TOEPLITZ AND HANKEL OPERATORS
AND DIXMIER TRACES ON THE UNIT BALL OF $\mathbb{C}^n$

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Abstract. We compute the Dixmier trace of pseudo-Toeplitz operators on the Fock space. As an application we find a formula for the Dixmier trace of the product of commutators of Toeplitz operators on the Hardy and weighted Bergman spaces on the unit ball of $\mathbb{C}^d$. This generalizes an earlier work of Helton-Howe for the usual trace of the anti-symmetrization of Toeplitz operators.

1. Introduction

In the present paper we will study the Dixmier trace of a class of Toeplitz and Hankel operators on the Hardy and weighted Bergman spaces on the unit ball of $\mathbb{C}^d$. We give a brief account of our problem and explain some motivations. Consider the Bergman space $L^2_a(D)$ of holomorphic functions on the unit disk $D$ in the complex plane. For a bounded function $f$ let $T_f$ be the Toeplitz operator on $L^2_a(D)$. It is well-known that for a holomorphic function $f$ in a neighborhood of $D$ the commutator $[T^*_f, T_f]$ is of trace class and the trace is given by the square of the Dirichlet norm of $f$,

$$\text{tr}[T^*_f, T_f] = \int_D |f'(z)|^2 \, dm(z),$$

which is one of the best known Möbius invariant integrals. This formula actually holds for Toeplitz operators on any Bergman space on a bounded domain with the area measure replaced by any reasonable measure [2]. There is a significant difference between Toeplitz operators on the unit disk and on the unit ball $B = B^d$ in $\mathbb{C}^d$, $d > 1$. Let $L^p$ be the Schatten - von Neumann class of $p$-summable operators. The commutator $[T^*_f, T_f]$ on the weighted Bergman space, say for holomorphic functions $f$ in a neighborhood of the closed unit disk, is in the Schatten - von Neumann class $L^p$, for $p > \frac{1}{2}$, and is zero if it is in $L^p$, for $p \leq \frac{1}{2}$, $\frac{1}{2}$ being called the cut-off; on the Hardy space $[T^*_f, T_f]$ can be in any Schatten - von Neumann class.
the Macaev class

Theorem 4.1 can also be obtained from [4] provided one proves that the lower order transforms and they differ from pseudo-differential operators of lower order, so that pseudo-differential operators of a certain class is computed; here we use the Weyl our Theorem 4.1 is closely related to the results in [4], where the residue trace of and an application of the ideas in [7] of computing Dixmier traces. In particular be viewed as a generalization of the compactification to weighted Bergman spaces. However for \( d > 1 \), it is in \( L^d \) for \( p > d \), with \( p = d \) being the cut-off, both on the weighted Bergman spaces and on the Hardy space. Thus no trace formula was expected for the commutators. Nevertheless Helton and Howe [9] were able to find an analogue of the previous formula. They showed, for smooth functions \( f_1, \ldots, f_{2d} \) on the closed unit ball, that the anti-symmetrization \([T_{f_1}, T_{f_2}, \ldots, T_{f_{2d}}]\) of the \( 2d \) operators \( T_{f_1}, T_{f_2}, \ldots, T_{f_{2d}} \) is of trace class and found that

\[
\text{tr}[T_{f_1}, \ldots, T_{f_{2d}}] = \int_B df_1 \wedge df_2 \cdots \wedge df_{2d}.
\]

On the other hand, we observe that \([T_f, T_g]\) is, for smooth functions \( f \) and \( g \), in the Macaev class \( L^{d, \infty} \) (which is an analogue of the Lorentz space \( L^{d, \infty} \)); thus the product of \( d \) such commutators \([T_{f_1}, T_{g_1}][T_{f_2}, T_{g_2}] \cdots [T_{f_d}, T_{g_d}]\) is in \( L^{1, \infty} \) and hence has a Dixmier trace. One of the goals of the present paper is to prove the following formula for the Dixmier trace of this product of commutators:

\[
\text{tr}_w[T_{f_1}, T_{g_1}] \cdots [T_{f_d}, T_{g_d}] = \frac{1}{d!} \int_S \{f_1, g_1\} \cdots \{f_d, g_d\}.
\]

Here \( \{f, g\} \) is the Poisson bracket of \( f \) and \( g \); its restriction to the boundary \( S \) of \( B \) depends only on the boundary values of \( f \) and \( g \) and can be expressed in terms of the boundary \( CR \) operators. This can be viewed as a generalization of the Helton-Howe theorem. We apply our result also to Hankel operators and obtain a formula for the Dixmier trace of the \( d \)-th power of the square modulus of the Hankel operators \( H_T^* H_T \) for holomorphic functions \( f \). Namely we have

\[
\text{tr}_w |H_f|^2 = \text{tr}_w([T_f, T_f]^d) = \frac{1}{d!} \int_S (|\nabla f|^2 - |Rf|^2)^d.
\]

This provides a boundary \( L^{d, \infty} \) result for the Schatten - von Neumann \( L^p \) (\( p > d \)) properties of the square modulus of the Hankel operators (see [3], [5] and [19]). We mention also, besides the above results on exact norms, that there are exact formulas proved by Jaason, Upmeier and Wallstén [12] on the Schatten - von Neumann \( L^p \)-norm of the Hankel operators on the unit circle for \( p = 2, 4, 6 \), and by Peetre [18] on \( L^1 \)-norms of Hankel forms on Fock spaces [11].

There has been an intensive study of Dixmier trace and residue trace of pseudo-differential operators, mostly on compact manifolds where the analysis is relatively easier; see e.g. [4] and [15] and the references therein. Thus the Toeplitz operators on Hardy spaces on the boundary of a bounded strictly pseudo-convex domain can be treated using the techniques developed there. The Hankel and Toeplitz operators on Bergman spaces, generally speaking, behave rather differently from those on a Hardy space, and the result of Howe [10] roughly speaking proves that Toeplitz operators of certain classes can be treated similarly to those in the Hardy space case (also called the de Monvel - Howe compactification [8]). Our result can thus be viewed as a generalization of the compactification to weighted Bergman spaces and an application of the ideas in [7] of computing Dixmier traces. In particular our Theorem 4.1 is closely related to the results in [3], where the residue trace of pseudo-differential operators of a certain class is computed; here we use the Weyl transforms and they differ from pseudo-differential operators of lower order, so that Theorem 4.1 can also be obtained from [3] provided one proves that the lower order terms are of trace class.
In another paper we will study the Dixmier trace for Toeplitz operators on a general strongly pseudo-convex domain. One of the authors, G. Zhang, would also like to thank Professor Richard Rochberg and Professor Harald Upmeier for introducing him to the work of Connes [8, Chapter IV.2] on Dixmier traces of pseudo-differential operators.

2. Toeplitz operators on Bergman spaces and their realization as pseudo-Toeplitz operators on Fock spaces

Let $d\mu_\nu = C_\nu (1 - |z|^2)^{\nu - d - 1} dm(z)$, where $C_\nu$ is the normalizing constant to make $d\mu_\nu$ a probability measure and $\nu > d$. We let $H_\nu$ be the corresponding Bergman space of holomorphic functions on $B$. We will also consider the Hardy space of square integrable functions on $S$ which are holomorphic on $B$. This can be viewed as the analytic continuation of $H_\nu$ at $\nu = d$.

Thus we assume throughout this paper that $\nu \geq d$.

Let $f$ be a bounded smooth function on $\overline{B}$, the closure of $B$. The Toeplitz operator $T_f$ on $H_\nu$ with symbol $f$ is defined by

$$T_f g = P(fg),$$

where $P$ is the Bergman or the Hardy projection for $\nu > d$ and $\nu = d$, respectively.

As was shown by Howe [10] there is a more flexible and effective way of studying the spectral properties of Toeplitz operators with smooth symbol, by using the theories of representations of the Heisenberg group and of pseudo-differential operators. We will adopt that approach. We will be very brief and refer to [10] and [18, Chapter XII] for details. So let $H_n = C^d \times T$ be the Heisenberg group as in loc. cit. The Heisenberg group has an irreducible representation, $\rho$, on the Fock space $F$ consisting of entire functions $f$ on $C^d$ such that

$$\int_{C^d} |f(z)|^2 e^{-\pi|z|^2} dm(z) < \infty.$$

The action of the Heisenberg group is explicitly given as follows. For $w \in C^d$ viewed as an element in $H_d$,

$$\rho(w)f(w') = e^{-\pi/2|w|^2 + \pi w' \cdot \overline{w}} f(w' - w),$$

where $w' \cdot \overline{w}$ is the Hermitian inner product on $C^d$. The action of $T$ is by rotation.

Identifying the Lie algebra $\mathfrak{h}$ of the Heisenberg group with $R^{2n} \oplus R$ and thus $R^{2n}$ with a subspace of the Lie algebra, we get an action of $R^{2n}$ as holomorphic differential operators on $F$, which extends from $\mathfrak{h}$ to the whole enveloping algebra $\mathfrak{u}(\mathfrak{h})$ and which will also be denoted by $\rho$. In particular, taking the basis elements $\partial_j = \partial / \partial w_j$ and $\overline{\partial}_j = \partial / \partial \overline{w}_j$ of $R^{2n}$ we have

$$\rho(\partial_j) f(w) = -\partial_j f(w), \quad \rho(\overline{\partial}_j) f(w) = \pi w_j f(w).$$

Following the notation in [10], let $\Delta \in \mathfrak{u}(\mathfrak{h})$ be the element

$$\Delta = \frac{1}{2}(\partial_j \cdot \overline{\partial}_j + \overline{\partial}_j \cdot \partial_j).$$
Then $\rho(\Delta)$ acts on $\mathcal{F}$ as a diagonal self-adjoint operator \cite{10}, under the orthogonal basis $\{w^\alpha, \alpha = (\alpha_1, \cdots, \alpha_d)\}$, viz.

\begin{equation}
\rho(\Delta)w^\alpha = -\pi (|\alpha| + \frac{d}{2})w^\alpha.
\end{equation}

Let $F(z)$ be a function on $\mathbb{C}^d$ (viewed as a function on the Heisenberg group). The Weyl transform $\rho(F)$ of $F$ is defined by

$$
\rho(F) = \int_{\mathbb{C}^d} F(w)\rho(w)dm(w).
$$

To understand the operator theoretic properties of $\rho(F)$ we will need the Fourier transform of $F$. Let $\hat{F}$ be the (symplectic-) Fourier transform of $F$,

$$
\hat{F}(w') = 2^{-d} \int_{\mathbb{C}^d} F(w)e^{\pi i \operatorname{Im} w \cdot w'}dm(w),
$$

and let $F \ast G$ be the symplectic convolution

$$
F \ast G(w) = \int_{\mathbb{C}^d} F(z)G(w-z)e^{\pi i \operatorname{Im} w \cdot z}dm(z).
$$

We recall that

$$
\hat{F} \ast \hat{G} = F \ast \hat{G}
$$

and

$$
\rho(F)\rho(G) = \rho(F \ast G)
$$

for an appropriate class of functions. A well-known theorem of Calderón-Vaillancourt states that if $\hat{F}$ and all its derivatives are bounded, then $\rho(F)$ can be defined as a bounded operator on $\mathcal{F}$.

We will need a finer class of symbols introduced by Howe, corresponding to the so-called pseudo-Toeplitz operators. Let

$$
\mathcal{PT}(m, \mu) = \{ F \in S^*(\mathbb{C}^d) : \|\partial^\beta \hat{F}\| \leq C_{\alpha, \beta}(1 + |w|)^{m-\mu(|\alpha|+|\beta|)} \}
$$

and

$$
\mathcal{PT}_{\text{rad}}(m, \mu) = \{ F \in \mathcal{PT}(m, \mu) : 
\hat{F} = (1 - g(|w|))\psi(\frac{w}{|w|})|w|^m + D_1, D_1 \in \mathcal{PT}(m - \mu, \mu) \}.
$$

Here $g$ is a smooth function on $\mathbb{R}$ such that $0 \leq g(t) \leq 1$ on $\mathbb{R}$, $g(t) = 0$ for $|t| \geq 2$ and $g(t) = 1$ for $0 \leq t \leq 1$.

For $F \in \mathcal{PT}_{\text{rad}}(m, \mu)$ we will call

$$
\sigma_m(F) := \psi(\frac{w}{|w|})|w|^m
$$

its principal symbol. It can be obtained, up to the factor $|w|^m$, by

$$
\psi(w) = \lim_{t \to \infty} t^{-m}\hat{F}(tw), \quad w \in S.
$$

Following Howe \cite{10}, we will call $\rho(F), F \in \mathcal{PT}(m, \mu)$, a pseudo-Toeplitz operator of order $m$ and smoothness $\mu$. One has \cite{10} Lemma 4.2.2

\begin{equation}
F \in \mathcal{PT}(m_1, \mu), \ G \in \mathcal{PT}(m_2, \mu) \implies F \ast G \in \mathcal{PT}(m_1 + m_2, \mu).
\end{equation}
Thus the map

\[ L(2.5) \]

is a unitary operator. First we will find the action of the elementary Toeplitz operators \( T \) under the intertwining map \( U \).

**Lemma 2.1.** The operator \( UT_{z^0}U^* \) on \( \mathcal{F} \) is given by

\[ U T_{z^0} U^* = \rho(z)^\alpha \rho \left( \pi^{\alpha i} \nu - \frac{d}{2} \frac{1}{\pi} \Delta \right)^{-1/2}. \]

This can be proved by direct computation. Indeed we have

\[ T_{z^0} e_{\beta} = \left( \frac{\beta}{\nu + |\beta|} \right)^\alpha e_{\beta + \alpha}, \]

and the right hand side (2.6) can be easily computed by (2.1) and (2.2).

By using the previous lemma we have then the following result, which was proved by Howe \[10\] Proposition 4.2.3] in the case when \( \nu = d + 1 \); the general case of \( \nu \geq d \) is essentially the same.

**Proposition 2.2.** Let \( f \in C^\infty(S) \) and let \( \tilde{f} \) be a \( C^\infty \) extension to \( B \) and \( T_j \) the Toeplitz operator on \( \mathcal{H}_\nu \). Then under the unitary equivalence of \( \mathcal{H}_\nu \) and the Fock space \( \mathcal{F} \) on \( \mathbb{C}^d \), the Toeplitz operators are pseudo-Toeplitz operators with radial asymptotic limits \( \mathcal{PT}_\text{rad}(0,1) \). More precisely, there exists \( F \in \mathcal{PT}_\text{rad}(0,1) \) such that \( U T_j U^* = \rho(F) \), and \( f(\zeta) = \lim_{\nu \to \infty} \tilde{F}(t \zeta) \) for each \( \zeta \in S \).

3. **Schatten - von Neumann properties of pseudo-Toeplitz operators**

Recall that the Schatten - von Neumann class \( \mathcal{L}^p, p \geq 1 \), consists of compact operators \( T \) such that the eigenvalues \( \{ \mu_n \} \) of \( |T| = (T^* T)^{1/2} \) are \( p \)-summable, \( \sum \mu_n^p < \infty \). In particular \( \mathcal{L}^2 \) is the Hilbert-Schmidt class, \( \mathcal{L}^1 \) the trace class, and \( \mathcal{L}^\infty \) are the compact operators. For \( 1 < p < \infty, 1 \leq q \leq \infty \), the Macaev class \( \mathcal{L}^{p,q} \) is obtained by the real interpolation between \( \mathcal{L}^1 \) and \( \mathcal{L}^\infty \). However, we will need the Macaev class \( \mathcal{L}^{1,\infty} \), which consists\(^1\) of all compact operators such that, if \( \mu_1 \geq \mu_2 \geq \ldots \),

\[ \sum_{n=1}^N \mu_n = O(\log N). \]

There exists a linear functional on the space \( \mathcal{L}^{1,\infty} \) that resembles the usual trace, called the Dixmier trace. Its definition is rather involved and we refer to \[8\].

\(^1\) Sometimes this ideal is denoted by \( \mathcal{L}_{bt} \), the notation \( \mathcal{L}^{1,\infty} \) being reserved for the (smaller) class of operators for which \( \mu_n = O(1/n) \). Our notation follows Connes’ book \[6\].
Chapter IV for details. Let \( C_b(\mathbb{R}_+) \) be the space of bounded continuous functions on \( \mathbb{R}_+ \) and \( C_0(\mathbb{R}_+) \) the subspace of functions vanishing at \( \infty \). Let \( \omega \) be a positive linear functional on the quotient space \( C_b(\mathbb{R}_+)/C_0(\mathbb{R}_+) \) such that \( \omega(1) = 1 \). For a positive compact operator \( T \in \mathcal{L}^{1,\infty} \) with eigenvalues \( \{\mu_n\} \), extend \( \mu_n \) to a step function on \( \mathbb{R}_+ \) and let \( M_T(\lambda) \) be its Cesáro mean, which is a bounded continuous function on \( \mathbb{R}_+^+ \). The Dixmier trace of \( T \) is then defined by

\[
\text{tr}_\omega T = \omega(M_T).
\]

It is then extended to all of \( \mathcal{L}^{1,\infty} \) by linearity. In particular it is bounded and vanishes on trace class operators. The fact that we will need is that

\[
\text{tr}_\omega T = \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \mu_n(T)
\]

if \( T \) is a positive operator and if the right hand side exists.

**Lemma 3.1.** For any \( c \geq 0 \) the operator \( (c - \rho(\Delta))^{-d} = (c\delta_0 - \Delta)^{-d} \) is in the Macaev class \( \mathcal{L}^{1,\infty} \).

**Proof.** It follows from (2.2) that the eigenvalues of \( (c - \rho(\Delta))^d \) are \( (c + \pi(m + \frac{d}{2}))^d \), \( m = 0, 1, \cdots \), each of multiplicity \( d_m := \dim\{w^\alpha, |\alpha| = m\} = \binom{d+m-1}{d-1} \approx m^{d-1} \).

The partial sums thus satisfy

\[
\sum_{m \leq N} (c + \pi(m + \frac{d}{2}))^{-d} d_m \approx \frac{1}{(d-1)!} \sum_{m \leq N} (c + \pi(m + \frac{d}{2}))^{-d} m^{d-1} \approx \frac{1}{(d-1)!} \log N,
\]

completing the proof. \( \square \)

**Proposition 3.2.** Let \( F \in \mathcal{PT}(-2d,1) \). Then the Weyl transform \( \rho(F) \) is in the Macaev class \( \mathcal{L}^{1,\infty} \).

**Proof.** By (3.5.6) in [10],

\[
\hat{\Delta} = -\frac{\pi^2}{4} |w|^2,
\]

so \( -\Delta \in \mathcal{PT}(2,1) \), whence by (2.4) \( (-\Delta)^{\ast d} \in \mathcal{PT}(2d,1) \) and \( (-\Delta)^{\ast d} F \in \mathcal{PT}(0,1) \).

By the Calderón-Vaillancourt theorem [10 Theorem 3.1.3], the corresponding Weyl transform, \( \rho(-\Delta)^{\ast d}(\rho(F)) \), is bounded. Hence by the previous lemma, \( \rho(F) \in \mathcal{L}^{1,\infty} \), since the Macaev class \( \mathcal{L}^{1,\infty} \) is an ideal. \( \square \)

4. Dixmier trace formula for Toeplitz operators

**Theorem 4.1.** Let \( F \in \mathcal{PT}_{rad}(-2d,1) \) with the principal symbol \( \sigma_{-2d}(\hat{F}) \) as defined in (2.3). Then the Dixmier trace \( \text{tr}_\omega \rho(F) \) is independent of \( \omega \) and is given by

\[
\text{tr}_\omega \rho(F) = \frac{\pi^d}{4^d d!} \int_S \sigma_{-2d}(F)(w),
\]

where \( \int_S \) is the normalized integral over the unit sphere.
Proof. The proof is quite similar to that of Connes [7] for pseudo-differential operators on compact manifolds. Namely, by [10, Theorem 4.2.5] and the definition of $\mathcal{PT}_{rad}$, the Dixmier trace $\text{tr}_\omega \rho(\hat{F})$ depends only on the leading symbol of $\sigma_{-2d}(\hat{F})$ and defines a positive measure on the unit sphere $S$ in $\mathbb{C}^d$. By the unitary invariance of $\rho(F)$ the measure has to be a constant multiple of the area measure. To find the constant we note that the symbol of $c\delta_0 - \Delta, c > 0$, is absolutely elliptic in the sense of (4.2.20) in [10], and thus by pp. 246–247 in [10] we can construct $F_0 \in \mathcal{PT}_{rad}(-2d, 1)$ such that $\rho(F_0) = (c - \rho(\Delta))^{-d}$. The eigenvalue of $\rho(F_0)$ on the space of all $m$-homogeneous polynomials is, by the proof of Lemma 3.1,

$$\frac{1}{(c + \pi(m + \frac{d}{2}))^d}. $$

Its Dixmier trace exists and is (noticing that the dimension of the space of homogeneous polynomials of degree $m \leq N$ is $\approx N^d$)

$$\text{tr}_\omega \rho(F_0) = \frac{1}{\pi^d d!}. $$

On the other hand, the principal symbol $\sigma_{-2d}(F_0)$ is the constant function $(4/\pi^2)^d |w|^{-2d}$ by the definition (cf. (3.1)), whose integration over the sphere is $(4/\pi^2)^d d$. This completes the proof. □

To apply our result to Toeplitz operators we need to introduce some more notation. We let

$$\partial_j^b = \partial_j - \bar{z}_j R, \quad \bar{\partial}_j^b = \bar{\partial}_j - z_j \bar{R}$$

be the boundary Cauchy-Riemann operators [17], where $R = \sum_{j=1}^d z_j \partial_j$ is the holomorphic radial derivative. As vector fields they are linearly dependent, to wit,

$$\sum_{j=1}^d z_j \partial_j^b = 0, \quad \sum_{j=1}^d \bar{z}_j \bar{\partial}_j^b = 0. $$

Definition 4.2. We define a bracket $\{f, g\}_b$ for smooth functions $f$ and $g$ on $S$ by

$$\{f, g\}_b := \sum_{j=1}^d (\partial_j^b f \bar{\partial}_j^b g - \bar{\partial}_j^b f \partial_j^b g)$$

and call it the boundary Poisson bracket.

Lemma 4.3. Let $F$ and $G$ be two functions in $\mathcal{PT}_{rad}(0, \mu)$ with principal symbols

$$\sigma_0(F)(z) = f\left(\frac{z}{|z|}\right), \quad \sigma_0(G)(z) = g\left(\frac{z}{|z|}\right)$$

for $f$ and $g$ in $C^\infty(S)$. Then the principal symbol of $F \ast G - G \ast F$ is given by

$$\sigma_{-2}(F \ast G - G \ast F)(z) = \frac{4}{\pi} \{f, g\}_b(\frac{z}{|z|})|z|^{-2}. $$
Proof. By the general result for the symbol calculus for pseudo-Toeplitz operators (cf. (2.2.5) in [10]), we have $F \ast G - G \ast F \in \mathcal{PT}_{rad}(-2\mu, \mu)$ with the principal symbol
\[ \sigma_{-2}(F \ast G - G \ast F)(z) = \frac{4}{\pi} \{\sigma_0(F), \sigma_0(G)\}(z), \]
where $\{\cdot, \cdot\}$ is the ordinary Poisson bracket in complex coordinates
\[ \{\Psi, \Phi\} := \sum_{j=1}^{d} (\partial_j \Psi \overline{\partial_j \Phi} - \partial_j \overline{\Phi} \partial_j \Phi). \]
The function $\sigma_{-2}(F \ast G - G \ast F)(z)$ is positive homogeneous of degree $-2$. We need only to compute it for $z \in S$. Defining the Reeb vector field $E$ and the outward normal vector field $N$ in terms of the radial derivative $R$,
\[ E := \frac{1}{2}(\bar{R} - R), \quad N := \bar{R} + R, \]
we can write
\[ R = -E + \frac{N}{2}. \]
Note that $E$ is well-defined on $S$. The vector field $\overline{\partial_j \Phi}(z) = (\partial_j^b - \overline{z_j}E)\Phi(z)$ is thus a well-defined vector field on $S$, and for any function $\Phi(z) = \phi(\frac{1}{|z|})$ we have
\[ \partial_j \Phi(z) = (\partial_j^b + \overline{z_j}R)\Phi(z) = (\partial_j^b - \overline{z_j}E + \frac{\overline{z_j}}{2}N)\Phi(z) = (\partial_j^b - \overline{z_j}E)\phi(z), \]
since $N\Phi(z) = 0$ by homogeneity. Similarly $\overline{\partial_j \Phi} = (\partial_j^b + z_j E)\phi$ on $S$. From this it follows that for $z \in S$,
\[ \{\sigma_0(F), \sigma_0(G)\}(z) = \sum_{j=1}^{d} \left( (\partial_j^b f(z) - \overline{z_j}E f(z))(\overline{\partial_j^b g(z)} + z_j E g(z)) - (\overline{\partial_j^b f(z)} - \overline{z_j}E f(z))(\partial_j^b g(z) + z_j E g(z)) \right) = \{f, g\}_b. \]
by using (4.1). \hfill \Box

Theorem 4.4. Let $f_1, g_1, \cdots, f_d, g_d$ be smooth functions on $S$, $\tilde{f}_1, \tilde{g}_1, \cdots, \tilde{f}_d, \tilde{g}_d$ their smooth extensions to $B$ and $T_{\tilde{f}_1}, T_{\tilde{g}_1}, \cdots, T_{\tilde{f}_d}, T_{\tilde{g}_d}$ the associated Toeplitz operators on $\mathcal{H}_\nu$ for $\nu \geq d$. Then the product $\prod_{j=1}^{d} [T_{\tilde{f}_j}, T_{\tilde{g}_j}]$ is in the Macaev class and its Dixmier trace is given by
\[ \text{tr}_\omega \prod_{j=1}^{d} [T_{\tilde{f}_j}, T_{\tilde{g}_j}] = \frac{1}{d!} \int_S \prod_{j=1}^{d} \{f_j, g_j\}_b. \]
Proof. The proof is straightforward from the preceding lemma, formula (2.2.5) in [10] and Theorem 4.1. \hfill \Box

We apply our result to Hankel operators with anti-holomorphic symbols. Let $f$ be a holomorphic function in a neighborhood of $B$ and $H_f g = (I + P)\tilde{f}g$, $g \in \mathcal{H}_\nu$ the Hankel operator. Then
\[ [T_f, T_f] = [T_f^*, T_f] = |H_f|^2 = H_f^* H_f. \]
Corollary 4.5. Let \( f \) be as above. Then the Hankel operator is in \( \mathcal{L}^{2d, \infty} \); equivalently the commutator \([\bar{T}_f, T_f]\) is in \( \mathcal{L}^{d, \infty} \) and we have

\[
\operatorname{tr}_\omega (|H_f|^2) = \operatorname{tr}_\omega (|\bar{T}_f|^2) = \frac{1}{d!} \int_S (|\nabla f|^2 - |Rf|^2)^d.
\]

Notice that \( H_f \) is in the Schatten class \( \mathcal{L}^p \) for \( p > 2d \) and that its Schatten norm is

\[
||H_f||_p^p \approx \int_B (1 - |z|^2)^p (|\nabla f|^2 - |Rf|^2)^{\frac{p}{2}} dm(z);
\]

see [3] and [19] for the Bergman space case (\( \nu = d + 1 \)) and the Hardy space case (\( \nu = d \)). Our resulting formula thus provides a limiting result of the above estimates, and it is interesting to note that the estimate has an equality as its limit for \( p \to 2d \).

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