (q + 2^{2n}).$$

Proof. Let x_l be a root of $K_n(x)$ (we suppress the parameter q for a moment). Then

$$\begin{aligned} (1 + x_l) K_n'(x_l) &= 2n(1 + x_l)^{2n} + nqx_l^{n-1}(1 + x_l) \\ &= -2nqx_l^n + nqx_l^{n-1}(1 + x_l) \\ &= nqx_l^{n-1}(1 - x_l). \end{aligned}$$

Since $(-1)^{2n(2n-1)/2} = (-1)^n$, it follows that

$$(-1)^n \Delta_x(K_n(x,q)) \prod_{l=1}^{2n}(1+x_l) = n^{2n} q^{2n} \prod_{l=1}^{2n}(1-x_l).$$

Moreover we have

$$\prod_{l=1}^{2n}(1+x_l) = K_n(-1,q) = (-1)^n q$$

and

$$\prod_{l=1}^{2n}(1-x_l) = K_n(1,q) = q + 4^n,$$

and the proof follows. □

Proposition 2.2. *The discriminant of $f_m(x)$ is*

$$\Delta_x f_m(x) = (-1)^m (2m+1)^{2m-1}.$$

Proof. According to (1) we have

$$\begin{aligned} \Delta_x f_m(x) &= (-1)^{2m(2m-1)/2} \prod_{k=1}^{2m} f'(e^{ik\theta_m}) \\ &= (-1)^m \prod_{k=1}^{2m} \frac{(2m+1)}{e^{ik\theta_m}-1}. \end{aligned}$$

Since the factors in the denominator are the nonzero roots of $(x+1)^{2m+1} = 1$, their product is $2m+1$, and the proposition follows. □

Proposition 2.3. *The discriminant of $K_n(x,q) f_m(x)$ is*

$$\begin{aligned} \Delta_x(K_n f_m) &= C_m^{(n)} q^{2n-1}(q+2^{2n}) \prod_{k=1}^{m} (q + (2\cos k\theta_m + 2)^n)^4 \\ &= C_m^{(n)} q^{2n-1}(q+2^{2n}) H_m^{(n)}(q)^4, \end{aligned}$$

where

$$C_m^{(n)} = (-1)^m (2m+1)^{2m-1} n^{2n}.$$

Proof. From the definition of the discriminant we note that

$$\Delta_x(K_n f_m) = \Delta_x(K_n)\Delta_x(f_m) \prod_{\substack{1\le k\le 2m \\ 1\le l\le 2n}} (x_l - e^{ik\theta_m})^2,$$

where the x_l's are roots of K_n. Thus by Propositions 2.1 and 2.2 it suffices to show that

$$\begin{aligned} \prod_{l, 1\le k\le 2m} (x_i - e^{ik\theta_m})^2 &= \prod_{k=1}^{m} (q + (2\cos k\theta + 2)^n)^4 \\ &= \prod_{k=1}^{2m} (q + (e^{ik\theta_m} + e^{-ik\theta_m} + 2)^n)^2. \end{aligned}$$

From the definition of K_n, we can list all roots of this polynomial in pairs (x_l, x_l^{-1}) so that

$$e^{il\phi_n + i\pi/n} \sqrt[n]{|q|} = x_l + x_l^{-1} + 2,$$

where $\phi_n = 2\pi/n$ and $1 \le l \le n$. Thus

$$\begin{aligned} q + (e^{ik\theta_m} + e^{-ik\theta_m} + 2)^n &= (e^{ik\theta_m} + e^{-ik\theta_m} + 2)^n - (e^{i\pi/n}\sqrt[n]{|q|})^n \\ &= \prod_{l=1}^{n}((e^{ik\theta_m} + e^{-ik\theta_m} + 2) - e^{il\phi_n + i\pi/n}\sqrt[n]{|q|}) \\ &= \prod_{l=1}^{n}((e^{ik\theta_m} + e^{-ik\theta_m} + 2) - (x_l + x_l^{-1} + 2)) \\ &= \prod_{l=1}^{n}\frac{(x_l - e^{ik\theta_m})(x_l^{-1} - e^{ik\theta_m})}{e^{ik\theta_m}}. \end{aligned}$$

Since $\prod_{1\le k\le 2m} e^{ik\theta_m} = 1$, the proof follows. □

It follows that all roots of $\Delta_x(K_n f_m)$ stay in the range $[-2^n, 0]$. This is a special case of Stolarsky's proposition stated in the introductory section.

Corollary. *For any m, the $H^{(n)}_{3m+1}(q)$ polynomials are divisible by $(q+1)$.*

Proof. Consider

$$H^{(n)}_{3m+1}(q) = \prod_{k=1}^{3m+1}(q + (2\cos k\theta_{3m+1} + 2)^n),$$

where

$$\theta_{3m+1} = \frac{2\pi}{6m+3}\ .$$

When $k = 2m+1$ we have $(2\cos k\theta_{3m+1} + 2)^n = 1$. Thus $(q+1) \mid H^{(n)}_{3m+1}(q)$. □

Corollary. *For any n,*

$$H^{(n)}_m(q) \mid H^{(n)}_{3m+1}(q).$$

Proof. This is clear since the terms $k = 3, 6, 9, \ldots, 3m$ in the product for $H^{(n)}_{3m+1}(q)$ give $H^{(n)}_m(q)$. □

3. Generating function for $H^{(1)}_m(q)$

It is not hard to show that $H^{(1)}_m(q)$ has a generating function similar to that of the closely related Chebyshev polynomials. We give the details both for the sake of completeness and because they are useful in the rather harder analysis required for $H^{(2)}_m(q)$.

Proposition 3.1. *The polynomials $H^{(1)}_m(x)$ satisfy*

$$\frac{1-t}{(1-t)^2 - xt} = 1 + \sum_{m=1}^{\infty} H^{(1)}_m(x)t^m\ .$$

Proof. Recall the generating function definition of the Chebyshev T-polynomial $T_n(x)$:

$$\frac{1-xt}{1+t^2-2tx} = 1 + \sum_{m=1}^{\infty} T_m(x)t^m.$$

By replacing x by $(x+2)/2$, multiplying both sides by 2 and subtracting 1 from each side, we obtain

$$\frac{1-t}{(1-t)^2-xt} = \frac{1}{1+t}\left(1+\sum_{m=1}^{\infty} 2T_m\left(\frac{x+2}{2}\right)t^m\right).$$

So it remains to prove that $H_m^{(1)}(x)$ is twice an alternating sum of Chebyshev polynomials $T_m((x+2)/2)$ plus or minus 1 depending on the parity of m, i.e.

(2)

$$H_m^{(1)}(x) = 2T_m\left(\frac{x+2}{2}\right) - 2T_{m-1}\left(\frac{x+2}{2}\right) + 2T_{m-2}\left(\frac{x+2}{2}\right) - \cdots \mp 2T_1\left(\frac{x+2}{2}\right) \pm 1.$$

By writing $2\cos m\theta = z^m + z^{-m}$, where $z = e^{i\theta}$, we obtain

$$S_m(z) := 2\cos m\theta - 2\cos(m-1)\theta + 2\cos(m-2)\theta - \cdots \mp 2\cos\theta \pm 1 = \frac{z^{2m+1}+1}{z^m(z+1)}.$$

$S_m(z)$ has roots $e^{-ik\theta_m}$, so twice the alternating sum of the Chebyshev polynomials plus or minus 1 has roots x such that

$$\frac{x+2}{2} = -\cos k\theta_m$$

or $x = -2\cos k\theta_m - 2$. These roots correspond to roots of $H_m^{(1)}(x)$. This completes the proof by noting that both sides of (2) are monic polynomials in x. □

Proposition 3.1 leads to an explicit formula for the coefficients of $H_m^{(1)}(x)$. This formula is not new, but will be useful in finding the generating function for $H_m^{(2)}(x)$, which will be discussed in Section 3.

Proposition 3.2. *The polynomial $H_m^{(1)}(x)$ is given by the formula*

$$H_m^{(1)}(x) = \sum_{k=0}^{m} \binom{m+k}{m-k} x^k.$$

Proof. Let $a_{k,m}$ be the coefficients of the polynomial $H_m^{(1)}(x)$. Then by interchanging summation, we have the following identity:

$$\begin{aligned}\sum_{m=0}^{\infty} H_m^{(1)}(x)t^m &= \sum_{m=0}^{\infty}\sum_{k=0}^{m} a_{k,m}x^k t^m \\ &= \sum_{k=0}^{\infty} x^k t^k \sum_{m=0}^{\infty} a_{k,m+k}t^m.\end{aligned}$$

On the other hand, by expanding the generating function for $H_m^{(1)}(x)$ in terms of x, we obtain

$$\begin{aligned}\frac{1-t}{(1-t)^2-xt} &= \frac{1}{1-t}\frac{1}{1-xt/(1-t)^2} \\ &= \sum_{k=0}^{\infty} x^k t^k/(1-t)^{2k+1} \\ &= \sum_{k=0}^{\infty} x^k t^k \sum_{m=0}^{\infty}\binom{m+2k}{m} t^m.\end{aligned}$$

By equating the two double summations, we find that

$$a_{k,m+k} = \binom{m+2k}{m},$$

and the proposition follows. □

4. Generating function for $H_m^{(2)}(x)$

Stolarsky [9] conjectures that the generating function of $H_m^{(2)}(x)$ will have the form

$$\frac{(1-t)^3}{(1-t)^4 - xt(t+1)^2}.$$

Our main approach to prove this formula is to express $H_m^{(2)}(x)$ in terms of $H_m^{(1)}(x)$ and apply Proposition 3.2 in the previous section. In particular, the connection between $H_m^{(2)}(x)$ and $H_m^{(1)}(x)$ is given by

$$H_m^{(2)}(-x^2) = H_m^{(1)}(x)H_m^{(1)}(-x).$$

This equation easily follows from the definition of $H_m^{(n)}(x)$. For the rest of this section, we use the following identity:

$(*)$

$$\sum_{i=0}^{2k}(-1)^i \binom{m+k+i}{m+k-i}\binom{m+3k-i}{m-k+i} = (-1)^k \sum_{i=0}^{m}\binom{2k}{m-i}\binom{4k+i}{i}.$$

We will provide a proof for this identity in Section 5.

Proposition 4.1. *The polynomials $H_m^{(2)}(x)$ satisfy*

$$\frac{(1-t)^3}{(1-t)^4 - xt(t+1)^2} = 1 + \sum_{m=1}^{\infty} H_m^{(2)}(x)t^m.$$

Proof. Recall that $H_m^{(2)}(-x^2) = H_m^{(1)}(x)H_m^{(1)}(-x)$. So $H_m^{(2)}(-x^2)$ is an even polynomial of degree $2m$ whose k^{th}-coefficient is given by

$$\sum_{i=0}^{k}(-1)^i \binom{m+i}{m-i}\binom{m+k-i}{m-k+i}.$$

By interchanging summations as in the proof of Proposition 3.2, we obtain

$$\sum_{m=0}^{\infty} H_m^{(2)}(-x^2)t^m = \sum_{k=0}^{\infty} x^{2k}t^k \sum_{m=0}^{\infty}\sum_{i=0}^{2k}(-1)^i \binom{m+k+i}{m+k-i}\binom{m+3k-i}{m-k+i} t^m.$$

Upon expanding the function

$$\frac{(1-t)^3}{(1-t)^4 + x^2t(t+1)^2}$$

in terms of x first and then in terms of t, one obtains

$$\begin{aligned}\frac{(1-t)^3}{(1-t)^4+x^2t(t+1)^2} &= \sum_{k=0}^{\infty}(-1)^k x^{2k}t^k(t+1)^{2k}/(1-t)^{4k+1}\\ &= \sum_{k=0}^{\infty}(-1)^k x^{2k}t^k\sum_{i=0}^{2k}\binom{2k}{i}t^i\sum_{j=0}^{\infty}\binom{4k+j}{j}t^j\\ &= \sum_{k=0}^{\infty}(-1)^k x^{2k}t^k\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{2k}{m-i}\binom{4k+i}{i}t^m.\end{aligned}$$

The proposition follows from the identity $(*)$ above. □

Remark. It is interesting to note that since $H_m^{(2)}(-x^2)$ is an even function, the following identity holds for any odd integer k:

$$\sum_{i=0}^{2m}(-1)^i\binom{m+i}{m-i}\binom{m+k-i}{m-k+i}=0.$$

5. A hypergeometric identity via Euler's contiguous relation and the Wilf-Zeilberger algorithm

In this section we will derive a proof for the identity

$(*)$

$$\sum_{i=0}^{2k}(-1)^i\binom{m+k+i}{m+k-i}\binom{m+3k-i}{m-k+i}=(-1)^k\sum_{i=0}^{m}\binom{2k}{m-i}\binom{4k+i}{i}.$$

The method of proving this identity is similar to that of Vidūnas [10]. We first express the right-hand side in terms of hypergeometric ${}_2F_1$ functions.

Proposition 5.1. *The following equations hold:*

$$\sum_{i=0}^{m}\binom{2k}{m-i}\binom{4k+i}{i}=\binom{2k}{m}{}_2F_1\left(\begin{matrix}4k+1,-m\\2k-m+1\end{matrix};-1\right)$$

if $2k\geq m$ *and*

$$\sum_{i=0}^{m}\binom{2k}{m-i}\binom{4k+i}{i}=\binom{m+2k}{m-2k}{}_2F_1\left(\begin{matrix}-2k,2k+m+1\\m-2k+1\end{matrix};-1\right)$$

if $2k\leq m$.

Proof. The first equation follows directly from the fact that

$$\binom{4k+i}{i}=\frac{(4k+1)_i}{i!}$$

and

$$\binom{2k}{m-i}=\frac{(-1)^i(2k)!(-m)_i}{m!(2k-m)!(2k-m+1)!}.$$

For the second equation, we note that the summand on the left side equals 0 when $i < m - 2k$. Thus the left side equals

$$\begin{aligned}\sum_{i=m-2k}^{m} \binom{2k}{m-i}\binom{4k+i}{4k} &= \sum_{i=0}^{2k} \binom{2k}{i}\binom{4k+m-i}{4k} \\ &= \sum_{i=0}^{2k} \binom{2k}{i}\binom{m+2k+i}{4k}.\end{aligned}$$

To complete the proof note that

$$\binom{2k}{i} = \frac{(-2k)_i(-1)^i}{i!}$$

and

$$\binom{2k+m+i}{4k} = \frac{(2k+m)!(2k+m+1)_i}{(4k)!(m-2k)!(m-2k+1)_i}.$$

To continue the proof of $(*)$, we recall Euler's contiguous relation:

$$\begin{aligned}{}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};u\right) &= \frac{c+(a-b+1)u}{c}\,{}_2F_1\left(\begin{matrix} a+1,b \\ c+1 \end{matrix};u\right) \\ &\quad -\frac{(a+1)(c-b+1)u}{c(c+1)}\,{}_2F_1\left(\begin{matrix} a+2,b \\ c+2 \end{matrix};u\right).\end{aligned}$$

Applying this relation to

$${}_2F_1\left(\begin{matrix} 4k+1,-m \\ 2k-m+1 \end{matrix};-1\right)$$

with $a = -m$, $b = 4k+1$ and $c = 2k-m+1$, we obtain the identity

$$\begin{aligned}{}_2F_1\left(\begin{matrix} 4k+1,-m \\ 2k-m+1 \end{matrix};-1\right) &= \frac{6k+1}{2k-m+1}\,{}_2F_1\left(\begin{matrix} 4k+1,-m+1 \\ 2k-m+2 \end{matrix};-1\right) \\ &\quad + \frac{(m-1)(m-1+2k)}{(2k-m+1)(2k-m+2)}\,{}_2F_1\left(\begin{matrix} 4k+1,-m+2 \\ 2k-m+3 \end{matrix};-1\right).\end{aligned}$$

Also applying the same contiguous relation to

$${}_2F_1\left(\begin{matrix} -2k,2k+m+1 \\ m-2k+1 \end{matrix}\right)$$

with $a = 2k+m+1$, $b = -2k$ and $c = m-2k+1$, we obtain

$$\begin{aligned}{}_2F_1\left(\begin{matrix} -2k,2k+m+1 \\ m-2k+1 \end{matrix};-1\right) &= -\frac{6k+1}{m-2k+1}\,{}_2F_1\left(\begin{matrix} -2k,2k+m+2 \\ m-2k+2 \end{matrix};-1\right) \\ &\quad + \frac{(m+2)(2k+m+2)}{(m-2k+1)(m-2k+2)}\,{}_2F_1\left(\begin{matrix} -2k,2k+m+3 \\ m-2k+3 \end{matrix};-1\right).\end{aligned}$$

Fix k and denote by S_m the right side of $(*)$. From the two identities above and Proposition 5.1, we have the recursive relation

$$(m+2)S_{m+2} = (6k+1)S_{m+1} + (m+1+2k)S_m,$$

where $S_0 = (-1)^k$ and $S_1 = (-1)^k(6k+1)$.

Let T_m be the left side of $(*)$. It is easy to check that $T_0 = (-1)^k$ and

$$\begin{aligned} T_1 &= (-1)^{k-1}\binom{2k}{2} + (-1)^k(2k+1)^2 + (-1)^{k+1}\binom{2k}{2} \\ &= (-1)^k(6k+1). \end{aligned}$$

So it suffices to show that the sequence T_m also satisfies the same recursive relation, i.e.

$$(m+2)T_{m+2} - (6k+1)T_{m+1} - (m+1+2k)T_m = 0.$$

By definition of T_m, the left-hand side of the relation above is a finite summation of several terms. We denote by $f(m,i)$ its summand. Also let $F(m,i) = f(m,i)/(2k+m+1)!$. Following the WZ-algorithm we can define the certificate function

$$R(m,i) := \frac{i(2i-1)(i-3k-m-1)}{2(-3+i-k-m)(2+2k+m)}\frac{R_1(m,i)}{R_2(m,i)},$$

where

$$\begin{aligned} R_1(m,i) = {} & -10 - i + i^2 - 5k - 4ik + 2i^2k + 7k^2 - 4ik^2 \\ & +2k^3 - 13m - im + i^2m - 5km \\ & -2ikm + k^2m - 6m^2 - 2km^2 - m^3 \end{aligned}$$

and

$$\begin{aligned} R_2(m,i) = {} & -5i^2 + 2i^4 + 2k + 10ik - 3i^2k - 8i^3k \\ & -5k^2 + 6ik^2 + 6i^2k^2 - 11k^3 + 4ik^3 - 4k^4 \\ & -6i^2m + 3km + 12ikm - 4i^2km - 10k^2m + 8ik^2m \\ & -8k^3m - 2i^2m^2 + km^2 + 4ikm^2 - 4k^2m^2. \end{aligned}$$

With computer algebra one can check that

$$F(m+1,i) - F(m,i) = G(m,i+1) - G(m,i),$$

where $G(m,i) = R(m,i)F(m,i)$. The equation above can be checked easily by Mathematica using the FactorialSimplify function in the aisb.m package (see [6]). We note that $G(m,0) = 0$ since $R(m,0) = 0$ by definition of $R(m,i)$. Also $G(m,2k+1) = 0$ since $F(m,2k+1) = 0$. Therefore

$$\sum_{i=0}^{2k} F(m,i)$$

is a constant, and it is easy to check that this constant is 0 by the initial condition. Thus

$$\sum_{i=0}^{2k} f(m,i) = 0,$$

and T_m satisfies the recursive relation. □

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