GENERALIZED DIMENSION DISTORTION
UNDER PLANAR SOBOLEV HOMEOMORPHISMS

PEKKA KOSKELA, ALEKSANDRA ZAPADINSKAYA, AND THOMAS ZÜRCHER

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ABSTRACT. We prove essentially sharp dimension distortion estimates for planar Sobolev-Orlicz homeomorphisms.

1. Introduction

Let \( \Omega, \Omega' \subset \mathbb{R}^2 \) be open and connected. We consider homeomorphisms \( f : \Omega \to \Omega' \) that belong to the Sobolev class \( W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^2) \), which means that both component functions of \( f \) have locally integrable distributional partial derivatives. It is by now well-known that the Luzin condition (\( N \)), which requires that \( f \) map Lebesgue null sets to Lebesgue null sets, holds if we additionally assume that \( |Df| \in L^2_{\text{loc}}(\Omega) \) \([18, 15, 14]\), but may fail if \( |Df| \in L^p_{\text{loc}}(\Omega) \) for some \( p < 2 \) \([16, 17]\). On the other hand, if \( |Df| \in L^p_{\text{loc}}(\Omega) \) for some \( p > 2 \), then the image of any set of Hausdorff dimension strictly less than two is also of Hausdorff dimension strictly less than two \([4, 10]\). Recently it was proven \([11]\) that local integrability of \( |Df|^2 \log^{-1}(e + |Df|) \) already suffices for the Luzin condition (\( N \)) to hold. The motivation for this result and our results below arises in part from the theory of mappings with finite distortion, where the natural regularity assumption is that \( |Df|^2 \log^{\lambda-1}(e+|Df|) \in L^1_{\text{loc}} \) for some \( \lambda > 0 \) \([2, 1, 7, 8, 3]\).

Analogously to the \( L^p \)-scale setting, one expects some kind of dimension distortion estimate to hold when \( \lambda \) as above is strictly positive. However, it is rather easy to map, for example, a subset of the real line onto a set of Hausdorff dimension two \([6, 19]\), and thus we have to work with a refined scale. Towards this end, we consider the gauge functions \( h_\lambda(t) = t^2 \log^{\frac{1}{\lambda}}(1/e) \), \( \lambda > 0 \). In Section 2 we describe a homeomorphism \( f \) that maps a Cantor set \( E \) of Minkowski (and so also Hausdorff) dimension strictly less than two to a set of positive \( H^{\lambda}\)-measure, with \( |Df|^2 \log^{-1}(e + |Df|) \in L^1_{\text{loc}} \) for all \( t < \lambda \).

Our main result shows that this homeomorphism is critical for our generalized dimension distortion.

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3815
Theorem 1.1. Let \( \Omega \) and \( \Omega' \) be open sets in \( \mathbb{R}^2 \) and \( f: \Omega \to \Omega' \) a homeomorphism of class \( W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^2) \) with

\[
|Df|^2 \log^{\lambda-1}(e + |Df|) \in L^1_{\text{loc}}(\Omega)
\]

for some \( \lambda > 0 \). Then

\[
\mathcal{H}^{h \chi}(f(E)) = 0
\]

for every set \( E \subset \Omega \) of lower Minkowski dimension \( \dim_{\mathcal{M}}(E) \) strictly less than two.

We conjecture that one may replace the Minkowski dimension in Theorem 1.1 with the Hausdorff dimension. For a related, weaker result in this direction, see [12].

This note is organized as follows. In Section 2 we recall the necessary definitions and describe the construction of the homeomorphism referred to above. Section 3 contains the proof of Theorem 1.1.

2. Preliminaries

Let \( U \subset \mathbb{R}^2 \) be open and connected. We say that a mapping \( f \in L^1(U; \mathbb{R}^2) \) has bounded variation, \( f \in BV(U) \), if the component functions \( f_1 \) and \( f_2 \) of \( f \) are of bounded variation, that is,

\[
\sup \left\{ \int_U f_i \, \text{div} \, \phi \, dx \mid \phi \in C^0_c(U; \mathbb{R}^2), \ |\phi| \leq 1 \right\} < \infty, \ i = 1, 2.
\]

We write \( f \in BV_{\text{loc}}(U) \) if \( f \in BV(G) \) for each open and connected \( G \) compactly contained in \( U \). For each function \( g \in BV(U; \mathbb{R}) \) of bounded variation we can define a Radon measure \( ||Dg|| \) in the following way: for an open set \( V \subset U \) we put

\[
||Dg||(V) = \sup \left\{ \int_V g \, \text{div} \, \phi \, dx \mid \phi \in C^0_c(V; \mathbb{R}^2), \ |\phi| \leq 1 \right\},
\]

and for \( A \subset U \) not necessarily open,

\[
||Dg|||(A) = \inf \left\{ ||Dg||(V) \mid A \subset V \subset U, \ V \text{ is open} \right\}.
\]

For a set \( V \) and a number \( \delta > 0 \), \( V + \delta \) denotes the set \( \{ y \mid \text{dist}(y, V) < \delta \} \).

We write \( \mathcal{H}^h(A) \) for the generalized Hausdorff measure of a set \( A \), given by

\[
\mathcal{H}^h(A) = \lim_{\delta \to 0} \mathcal{H}^h_{\delta}(A) = \lim_{\delta \to 0} \left[ \inf \left\{ \sum_{i=1}^{\infty} h(\text{diam} U_i) : A \subset \bigcup_{i=1}^{\infty} U_i, \text{diam} U_i \leq \delta \right\} \right],
\]

where \( h \) is a dimension gauge (non-decreasing, with \( h(0) = 0 \)). If \( h(t) = t^\alpha \) for some \( \alpha \geq 0 \), we write simply \( \mathcal{H}^\alpha \) for \( \mathcal{H}^t^\alpha \) and call it the Hausdorff \( \alpha \)-dimensional measure; the Hausdorff dimension \( \dim_{\mathcal{H}}(A) \) of the set \( A \) is the smallest \( \alpha_0 \geq 0 \) such that \( \mathcal{H}^\alpha(A) = 0 \) for any \( \alpha > \alpha_0 \). The lower Minkowski dimension \( \dim_{\mathcal{M}}(A) \) of a bounded set \( A \subset \mathbb{R}^2 \) is defined as

\[
\dim_{\mathcal{M}}(A) = \inf \{ s : \lim_{\varepsilon \to 0^+} \inf N(A, \varepsilon) \varepsilon^s = 0 \},
\]

where \( N(A, \varepsilon), \varepsilon > 0 \), denotes the smallest number of balls of radius \( \varepsilon \) needed to cover \( A \):

\[
N(A, \varepsilon) = \min \{ k : A \subset \bigcup_{i=1}^{k} B(x_i, \varepsilon) \text{ for some } x_i \in \mathbb{R}^2 \}.
\]

Finally, let \( a \lesssim b \) mean that there exists some constant \( C > 0 \) such that \( a \leq Cb \).
In [6] a homeomorphism \( h: \mathbb{R}^n \to \mathbb{R}^n \) was constructed which maps a set \( C \) of Minkowski and Hausdorff dimension \( n \log 2/\log (1/\sigma) \) for some \( 0 < \sigma < 1/2 \) onto a set \( C' \) of positive \( \mathcal{H}^b \)-measure with \( h(t) = t^n(\log(1/t))^{p_n} \) for given \( p > 0 \). This mapping is the identity outside the cube \([0,1]^n\) and satisfies \( |Dh(x)| \leq \frac{\tau_1 \cdots \tau_k}{\sigma^k} \) in \( A_{k,i} \). Here \( A_{k,i} \), \( k = 1, 2, \ldots \) and \( i = 1, \ldots, 2^{kn} \), are the open "cubical frames" needed to construct the Cantor set \( C \). They are pairwise disjoint with respect to both \( i \) and \( k \); that is, \( \text{int}(A_{k,i}) \cap \text{int}(A_{k,j}) = \emptyset \) when \((k,i) \neq (l,j)\), they cover the set \([0,1]^n\) up to a set of zero \( n \)-Lebesgue measure, and each \( A_{k,i} \) is contained in a cube of edge length \((1/2)\sigma^{k-1}\). The numbers \( \tau_k \), \( k = 1, 2, \ldots \), used to construct the image Cantor-type set are defined as follows:

\[
\tau_1 = \frac{1}{2} \frac{1}{\log^p 4} \quad \text{and} \quad \tau_k = \frac{1}{2} \left( 1 - \frac{1}{k} \right)^p \quad \text{for} \quad k = 2, 3, \ldots.
\]

Note that

\[
\tau_1 \cdots \tau_k = \frac{1}{2^k} \frac{1}{\log^p 4} \frac{1}{k^p},
\]

so in the case \( n = 2 \), we have

\[
\int_{[0,1]^2} |Dh|^2 \log^s(e + |Dh|) = \sum_{k=1}^{\infty} \sum_{i=1}^{4^k} \int_{A_{k,i}} |Dh|^2 \log^s(e + |Dh|)
\leq \sum_{k=1}^{\infty} 4^k \frac{1}{4} \sigma^{2k-2} \left( \frac{\tau_1 \cdots \tau_k}{\sigma^{2k}} \right)^2 \log^s(e + \frac{\tau_1 \cdots \tau_k}{\sigma^k})
= \sum_{k=1}^{\infty} \frac{1}{4 \sigma^{2k} k^{2p} \log^{2p} 4} \log^s(e + \frac{1}{(2\sigma)^k k^p \log^p 4})
\leq \sum_{k=1}^{\infty} k^{s-2p} < \infty
\]

when \( s + 1 < 2p \).

3. Proofs

Clearly, we may assume in the rest of this paper that \( \Omega \) is an open and connected subset of \( \mathbb{R}^2 \). We begin with the following lemma.

Lemma 3.1. Let \( f: \Omega \to f(\Omega) \subset \mathbb{R}^2 \) be a homeomorphism in \( W^{1,1}_{loc}(\Omega, \mathbb{R}^2) \). Then there exists a set \( F \subset f(\Omega) \) with \( H^{1/2}(F) = 0 \) such that for all \( y \in f(\Omega) \setminus F \) there exist constants \( C_y > 0 \) and \( r_y > 0 \) such that

\[
(3.1) \quad \text{diam}(f^{-1}(B(y, r))) \leq C_y r^{1/2}
\]

for all \( 0 < r < r_y \).

Proof. First, note that by Theorem 1.2 in [5], \( f^{-1} \) is in \( BV_{loc}(f(\Omega)) \). Next, fix \( y \in f(\Omega) \) and \( r > 0 \) such that \( B(y, 3r) \subset f(\Omega) \). Let \( Q(y, t) \) be the square centered at \( y \) and having edge length \( 2t \). As \( f^{-1} \) is a homeomorphism, for \( t \in (r, 2r) \) we have

\[
\text{diam } f^{-1}(B(y, r)) < \text{diam } f^{-1}(Q(y, t)) \leq \text{diam } f^{-1}(\partial Q(y, t))
\leq \text{diam } f^{-1}_1(\partial Q(y, t)) + \text{diam } f^{-1}_2(\partial Q(y, t)),
\]

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where \( f_i^{-1}, i = 1, 2 \), denotes the \( i \)-th component function of \( f^{-1} \). Integrating this inequality over the interval \([r, 2r]\) with respect to \( t \), we obtain

\[
(3.2) \quad r \operatorname{diam} f^{-1}(B(y, r)) < \sum_{i=1}^{2} \int_{[r, 2r]} \operatorname{diam} f_i^{-1}(\partial Q(y, t))dt.
\]

Let us consider the smooth approximation \( g_i^\varepsilon = \eta_\varepsilon * f_i^{-1} \) of \( f_i^{-1}, i = 1, 2 \), on the cube \( Q(y, 2r) \). Here \( \eta_\varepsilon \) is a standard bump function. As \( f^{-1} \) is continuous, the convergence \( g_i^\varepsilon \to f_i^{-1} \) is pointwise and uniform on each compact set \( K \subset Q(y, 2r) \). So, for \( t \in (r, 2r) \) and \( i = 1, 2 \) we have \( \operatorname{diam} f_i^{-1}(\partial Q(y, t)) = \lim_{\varepsilon \to 0} \operatorname{diam} g_i^\varepsilon(\partial Q(y, t)) \).

Put \( a_i = y_i - 2r \) and \( b_i = y_i + 2r \), \( i = 1, 2 \), where \( y = (y_1, y_2) \). Fatou’s Lemma implies that

\[
(3.3) \quad \int_{[r, 2r]} \operatorname{diam} f_i^{-1}(\partial Q(y, t))dt = \int_{[r, 2r]} \lim_{\varepsilon \to 0} \operatorname{diam} g_i^\varepsilon(\partial Q(y, t))dt
\]

\[
\leq \liminf_{\varepsilon \to 0} \int_{[r, 2r]} \operatorname{diam} g_i^\varepsilon(\partial Q(y, t))dt.
\]

We use the fundamental theorem of calculus and Fubini’s theorem to obtain

\[
(3.4) \quad \int_{[r, 2r]} \operatorname{diam} g_i^\varepsilon(\partial Q(y, t))dt \leq \int_{[r, 2r]} \left\{ \int_{[a_2, b_2]} \left| \frac{\partial g_i^\varepsilon}{\partial \xi}(y_1 - t, \xi) \right| d\xi + \int_{[a_1, b_1]} \left| \frac{\partial g_i^\varepsilon}{\partial \xi}(\xi, y_2 - t) \right| d\xi \right. \\
+ \int_{[a_1, b_1]} \left| \frac{\partial g_i^\varepsilon}{\partial \xi}(\xi, y_2 + t) \right| d\xi \left. \right\} dt = \int_{[a_1, y_1 - r] \times [a_2, b_2]} \left| \frac{\partial g_i^\varepsilon}{\partial x_2}(x) \right| dx \\
+ \int_{[y_1 + r, b_1] \times [a_2, b_2]} \left| \frac{\partial g_i^\varepsilon}{\partial x_2}(x) \right| dx + \int_{[a_1, b_1] \times [a_2, y_2 - r]} \left| \frac{\partial g_i^\varepsilon}{\partial x_1}(x) \right| dx \\
+ \int_{[a_1, b_1] \times [y_2 + r, b_2]} \left| \frac{\partial g_i^\varepsilon}{\partial x_1}(x) \right| dx \leq \sum_{j=1}^{2} \int_{Q(y, 2r)} \left| \frac{\partial g_i^\varepsilon}{\partial x_j}(x) \right| dx
\]

for \( i = 1, 2 \). Let us show that

\[
(3.5) \quad \int_{Q(y, 2r)} \left| \frac{\partial g_i^\varepsilon}{\partial x_j}(x) \right| dx \leq ||Df_i^{-1}||(Q(y, 2r))
\]

for \( i, j = 1, 2 \). Given \( \varphi \in C_0^1(Q(y, 2r)), |\varphi| \leq 1 \), we may write

\[
\int_{Q(y, 2r)} \frac{\partial g_i^\varepsilon}{\partial x_j}(x) \varphi dx = - \int_{Q(y, 2r)} g_i^\varepsilon \frac{\partial \varphi}{\partial x_j} dx = - \int_{Q(y, 2r)} (\eta_\varepsilon * f_i^{-1}) \frac{\partial \varphi}{\partial x_j} dx \\
= - \int_{Q(y, 2r)} f_i^{-1} \frac{\partial \eta_\varepsilon * \varphi}{\partial x_j} dx \leq ||Df_i^{-1}||(Q(y, 2r)).
\]

This implies (3.3), and combining it with (3.2), (3.3) and (3.4), we finally obtain

\[
\operatorname{diam} f^{-1}(B(y, r)) < \frac{2}{r}(||Df_1^{-1}||(Q(y, 2r)) + ||Df_2^{-1}||(Q(y, 2r)))
\]

for all \( y \in f(\Omega) \) and \( r > 0 \) such that \( B(y, 3r) \subset f(\Omega) \). That is, the inequality (3.1) holds for all \( y \in f(\Omega) \) such that

\[
(3.6) \quad \frac{||Df_i^{-1}||(Q(y, 2r))}{r^{3/2}} < M_y
\]
is valid for \( i = 1, 2 \), all small enough \( r > 0 \) and some constant \( M_y \) depending on \( y \). Let \( F_1 \) be the set of those \( y \) for which (5.6) does not hold for \( i = 1 \). Let \( K \subset f(\Omega) \) be a compact set and fix some \( \delta > 0 \) such that \( \text{dist}(K, \partial f(\Omega)) > \delta \).

For every \( k \in \mathbb{N} \) and every \( y \in F_1 \cap K \) there exists \( r_{k,y} < \delta \sqrt{2}/20 \) such that 
\[
\|Df_k^{-1}\|(Q(y, 2r_{k,y})) \geq k (r_{k,y})^{3/2}.
\]
Consider the collection of all balls 
\[
B_k = \{ B(y, 2\sqrt{2}r_{k,y}) : y \in F_1 \cap K \}
\]
for \( k \in \mathbb{N} \). Using Vitali’s covering theorem, we obtain for every \( k \in \mathbb{N} \) a countable subcollection of disjoint balls \( B_{k,j}, j = 1, 2, \ldots, \) centered in \( F_1 \cap K \), having radii \( 2\sqrt{2}r_{k,j} < \delta/5 \) and with \( 5B_{k,j} \) covering \( F_1 \cap K \). As \( Q(y, 2r_{k,j}) \subset B_{k,j} \), we have
\[
\mathcal{H}^{3/2}(F_1 \cap K) \leq \sum_{j=1}^{\infty} \left( 10\sqrt{2}r_{k,j} \right)^{3/2} \leq \frac{10\sqrt{2}}{k} \sum_{j=1}^{\infty} \|Df_k^{-1}\|(Q(y, 2r_{k,j}))
\]
\[
\leq \frac{10\sqrt{2}}{k} \sum_{j=1}^{\infty} \|Df_k^{-1}\|(B_{k,j}) \leq \frac{10\sqrt{2}}{k} \|Df_k^{-1}\|(K + \delta/5)
\]
for all \( k \in \mathbb{N} \). Letting \( k \to \infty \) and \( \delta \to 0 \), we obtain \( \mathcal{H}^{3/2}(F_1 \cap K) = 0 \).

The previous lemma implies the following result.

**Lemma 3.2.** Let \( E \subset \Omega \) and let \( f : \Omega \to f(\Omega) \subset \mathbb{R}^2 \) be a homeomorphism of class \( W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \). Then there exists a decomposition \( f(E) = \bigcup_{i=0}^{\infty} F_i \) where \( \mathcal{H}^{3/2}(F_0) = 0 \) and for each \( F_i, i = 1, 2, \ldots, \) there exist constants \( C_i < \infty \) and \( r_i > 0 \) such that 
\[
f^{-1}(F_i + r) \subset E + C_i r^{1/2}
\]
for every \( r \in (0, r_i) \).

**Proof.** We choose \( F_0 = F \), where \( F \) is the set from the previous lemma. Moreover, by this lemma we may represent the set \( f(E) \) as
\[
f(E) = F_0 \cup \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \{ y \in f(E) \mid \text{diam}(f^{-1}(B(y, r))) \leq kr_j^{1/2} \text{ for all } r \in (0, \frac{1}{j}) \}.
\]
So, putting \( C_i = C_{i(j,k)} = k \) and \( r_i = r_{i(j,k)} = \frac{1}{j} \), we complete the proof.

**Proof of Theorem 1.1.** As \( f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \) is a homeomorphism, its Jacobian \( J_f \) is either non-negative almost everywhere in \( \Omega \) or non-positive almost everywhere in \( \Omega \). We may assume that \( J_f \geq 0 \) almost everywhere in \( \Omega \). Recalling that
\[
|Df|^2 \geq \lambda^{-1}(\lambda + |Df|) \in L^1_{\text{loc}}(\Omega),
\]
by Corollary 9.1 in [10], we have \( J_f \log(\lambda + J_f) \in L^1_{\text{loc}} \). Next, as \( \dim_M(E) < 2 \), there exist constants \( C, \varepsilon > 0 \) and a sequence of numbers \( r_j, j = 1, 2, \ldots, \) tending to zero as \( j \to \infty \), such that \( L^2(E + r_j) \leq Cr_j^2 \) for all \( j = 1, 2, \ldots, \). By Lemma 3.2

we have \( f(E) = \bigcup_{i=0}^{\infty} F_i \) where \( \mathcal{H}^{3/2}(F_0) = 0 \) and \( f^{-1}(F_i + R_{i,j}) \subset E + r_j \) for all large enough \( j \) \( (j \geq j_i \text{ for some } j_i \in \mathbb{N}) \). Here \( R_{i,j} = (r_j/C_i)^2 \) and \( C_i \) are the constants from Lemma 3.2. It suffices to show that \( \mathcal{H}^H(F_i) = 0 \) for all \( i \in \mathbb{N} \). We use the fact
that $L^2(f(A)) \leq \int_A J_f$ for each open $A \subset \Omega$ \cite[Lemma 3.2]{KZ}. Thus, for a fixed $i \in \mathbb{N}$, we have

$$L^2(F_i + R_{i,j}) \leq \int_{f^{-1}(F_i + R_{i,j})} J_f(x) dx \leq \int_{E+\tau_j} J_f(x) dx$$

$$\leq \int_{\{x \in E+\tau_j: J_f(x) < r_j^{-\epsilon/2}\}} J_f + \int_{\{x \in E+\tau_j: J_f(x) \geq r_j^{-\epsilon/2}\}} J_f$$

$$\leq r_j^{-\epsilon/2} L^2(E + r_j) + \log^{-\lambda}(e + r_j^{-\epsilon/2}) \int_{E+\tau_j} \frac{1}{r_j} J_f \log^\lambda(e + J_f)$$

$$\leq C r_j^{\epsilon/2} + M(r_j) \log^{-\lambda} \frac{1}{r_j}$$

for big enough $j$, where $M(r) \to 0$ as $r \to 0$. In other words,

$$L^2(F_i + R_{i,j}) = o(\log^{-\lambda} \frac{1}{r_j})$$

as $j \to \infty$. Using the Besicovitch covering theorem, for each large enough $j \in \mathbb{N}$, we can cover the set $F_i$ with $N$ countable families of pairwise disjoint balls centered in $F_i$ and of radius $R_{i,j}$ ($N$ is independent of both $i$ and $j$). It is obvious that each of these families is finite. Let $l_{i,j}$ denote the total number of covering balls. We have $L^2(F_i + R_{i,j}) \geq Cl_{i,j} R_{i,j}^2$, where $C$ is a constant independent of $i$ and $j$. So, for each fixed $i \in \mathbb{N}$ and all big enough $j \geq j_i$, we have

$$\mathcal{H}^h_{R_{i,j}}(F_i) \leq l_{i,j} R_{i,j}^2 \log^\lambda(1/R_{i,j}) \leq \frac{2\lambda}{C} L^2(F_i + R_{i,j}) \log^\lambda(C/r_j),$$

and thus (3.7) shows that $\mathcal{H}^h(F_i) = 0$. It follows that $\mathcal{H}^h(f(E)) = 0$. \hfill \Box

**References**


