PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 10, October 2009, Pages 3399–3406
S 0002-9939(09)09911-0
Article electronically published on May 6, 2009

NON-COMMUTATIVE ARITHMETIC-GEOMETRIC MEAN INEQUALITY

TOMOHIRO HAYASHI

(Communicated by Marius Junge)

Abstract. In this paper we consider a non-commutative analogue of the arithmetic-geometric mean inequality

$$a^r b^{1-r} + (r-1)b \geq ra$$

for two positive numbers a, b and for $r > 1$. We show that under certain assumptions the non-commutative analogue of $a^r b^{1-r}$ which satisfies this inequality is unique and equal to the r-mean. The case $0 < r < 1$ is also considered. In particular, we give a new characterization of the geometric mean.

1. Introduction

For any two positive numbers a, b and for $r > 1$, we have the arithmetic-geometric mean inequality

$$a^r b^{1-r} + (r-1)b \geq ra.$$

In this paper we consider the non-commutative analogue of this inequality for bounded linear operators on a Hilbert space. In particular, we give a new characterization of the geometric mean. Recently, in their ingenious paper [4], Carlen and Lieb used a certain non-commutative analogue of this inequality. Their paper is a motivation for our considerations.

There is one obvious non-commutative analogue as follows. For a bounded positive operator X on a Hilbert space, we always have

$$X^r + (r-1) \geq rX.$$

For any two positive invertible operators A and B, set $X = B^{-1/2}AB^{-1/2}$. Then we have

$$(B^{-1/2}AB^{-1/2})^r + (r-1) \geq rB^{-1/2}AB^{-1/2}$$

and hence

$$B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2} + (r-1)B \geq rA.$$

Thus if we consider $B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2}$ (the so-called r-mean) as the non-commutative analogue of $a^r b^{1-r}$, we get the desired inequality.

Our main result can be considered as a characterization of the r-mean, in particular of the geometric mean. In the paper [3] T. Ando and K. Nishio gave a characterization of the harmonic mean.

Received by the editors May 1, 2008, and, in revised form, February 12, 2009.
2000 *Mathematics Subject Classification.* Primary 47A63, 47A64.
Key words and phrases. Operator inequality, operator mean, geometric mean.

2. Main result

Throughout this paper we assume that the reader is familiar with the basic notation and results of operator theory. We refer the reader to Conway's book [5] for background.

We denote by $\mathfrak{H}$ a (finite or infinite dimensional) complex Hilbert space and by $B(\mathfrak{H})$ all bounded linear operators on $\mathfrak{H}$. For each operator $A \in B(\mathfrak{H})$, its operator norm is denoted by $||A||$. We denote by $B(\mathfrak{H})^+$ the set of all positive invertible operators. For two vectors $\xi, \eta \in \mathfrak{H}$, their inner product and norm are denoted by $\langle \xi, \eta \rangle$ and $||\xi||$ respectively.

In this paper we consider the map $M(\cdot,\cdot)$ from $B(\mathfrak{H})^+ \times B(\mathfrak{H})^+$ to $B(\mathfrak{H})^+$.

We fix a positive number $r > 0$. For $A, B \in B(\mathfrak{H})^+$, define

$$M_r(A,B) = B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2}.$$

Here we remark that

$$(\star) \qquad B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2} = A^{1/2}(A^{-1/2}BA^{-1/2})^{1-r} A^{1/2}.$$

(This is well-known to specialists.) Indeed, if r is an integer, direct computations show this equality. Thus for any polynomial $p(t)$ with $p(0) = 0$, we have

$$B^{1/2} \cdot p(B^{-1/2}AB^{-1/2}) \cdot B^{1/2} = A^{1/2} \cdot p((A^{-1/2}BA^{-1/2})^{-1}) \cdot (A^{-1/2}BA^{-1/2}) \cdot A^{1/2}.$$

Then by continuity and Weierstrass approximation we get $(\star)$. The map M_r is the so-called r-mean, and usually the case $0 < r < 1$ is considered. (When $0 < r < 1$, $M_r(A,B)$ is one of the so-called operator means. In particular, in the case $r = 1/2$, $M_r(A,B)$ is said to be the geometric mean.)

First we shall consider the case $r > 1$. The following is our main result.

Theorem 2.1. *Assume $r > 1$. For any $A, B \in B(\mathfrak{H})^+$, if the map M satisfies*

(i) $M(A,B) \geq rA + (1-r)B$,
(ii) $M(tA,B) = t^r M(A,B)$ *for any positive number t, and*
(iii) $M(A,B)^{-1} = M(A^{-1}, B^{-1})$,

then we have $M = M_r$.

Before starting the proof, we would like to state a conjecture as follows.

Conjecture. Theorem 2.1 holds upon replacing condition (iii) by

$$\text{(iii)}' \quad M(A,B) = A^r B^{1-r} \quad \text{if } A \text{ commutes with } B$$

and assuming continuity of the map M.

Unfortunately we have no idea how to show this.

Now we would like to start the proof. We need some preparation to prove this theorem. The following lemma states that under assumptions (i) and (ii), the map $M_r(A,B)$ is "less" than $M(A,B)$ in a certain sense. (See Remark 2.1.)

Lemma 2.2. *For any $A, B \in B(\mathfrak{H})^+$, we assume that the map M satisfies*

(i) $M(A,B) \geq rA + (1-r)B$,
(ii) $M(tA,B) = t^r M(A,B)$ *for any positive number t.*

Then for any unit vector $\xi \in \mathfrak{H}$, if $r \geq 2$ we have

$$\langle A^{-1/2}M(A,B)A^{-1/2}\xi, \xi\rangle \langle (A^{-1/2}M_r(A,B)A^{-1/2})^{-1}\xi, \xi\rangle \geq 1.$$

On the other hand, if $1 < r \leq 2$ *we have*

$$\langle (A^{-1/2}M(A,B)A^{-1/2})^{1/(r-1)}\xi,\xi\rangle\langle (A^{-1/2}M_r(A,B)A^{-1/2})^{-1/(r-1)}\xi,\xi\rangle \geq 1.$$

Proof. By the assumptions we have

$$t^r M(A,B) \geq rtA + (1-r)B$$

and hence

$$(\dagger) \qquad A^{-1/2}M(A,B)A^{-1/2} \geq rt^{1-r} + (1-r)t^{-r}A^{-1/2}BA^{-1/2}.$$

For a unit vector $\xi \in \mathfrak{H}$, set

$$f(t) = rt^{1-r} + (1-r)t^{-r}\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle.$$

With this definition, (†) can be rewritten as $\langle A^{-1/2}M(A,B)A^{-1/2}\xi,\xi\rangle \geq f(t)$. Then it is easy to see that the maximum value of $f(t)$ on $(0,\infty)$ is equal to $\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle^{1-r}$. Thus we get

$$\langle A^{-1/2}M(A,B)A^{-1/2}\xi,\xi\rangle\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle^{r-1} \geq 1.$$

In the case $r \geq 2$, by the Jensen inequality and (†) we have

$$\begin{aligned}\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle^{r-1} &\leq \langle (A^{-1/2}BA^{-1/2})^{r-1}\xi,\xi\rangle \\ &= \langle (A^{-1/2}M_r(A,B)A^{-1/2})^{-1}\xi,\xi\rangle.\end{aligned}$$

So we are done.

Next we consider the case $1 < r \leq 2$. Let s be a positive number such that $\frac{1}{r} + \frac{1}{s} = 1$. Then since $(r-1) = 1/(s-1)$, we have

$$\langle A^{-1/2}M(A,B)A^{-1/2}\xi,\xi\rangle\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle^{1/(s-1)} \geq 1$$

and hence

$$(\ddagger) \qquad \langle A^{-1/2}M(A,B)A^{-1/2}\xi,\xi\rangle^{s-1}\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle \geq 1.$$

Since $s \geq 2$ and $(s-1) = 1/(r-1)$, we compute, as above,

$$\begin{aligned}&\langle (A^{-1/2}M(A,B)A^{-1/2})^{1/(r-1)}\xi,\xi\rangle\langle (A^{-1/2}M_r(A,B)A^{-1/2})^{-1/(r-1)}\xi,\xi\rangle \\ &= \langle (A^{-1/2}M(A,B)A^{-1/2})^{s-1}\xi,\xi\rangle\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle \\ &\geq \langle (A^{-1/2}M(A,B)A^{-1/2})\xi,\xi\rangle^{s-1}\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle \geq 1.\end{aligned}$$

In the third line of this computation, the first inequality follows from the Jensen inequality, and the second one is (‡). □

Theorem 2.3. *If two positive invertible operators* $X, Y \in B(\mathfrak{H})^+$ *satisfy*

$$\langle X\xi,\xi\rangle\langle Y^{-1}\xi,\xi\rangle \geq 1$$

and

$$\langle Y\xi,\xi\rangle\langle X^{-1}\xi,\xi\rangle \geq 1$$

for any unit vector $\xi \in \mathfrak{H}$*, then we have* $X = Y$*.*

In order to show this theorem, we need the following lemma.

Lemma 2.4. *Two operators $X, Y \in B(\mathfrak{H})^+$ satisfy*

$$\langle X\xi, \xi\rangle\langle Y^{-1}\xi, \xi\rangle \geq 1$$

for any unit vector $\xi \in \mathfrak{H}$ if and only if we have

$$tX + (tY)^{-1} \geq 2$$

for any positive number t.

Proof. Let $\xi \in \mathfrak{H}$ be a unit vector. We set $f(t) = t\langle X\xi, \xi\rangle + t^{-1}\langle Y^{-1}\xi, \xi\rangle$ for $t > 0$. Then it is easy to see that the minimum value of $f(t)$ is equal to $2\sqrt{\langle X\xi, \xi\rangle\langle Y^{-1}\xi, \xi\rangle}$. Hence we are done. □

Proof of Theorem 2.3. By the previous lemma we have

$$tX + (tY)^{-1} \geq 2$$

and

$$tY + (tX)^{-1} \geq 2$$

for any positive number t. Let $Z = Y^{1/2}XY^{1/2}$. Then we have $tZ + t^{-1} \geq 2Y$ and $t + (tZ)^{-1} \geq 2Y^{-1}$. So we get

$$\frac{2tZ}{t^2Z + 1} \leq Y \leq \frac{t^2Z + 1}{2t}. \tag{1}$$

First we assume that the Hilbert space is finite dimensional because in this case the proof becomes simpler. Take any projection P of rank one which reduces Z, that is, $ZP = \lambda P$ for some positive number λ. (Here we use the fact that Z is atomic, thanks to finite dimensionality.) Then we get

$$\frac{2t\lambda}{t^2\lambda + 1}P \leq PYP \leq \frac{t^2\lambda + 1}{2t}P.$$

Since P is of rank one, PYP is of the form $PYP = \alpha P$ for some $\alpha > 0$. Therefore upon taking the maximum in t on the left-hand side and the minimum on the right-hand side we have $PYP = \lambda^{1/2}P$. On the other hand, since we also have

$$\frac{t^2Z + 1}{2tZ} \geq Y^{-1} \geq \frac{2t}{t^2Z + 1}, \tag{2}$$

we get

$$\frac{t^2\lambda + 1}{2t\lambda}P \geq PY^{-1}P \geq \frac{2t}{t^2\lambda + 1}P$$

and hence $PY^{-1}P = \lambda^{-1/2}P$ as above.

Let $\xi \in \mathfrak{H}$ be a vector satisfying $P\xi = \xi$. Then we have

$$\begin{aligned} ||Y^{1/2}\xi|| \cdot ||Y^{-1/2}\xi|| &= \langle PYP\xi, \xi\rangle^{1/2}\langle PY^{-1}P\xi, \xi\rangle^{1/2} = \langle \lambda^{1/2}\xi, \xi\rangle^{1/2}\langle \lambda^{-1/2}\xi, \xi\rangle^{1/2} \\ &= ||\xi||^2 = \langle Y^{1/2}\xi, Y^{-1/2}\xi\rangle. \end{aligned}$$

By the equality condition for the Cauchy-Schwarz inequality, this implies that $Y^{1/2}\xi$ is a scalar-multiple of $Y^{-1/2}\xi$ or, in other words, that $Y\xi$ is a scalar-multiple of ξ. So we get $YP = PYP = Z^{1/2}P$. Since by the spectral theory for a positive matrix there are projections P_i of rank one such that $\sum_i P_i = 1$, we conclude that $Y = Z^{1/2} = (Y^{1/2}XY^{1/2})^{1/2}$ and hence $X = Y$.

Next we shall consider the general case. The following argument is due to a private communication with T. Ando [2]. The author would like to thank Professor Ando for permitting the author to include his argument in this paper.

It is easy to see that for any $t > 0$,

$$\frac{2tZ}{t^2Z+1} \leq Z^{1/2} \leq \frac{t^2Z+1}{2t}$$

and

$$\frac{t^2Z+1}{2tZ} \geq Z^{-1/2} \geq \frac{2t}{t^2Z+1}.$$

Combining these with (1) and (2), we have

$$\frac{2tZ}{t^2Z+1} - \frac{t^2Z+1}{2t} \leq Y - Z^{1/2} \leq \frac{t^2Z+1}{2t} - \frac{2tZ}{t^2Z+1}$$

and

$$\frac{t^2Z+1}{2tZ} - \frac{2t}{t^2Z+1} \geq Y^{-1} - Z^{-1/2} \geq \frac{2t}{t^2Z+1} - \frac{t^2Z+1}{2tZ}.$$

Next we compute

$$\frac{t^2Z+1}{2t} - \frac{2tZ}{t^2Z+1} = (t - Z^{-1/2})^2 \frac{Z^2(t+Z^{-1/2})^2}{2t(t^2Z+1)}$$

and

$$\frac{t^2Z+1}{2tZ} - \frac{2t}{t^2Z+1} = (t - Z^{-1/2})^2 \frac{Z(t+Z^{-1/2})^2}{2t(t^2Z+1)}.$$

Therefore there is a positive number γ such that for any λ in the spectrum of Z and a projection P which reduces Z we have

$$||PYP - Z^{1/2}P|| \leq \gamma ||(\lambda - Z^{-1/2})P||^2 \tag{3}$$

and

$$||PY^{-1}P - Z^{-1/2}P|| \leq \gamma ||(\lambda - Z^{-1/2})P||^2. \tag{4}$$

Here we remark that the constant γ depends only on the norms of X, X^{-1}, Y and Y^{-1}.

Let us use $(PY^{-1}P)^{-1}$ to denote the inverse of $PY^{-1}P$ on $P\mathfrak{H}$. Then we see that

$$PY^{-1}P - Z^{-1/2}P = (PY^{-1}P)(Z^{1/2}P - (PY^{-1}P)^{-1})Z^{-1/2}P$$

and hence by using (4) that there is a positive number γ' such that

$$||(PY^{-1}P)^{-1} - Z^{1/2}P|| \leq \gamma' ||(\lambda - Z^{-1/2})P||^2.$$

Combining this with (3), we conclude that there is a positive number γ'' such that for any spectral value λ of Z, and spectral projection P of Z,

$$||PYP - (PY^{-1}P)^{-1}|| \leq \gamma'' ||(\lambda - Z^{-1/2})P||^2. \tag{5}$$

For any integer n, take a partition of unity $\{P_i\}_{i=1}^n$ which consists of spectral projections of Z such that there exist spectral values $\{\lambda_i\}_{i=1}^n$ of $Z^{-1/2}$ satisfying

$$||(\lambda_i - Z^{-1/2})P_i|| \leq \frac{||Z^{-1/2}||}{n}.$$

Then it follows from (3) that

$$||\sum_{i=1}^n (P_iYP_i - Z^{1/2}P_i)|| \leq \frac{\gamma ||Z^{-1/2}||^2}{n^2}. \tag{6}$$

Similarly it follows from (5) that

$$||P_iYP_i - (P_iY^{-1}P_i)^{-1}|| \leq \frac{\gamma''||Z^{-1/2}||^2}{n^2}.$$

Recall the following formula, which is the so-called Schur complement:

$$(P_iY^{-1}P_i)^{-1} = P_iYP_i - P_iYP_i^{\perp}(P_i^{\perp}YP_i^{\perp})^{-1}P_i^{\perp}YP_i$$

where $P_i^{\perp} = 1 - P_i$. Indeed we can show that

$$\begin{aligned}
\{P_iYP_i - P_iYP_i^{\perp}(P_i^{\perp}YP_i^{\perp})^{-1}P_i^{\perp}YP_i\} \cdot P_iY^{-1}P_i \\
&= P_iYP_iY^{-1}P_i - P_iYP_i^{\perp}(P_i^{\perp}YP_i^{\perp})^{-1}P_i^{\perp}YP_iY^{-1}P_i \\
&= P_iYP_iY^{-1}P_i - P_iYP_i^{\perp}(P_i^{\perp}YP_i^{\perp})^{-1}(P_i^{\perp}Y - P_i^{\perp}YP_i^{\perp})Y^{-1}P_i \\
&= P_iYP_iY^{-1}P_i + P_iYP_i^{\perp}Y^{-1}P_i = P_i.
\end{aligned}$$

By using this formula we have

$$\begin{aligned}
||P_i^{\perp}YP_i||^2 &= ||(P_i^{\perp}YP_i^{\perp})^{1/2}(P_i^{\perp}YP_i^{\perp})^{-1/2}P_i^{\perp}YP_i||^2 \\
&\leq ||Y|| \cdot ||(P_i^{\perp}YP_i^{\perp})^{-1/2}P_i^{\perp}YP_i||^2 \\
&= ||Y|| \cdot ||P_iYP_i^{\perp}(P_i^{\perp}YP_i^{\perp})^{-1}P_i^{\perp}YP_i|| \\
&= ||Y|| \cdot ||P_iYP_i - (P_iY^{-1}P_i)^{-1}|| \\
&\leq \frac{\gamma''||Y|| \cdot ||Z^{-1/2}||^2}{n^2}.
\end{aligned}$$

Therefore for each unit vector $\xi \in \mathfrak{H}$, by using the Cauchy-Schwarz inequality we see that

$$\begin{aligned}
||\sum_{i=1}^{n} P_i^{\perp}YP_i\xi|| &\leq \sum_{i=1}^{n} ||P_i^{\perp}YP_i|| \cdot ||P_i\xi|| \\
&\leq \sqrt{\sum_{i=1}^{n} ||P_i^{\perp}YP_i||^2}\sqrt{\sum_{i=1}^{n} ||P_i\xi||^2} \\
&= \sqrt{\sum_{i=1}^{n} ||P_i^{\perp}YP_i||^2} \\
&\leq \sqrt{\sum_{i=1}^{n} \frac{\gamma''||Y|| \cdot ||Z^{-1/2}||^2}{n^2}} = \sqrt{\frac{\gamma''||Y|| \cdot ||Z^{-1/2}||^2}{n}}.
\end{aligned}$$

Thus we get

$$||\sum_{i=1}^{n} P_i^{\perp}YP_i|| \leq \sqrt{\frac{\gamma''||Y|| \cdot ||Z^{-1/2}||^2}{n}}. \tag{7}$$

By using (6) and (7) we see that

$$\begin{aligned}||Y - Z^{1/2}|| &\leq ||\sum_{i=1}^{n} P_i Y P_i - Z^{1/2} P_i|| + ||\sum_{i=1}^{n} P_i^{\perp} Y P_i|| \\ &\leq \frac{\gamma ||Z^{-1/2}||}{n^2} + \sqrt{\frac{\gamma'' ||Y|| \cdot ||Z^{-1/2}||^2}{n}}.\end{aligned}$$

By taking $n \to \infty$ we get $Y = Z^{1/2}$ and hence $X = Y$.

□

Now we can prove our main result.

Proof of Theorem 2.1. First we consider the case $r \geq 2$. Set

$$X = A^{-1/2} M(A, B) A^{-1/2} \text{ and } Y = A^{-1/2} M_r(A, B) A^{-1/2}.$$

By Lemma 2.2, for any unit vector $\xi \in \mathfrak{H}$ we have

$$\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1.$$

On the other hand, thanks to the two relations $M(A, B)^{-1} = M(A^{-1}, B^{-1})$ and $M_r(A, B)^{-1} = M_r(A^{-1}, B^{-1})$, applying Lemma 2.2 to the pair (A^{-1}, B^{-1}) gives

$$\begin{aligned}&\langle X^{-1}\xi, \xi \rangle \langle Y\xi, \xi \rangle \\ &\quad = \langle A^{1/2} M(A^{-1}, B^{-1}) A^{1/2}\xi, \xi \rangle \langle (A^{1/2} M_r(A^{-1}, B^{-1}) A^{1/2})^{-1}\xi, \xi \rangle \geq 1.\end{aligned}$$

Therefore by Theorem 2.3 we get $X = Y$ and hence $M = M_r$.

In the case $1 < r \leq 2$, set

$$X = (A^{-1/2} M(A, B) A^{-1/2})^{1/(r-1)} \text{ and } Y = (A^{-1/2} M_r(A, B) A^{-1/2})^{1/(r-1)}.$$

Then in the same way we conclude the desired fact. □

Remark 2.1. For positive invertible operators A, B, C, the block matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

is positive if and only if $A \geq BC^{-1}B$ [1]. Therefore for two positive invertible operators X, Y, the block matrix

$$\begin{pmatrix} X & 1 \\ 1 & Y^{-1} \end{pmatrix}$$

is positive if and only if $X \geq Y$. On the other hand, for any unit vector $\xi \in \mathfrak{H}$ the matrix

$$\begin{pmatrix} \langle X\xi, \xi \rangle & 1 \\ 1 & \langle Y^{-1}\xi, \xi \rangle \end{pmatrix}$$

is positive if and only if $\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1$. The condition $\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1$ is therefore weaker than $X \geq Y$. We do not know whether the condition $\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1$ defines a new order $X \text{“} \geq \text{”} Y$ or not. The author guesses that this relation does not satisfy transitivity. Here we remark that if $X \text{“} \geq \text{”} Y$, then we have $X^2 \text{“} \geq \text{”} Y^2$. Indeed if we have $\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1$, then we get

$$\langle X^2\xi, \xi \rangle \langle Y^{-2}\xi, \xi \rangle \geq \langle X\xi, \xi \rangle^2 \langle Y^{-1}\xi, \xi \rangle^2 \geq 1.$$

Thus this relation is not equivalent to the usual order. Theorem 2.3 states that if we have $X \text{“} \geq \text{”} Y$ and $Y \text{“} \geq \text{”} X$, then we can conclude that $X = Y$ (reflexivity).

Finally we shall prove the analogue of Theorem 2.1 in the case $0 < r < 1$.

Theorem 2.5. *Assume* $0 < r < 1$. *For any* $A, B \in B(\mathfrak{H})^+$, *if the map* M *satisfies*

(i) $M(A, B) \leq rA + (1 - r)B$,
(ii) $M(tA, B) = t^r M(A, B)$ *for any positive number* t, *and*
(iii) $M(A, B)^{-1} = M(A^{-1}, B^{-1})$,

then we have $M = M_r$.

Proof. The proof is essentially the same as that of Theorem 2.1, so we just give a sketch of the proof.

By the assumptions, for any positive number t we have

$$M(A, B) \leq rt^{r-1}A + (1 - r)t^r B$$

and

$$M(A, B)^{-1} \leq rt^{1-r}A^{-1} + (1 - r)t^{-r}B^{-1}.$$

Set

$$Y = B^{-1/2}M(A, B)B^{-1/2} \text{ and } Z = B^{-1/2}AB^{-1/2}.$$

Then we have

$$\frac{t^r Z}{rt + (1 - r)Z} \leq Y \leq \frac{rZ + (1 - r)t}{t^{1-r}}.$$

Then by almost the same arguments as those in the proof of Theorem 2.3, we can show that $Y = Z^r$. □

Acknowledgments

The author wishes to express his hearty gratitude to Professor Tsuyoshi Ando for valuable comments. The author is also grateful to Professor Yoshihiro Nakamura for helpful discussions. The author would like to thank Professors Hideki Kosaki, Mitsuru Uchiyama and Atsushi Uchiyama for useful advice and comments. The author would also like to express his hearty gratitude to the referee for valuable comments.

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Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya, Aichi, 466-8555, Japan
E-mail address: hayashi.tomohiro@nitech.ac.jp