EHRHART THEORY FOR LAWRENCE POLYTOPES AND ORBIFOLD COHOMOLOGY OF HYPERTORIC VARIETIES

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Abstract. We establish a connection between the orbifold cohomology of hypertoric varieties and the Ehrhart theory of Lawrence polytopes. More specifically, we show that the dimensions of the orbifold cohomology groups of a hypertoric variety are equal to the coefficients of the Ehrhart δ-polynomial of the associated Lawrence polytope. As a consequence, we deduce a formula for the Ehrhart δ-polynomial of a Lawrence polytope and use the injective part of the Hard Lefschetz Theorem for hypertoric varieties to deduce some inequalities between the coefficients of the δ-polynomial.

1. Introduction

Hypertoric varieties were introduced by Bielawski and Dancer [5] using a hyperkähler analogue of the construction of toric varieties by Kähler quotients. Hausel and Sturmfels [14, 15] showed that the cohomology of hypertoric varieties is intimately related to the combinatorics of matroids and hyperplane arrangements. Recently, Jiang and Tseng [21] and Goldin and Harada [11] independently gave explicit descriptions of the orbifold cohomology rings of hypertoric varieties. We will establish a relationship between the orbifold cohomology of hypertoric varieties and the lattice point enumeration of Lawrence polytopes.

Let N be a lattice of rank d and let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a configuration of n nonzero lattice points in N that generate N as an abelian group. Let $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ denote the dual lattice to N and let $\mathcal{R} = \{r_1, \ldots, r_n\}$ be a collection of real numbers. Consider the *weighted, co-oriented hyperplane arrangement* $\mathcal{H}$ in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ defined by the hyperplanes $H_i = \{u \in M_{\mathbb{R}} \mid \langle u, b_i \rangle = r_i\}$, for $i = 1, \ldots, n$, and the co-normal vectors $\mathcal{B}$. We may and will choose $\mathcal{R}$ so that the hyperplane arrangement $\mathcal{H}$ is *simple*. That is, the intersection of any l hyperplanes in $\mathcal{H}$ is either empty or has codimension l. The *Lawrence toric variety* $X_{\mathcal{H}}$ associated to $\mathcal{H}$ is a $(d + n)$-dimensional simplicial toric variety, and the corresponding hypertoric variety $\mathcal{M}_{\mathcal{H}}$ is a $2d$-dimensional complete intersection in $X_{\mathcal{H}}$, with a canonical orbifold structure [15, Section 6]. In Section 2, we will give a concrete local construction of $X_{\mathcal{H}}$ and $\mathcal{M}_{\mathcal{H}}$ in terms of $\mathcal{H}$, expanding on the descriptions in [15] and [21]. We refer the reader to [15] and [25] for an algebraic description of hypertoric varieties using Gale diagrams and for the construction of hypertoric varieties as hyperkähler quotients.

The theory of orbifold cohomology, developed by Chen and Ruan [7, 6], associates to an orbifold Y a finite-dimensional $\mathbb{Q}$-algebra $H^*_{\mathrm{orb}}(Y, \mathbb{Q})$, graded by $\mathbb{Q}$. In our case, the orbifold cohomology groups of $X_\mathcal{H}$ and $\mathcal{M}_\mathcal{H}$ are graded over $\mathbb{Z}$ and depend only on $\mathcal{B}$ and not on $\mathcal{H}$ [21, Theorem 1.1].

Theorem 1.1 ([20, Theorem 3.10]). *The orbifold cohomology groups $H^*_{\mathrm{orb}}(X_\mathcal{H}, \mathbb{Q})$ and $H^*_{\mathrm{orb}}(\mathcal{M}_\mathcal{H}, \mathbb{Q})$ are isomorphic as graded $\mathbb{Q}$-algebras and vanish in all degrees not in $2\mathbb{Z}$.*

If $e_1, \ldots, e_n$ denote the standard generators of $\mathbb{Z}^n$, then the *Lawrence polytope* $P_\mathcal{B}$ associated to $\mathcal{B}$ is the $(d + n - 1)$-dimensional lattice polytope in $N \times \mathbb{Z}^n$ given as the convex hull of $\{(b_i, e_i), (0, e_i) \mid i = 1, \ldots, n\}$. In fact, the lattice points $\{(b_i, e_i), (0, e_i) \mid i = 1, \ldots, n\}$ are the $2n$ vertices of $P_\mathcal{B}$. To see that (b_i, e_i) and $(0, e_i)$ are vertices, observe that, if $e_1^*, \ldots, e_n^*$ is the dual basis to $e_1, \ldots, e_n$, then (b_i, e_i) and $(0, e_i)$ are distinct lattice points in the affine hyperplane $\langle \ , (0, e_i^*) \rangle = 1$, while all (b_j, e_j) and $(0, e_j)$ with $i \neq j$ lie in the linear hyperplane $\langle \ , (0, e_i^*) \rangle = 0$. Lawrence polytopes have been crucial in the construction of non-rational polytopes in the work of Perles [12], Mnëv [24] and Ziegler [36, 35], and their combinatorial properties have been studied by Bayer and Sturmfels [3] and Santos [27]. Recently, they appeared in Batyrev and Nill's classification of degree 1 lattice polytopes [2] and in the construction of hypertoric varieties [15].

For every positive integer m, let $f_\mathcal{B}(m) := \#(mP_\mathcal{B} \cap (N \times \mathbb{Z}^n))$ denote the number of lattice points in the m'th dilate of $P_\mathcal{B}$. A famous theorem of Ehrhart [9] asserts that $f_\mathcal{B}(m)$ is a polynomial in m of degree $\dim P_\mathcal{B} = n + d - 1$, called the *Ehrhart polynomial* of $P_\mathcal{B}$. Equivalently, the generating series of $f_\mathcal{B}(m)$ can be written in the form

$$\sum_{m \geq 0} f_\mathcal{B}(m)\, t^m = \frac{\delta_\mathcal{B}(t)}{(1-t)^{n+d}}\,,$$

where $\delta_\mathcal{B}(t) = \delta_0 + \delta_1 t + \cdots + \delta_{n+d-1} t^{n+d-1}$ is a polynomial of degree at most $n + d - 1$ with integer coefficients, called the *Ehrhart δ-polynomial* of $P_\mathcal{B}$. Ehrhart δ-polynomials of lattice polytopes have been studied extensively over the last forty years by many authors, including Stanley [28, 29, 30, 31] and Hibi [16, 17, 18, 19]. In a recent paper [32], the author expressed the coefficients of the Ehrhart δ-polynomial of a lattice polytope as sums of dimensions of orbifold cohomology groups of a toric orbifold. We will use the following special case (cf. [22, Section 1]).

Theorem 1.2 ([32, Theorem 4.6]). *Let P be an m-dimensional lattice polytope in a lattice N' and let $\mathcal{T}$ be a lattice triangulation of P. If σ denotes the cone over $P \times \{1\}$ in $(N' \times \mathbb{Z})_\mathbb{R}$ and if $\triangle$ denotes the simplicial fan refinement of σ induced by $\mathcal{T}$, then we may consider the corresponding toric variety $Y = Y(\triangle)$, with its canonical orbifold structure. The Ehrhart δ-polynomial of P has the form*

$$\delta_P(t) = \sum_{i=0}^{m} \dim_\mathbb{Q} H^{2i}_{\mathrm{orb}}(Y, \mathbb{Q}) t^i.$$

We deduce the following geometric interpretation of $\delta_\mathcal{B}(t)$.

Theorem 1.3. *The Ehrhart δ-polynomial of the Lawrence polytope $P_\mathcal{B}$ has the form*

$$\delta_\mathcal{B}(t) = \sum_{i=0}^{n+d-1} \dim_\mathbb{Q} H^{2i}_{\mathrm{orb}}(X_\mathcal{H}, \mathbb{Q}) t^i = \sum_{i=0}^{n+d-1} \dim_\mathbb{Q} H^{2i}_{\mathrm{orb}}(\mathcal{M}_\mathcal{H}, \mathbb{Q}) t^i.$$

Proof. If $\psi : N \times \mathbb{Z}^n \to \mathbb{Z}$ is given by $\psi(\lambda + \sum_{i=1}^{n} \mu_i e_i) = \sum_{i=1}^{n} \mu_i$, for $\lambda \in N$ and integers μ_i, then we may choose an isomorphism $N \cong \psi^{-1}(0) \times \mathbb{Z}$ such that $P_{\mathcal{B}}$ is an $(n + d - 1)$-dimensional lattice polytope in $\psi^{-1}(0) \times \{1\}$. If $\Sigma_{\mathcal{H}}$ denotes the fan associated to the toric variety $X_{\mathcal{H}}$, then the hyperplane arrangement $\mathcal{H}$ induces a regular lattice triangulation $\mathcal{T}$ of $P_{\mathcal{B}}$ such that the cones of $\Sigma_{\mathcal{H}}$ are given by the cones over the faces of $\mathcal{T}$ [15, Proposition 4.2]. The result now follows from Theorem 1.1 and Theorem 1.2. $\qquad\square$

Theorem 1.3 and computations of the orbifold cohomology ring of $\mathcal{M}_{\mathcal{H}}$ by Jiang and Tseng [20, 21] give a combinatorial formula for $\delta_{\mathcal{B}}(t)$. In order to state the result, first recall that the *matroid* $M_{\mathcal{B}}$ associated to $\mathcal{B}$ is the collection of all linearly independent subsets of $\mathcal{B}$, as well as the origin $\{0\}$. The dimension of an element F in $M_{\mathcal{B}}$ is equal to the dimension of $\operatorname{span} F$, the span of the elements of F in $N_{\mathbb{R}}$. For any F in $M_{\mathcal{B}}$, consider the projection $\phi_F : N \to N/(N \cap \operatorname{span} F)$ and the induced configuration $\mathcal{B}_F = \{\phi_F(b_i) \mid b_i \notin \operatorname{span} F\}$ in $N/(N \cap \operatorname{span} F)$, with associated matroid M_F. The hyperplane arrangement $\mathcal{H}$ restricts to a simple hyperplane arrangement $\mathcal{H}^F$ on the affine space $\bigcap_{b_i \in F} H_i$, which we identify with $M_{F,\mathbb{R}} = \operatorname{Hom}_{\mathbb{R}}(N_{\mathbb{R}}/\operatorname{span} F, \mathbb{R})$ after translation. The h-vector of M_F is defined by

$$h_F(t) = \sum_{i=0}^{\operatorname{codim} F} f_i t^i (1 - t)^{\operatorname{codim} F - i},$$

where f_i equals the number of elements in M_F of dimension i. The hyperplane arrangement $\mathcal{H}^F$ divides $M_{F,\mathbb{R}}$ into locally closed cells (see Section 2), and we will consider the h-vector associated to the complex of bounded cells,

$$h_F^{\mathrm{bd}}(t) = \sum_{i=0}^{\operatorname{codim} F} f_i^{\mathrm{bd}}(t - 1)^i,$$

where f_i^{bd} equals the number of bounded cells in $\mathcal{H}^F$ of dimension i. Since $\mathcal{H}^F$ is simple, it follows from results of Zaslavsky that $h_F(t) = h_F^{\mathrm{bd}}(t)$ [34]. The following geometric interpretation of $h_F(t) = h_F^{\mathrm{bd}}(t)$ was observed by Hausel and Sturmfels after Konno had computed the cohomology ring of a hypertoric variety in [23].

Theorem 1.4 ([15, Theorem 1.1, Corollary 6.6])**.** *With the notation above, for any element F in $M_{\mathcal{B}}$, the varieties $X_{\mathcal{H}^F}$ and $\mathcal{M}_{\mathcal{H}^F}$ have no odd cohomology and*

$$h_F(t) = h_F^{\mathrm{bd}}(t) = \sum_{i=0}^{\operatorname{codim} F} \dim_{\mathbb{Q}} H^{2i}(X_{\mathcal{H}^F}, \mathbb{Q}) t^i = \sum_{i=0}^{\operatorname{codim} F} \dim_{\mathbb{Q}} H^{2i}(\mathcal{M}_{\mathcal{H}^F}, \mathbb{Q}) t^i.$$

If $F = \{b_{k_1}, \ldots, b_{k_r}\}$ lies in $M_{\mathcal{B}}$, then let $\operatorname{Box}(F) = \{v \in N_{\mathbb{R}} \mid v = \sum_{j=1}^{r} \alpha_j b_{k_j}, 0 < \alpha_j < 1\}$ and let $\operatorname{Box}(\{0\}) = \{0\}$. The following result follows from Theorem 1.4 and the work of Jiang and Tseng.

Theorem 1.5 ([21, Proposition 4.7])**.** *With the notation above,*

$$\sum_{i=0}^{n+d-1} \dim_{\mathbb{Q}} H_{\mathrm{orb}}^{2i}(X_{\mathcal{H}}, \mathbb{Q}) t^i = \sum_{i=0}^{d+n-1} \dim_{\mathbb{Q}} H_{\mathrm{orb}}^{2i}(\mathcal{M}_{\mathcal{H}}, \mathbb{Q}) t^i$$

$$= \sum_{F \in M} \#\left(\operatorname{Box}(F) \cap N\right) t^{\dim F} h_F(t).$$

We deduce the following result and refer the reader to Section 3 for a combinatorial proof.

Theorem 1.6. *The Ehrhart δ-polynomial of the Lawrence polytope $P_\mathcal{B}$ is equal to*

$$\delta_\mathcal{B}(t) = \sum_{F \in M} \# \left(\mathrm{Box}(F) \cap N \right) t^{\dim F} h_F(t) = \sum_{F \in M} \# \left(\mathrm{Box}(F) \cap N \right) t^{\dim F} h_F^{bd}(t).$$

In particular, if we write $b_i = a_i v_i$, for some primitive integer vector v_i and some positive integer a_i, and set R_F^{bd} to be the number of bounded regions of $M_{F,\mathbb{R}} \smallsetminus \mathcal{H}^F$, then $\delta_0 = 1$, $\delta_1 = \sum_{i=1}^n a_i - d$, $\delta_d = \sum_{F \in M} \# \left(\mathrm{Box}(F) \cap N \right) R_F^{bd}$ and $\delta_{d+1} = \cdots = \delta_{d+n-1} = 0$. If V_F denotes the number of vertices in $\mathcal{H}^F$, which equals the number of maximal elements of M_F, then $(d + n - 1)!$ times the volume of $P_\mathcal{B}$ is equal to $\sum_{F \in M} \# \left(\mathrm{Box}(F) \cap N \right) V_F$.

Remark 1.7. If we change the co-orientation of $\mathcal{B}$, i.e. replace some of the b_i by $-b_i$, then we obtain an isomorphic hypertoric variety $\mathcal{M}_\mathcal{H}$ [13, Theorem 2.2], and hence Theorem 1.3 implies that $\delta_\mathcal{B}(t)$ remains unchanged. This can also be deduced using the formula for $\delta_\mathcal{B}(t)$ in Theorem 1.6.

The matroid $M_\mathcal{B}$ is called *coloop free* if there does not exist an element b_i which lies in every maximal linearly independent subset of $\mathcal{B}$. One verifies that this condition holds if and only if $M_\mathbb{R} \smallsetminus \mathcal{H}$ contains a bounded region. Hausel and Sturmfels showed that if $M_\mathcal{B}$ is coloop free, then the injective part of the Hard Lefschetz theorem holds for the hypertoric variety $\mathcal{M}_\mathcal{H}$ and hence the h-vector of $M_\mathcal{B}$ is a *MacCauley vector* [15, Theorem 7.4] (cf. [33]). That is, if we write $h_{\{0\}}(t) = \sum_{i=0}^d h_i t^i$, then there exists a graded $\mathbb{Q}$-algebra $R = R_0 \oplus R_1 \oplus \cdots \oplus R_{\lfloor \frac{d}{2} \rfloor}$ generated by R_1 and with $g_i = h_i - h_{i-1} = \dim_\mathbb{Q} R_i$. We will use the following corollary and refer the reader to [14] for the definition of the g-inequalities.

Theorem 1.8 ([14, Corollary 4.2]). *If $M_\mathcal{B}$ is coloop free, then its h-vector satisfies $h_i \le h_j$, for $i \le j \le d - i$, and satisfies the g-inequalities.*

We deduce the following inequalities between the coefficients of the Ehrhart δ-polynomial of $P_\mathcal{B}$.

Theorem 1.9. *If $M_\mathcal{B}$ is coloop free, then the Ehrhart δ-polynomial of $P_\mathcal{B}$ satisfies $\delta_i \le \delta_j$, for $i \le j \le d - i$.*

Proof. One verifies that if $M_\mathcal{B}$ is coloop free, then M_F is coloop free for any element F in $M_\mathcal{B}$. By Theorem 1.6, $\delta_\mathcal{B}(t) = \sum_{F \in M} \# (\mathrm{Box}(F) \cap N) t^{\dim F} h_F(t)$. If we write $h_F(t) = \sum_{i=0}^{\mathrm{codim}\, F} h_{F,i} t^i$, then Theorem 1.8 implies that $h_{F,i-\dim F} \le h_{F,j-\dim F}$ for $i \le j \le d + \dim F - i$ and the result follows. $\square$

Remark 1.10. The theorem above fails if $M_\mathcal{B}$ is not coloop free. For example, if $N = \mathbb{Z}$ and $\mathcal{B} = \{1\}$, then P is an interval of length 1 and $\delta_0 = 1 > \delta_1 = 0$.

Example 1.11. Let $N = \mathbb{Z}^2$ and set $\mathcal{B} = \{b_1, b_2, b_3, b_4\} = \{(1,0), (0,1), (-2,0), (2,-1)\}$ and $\{r_1, r_2, r_3, r_4\} = \{0, 0, 2, -1\}$. The matroid $M_\mathcal{B}$ is given by $\{0, b_1, b_2, b_3, b_4, b_1 b_2, b_1 b_4, b_2 b_3, b_2 b_4, b_3 b_4\}$ and $h_{\{0\}}(t) = 1 + 2t + 2t^2$. We have $\mathrm{Box}(\{b_2 b_4\}) \cap N = \{(1,0)\}$, $M_{\{b_2 b_4\}} = \{0\}$ and $h_{\{b_2 b_4\}}(t) = 1$. Also $\mathrm{Box}(\{b_3\}) \cap N = \{(-1,0)\}$, $M_{\{b_3\}} = \{0, b_2, b_4\}$ and $h_{\{b_3\}}(t) = 1 + t$. We conclude that $P_\mathcal{B}$ is a 5-dimensional lattice polytope with 8 vertices and $\delta_P(t) = (1 + 2t + 2t^2) + t^2 + t(1 + t) = 1 + 3t + 4t^2$.
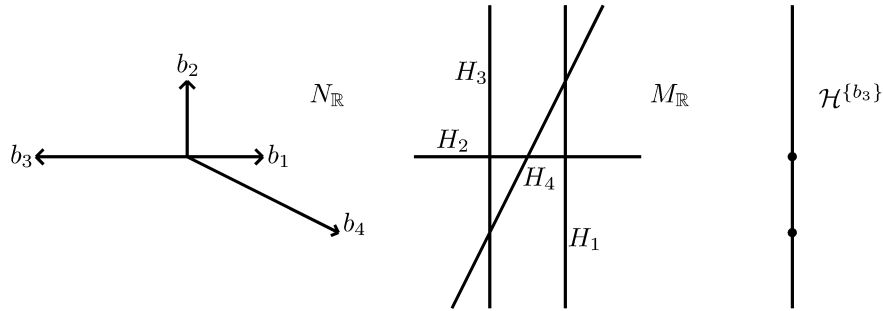
FIGURE 1

Remark 1.12. We have the following toric interpretation of the Ehrhart polynomial $f_{\mathcal{B}}(m)$ (see, for example, [8]). If P_i is the convex hull of the origin and b_i in $N_{\mathbb{R}}$, then one can consider the Minkowski sum $Q = P_1 + \cdots + P_n$ in $N_{\mathbb{R}}$. Each P_i determines a line bundle L_i on the d-dimensional toric variety Y corresponding to the polytope Q, and the toric variety corresponding to the polytope $P_{\mathcal{B}}$ is the $(d+n-1)$-dimensional projective bundle $\mathbb{P}_Y(L_1 \oplus \cdots \oplus L_n)$, with corresponding line bundle $\mathcal{O}(1)$. The Ehrhart polynomial is given by $f_{\mathcal{B}}(m) = \dim H^0(\mathbb{P}(L_1 \oplus \cdots \oplus L_n), \mathcal{O}(1)^{\otimes m})$, for positive integers m [10].

We conclude the introduction with a brief outline of the contents of the paper and note that all varieties and orbifolds will be over $\mathbb{C}$. In Section 2, we present a local construction of hypertoric varieties and use it to explain Theorem 1.5. In Section 3, we give a combinatorial proof of Theorem 1.6.

2. ORBIFOLD COHOMOLOGY OF HYPERTORIC VARIETIES

The goal of this section is to explain Theorem 1.5, which describes the dimensions of the orbifold cohomology groups of a hypertoric variety. We will first review the construction of Lawrence toric varieties and refer the reader to Section 4 in [15] for an equivalent presentation using Gale duality. We continue with the notation of the introduction and let $H_{i,+} = \{u \in M_{\mathbb{R}} \mid \langle u, b_i \rangle \geq r_i\}$ and $H_{i,-} = \{u \in M_{\mathbb{R}} \mid \langle u, b_i \rangle \leq r_i\}$, for $i = 1, \ldots, n$. The hyperplane arrangement $\mathcal{H}$ divides $M_{\mathbb{R}}$ into locally closed *cells* as follows: an element u in $M_{\mathbb{R}}$ lies in the cell given by the intersection of $\bigcap_{u \in H_{i,+}} H_{i,+}$ and $\bigcap_{u \in H_{i,-}} H_{i,-}$. If σ denotes the cone over the Lawrence polytope $P_{\mathcal{B}}$ in $N_{\mathbb{R}} \times \mathbb{R}^n$, then $\mathcal{H}$ determines a simplicial fan refinement $\Sigma_{\mathcal{H}}$ of σ [15]. If C is a cell of $\mathcal{H}$, then there is a corresponding simplicial cone σ_C in $\Sigma_{\mathcal{H}}$ with codimension equal to $\dim C$ and rays passing through the primitive integer vectors $\{(b_i, e_i) \mid C \subseteq H_{i,-}\}$ and $\{(0, e_i) \mid C \subseteq H_{i,+}\}$. The cones in $\Sigma_{\mathcal{H}}$ not contained in the boundary of σ are in bijection with the bounded cells of $\mathcal{H}$ [15, Theorem 4.7]. In particular, the vertices of $\mathcal{H}$ correspond to the maximal cones of $\Sigma_{\mathcal{H}}$. The simplicial toric variety $X_{\mathcal{H}} = X(\Sigma_{\mathcal{H}})$ is the *Lawrence toric variety* associated to $\mathcal{H}$. The proper torus-invariant subvarietes V_C of $X_{\mathcal{H}}$ are in bijective correspondence with the bounded cells C of $\mathcal{H}$. The union of the proper torus-invariant subvarieties of $X_{\mathcal{H}}$ is called the *core* of $X_{\mathcal{H}}$.

Fix a vertex γ of $\mathcal{H}$ corresponding to a maximal element F in $M_{\mathcal{B}}$ and to the maximal cone σ_γ in $\Sigma_{\mathcal{H}}$. We will describe the corresponding open toric subvariety

U_γ of $X_\mathcal{H}$. If N_F denotes the sublattice of N generated by $\{b_i \mid b_i \in F\}$, then the elements of the finite group $N(F) = N/N_F$ are in bijective correspondence with the elements of $\prod_{G \subseteq F} \mathrm{Box}(G) \cap N$. Consider $\mathbb{A}^{d+n}$ with co-ordinates $\{z_i \mid \gamma \in H_{i,-}\}$ and $\{w_i \mid \gamma \in H_{i,+}\}$ and consider the embedding $\iota : N(F) \hookrightarrow (\mathbb{C}^*)^{d+n}$ such that if g in $N(F)$ corresponds to $v = \sum_{b_i \in G} \alpha_i b_i \in \mathrm{Box}(G) \cap N$ with $0 < \alpha_i < 1$, then, for every b_i in G, $\iota(g)$ equals $e^{2\pi i \alpha_i}$ in the co-ordinate corresponding to z_i and equals $e^{2\pi i(1-\alpha_i)}$ in the co-ordinate corresponding to w_i, and $\iota(g)$ equals 1 in all other co-ordinates. The action of $(\mathbb{C}^*)^{d+n}$ on $\mathbb{A}^{d+n}$ induces an action of $N(F)$ on $\mathbb{A}^{d+n}$ such that the age of g (see Subsection 7.1 in [1]) is given by $\mathrm{age}(g) = \sum_{b_i \in G} \alpha_i + (1 - \alpha_i) = \dim G$. The corresponding open toric subvariety of $X_\mathcal{H}$ is given by $U_\gamma = \mathbb{A}^{d+n}/N(F)$ and the orbifold structure of $X_\mathcal{H}$ is locally induced by this quotient. If C is a bounded cell of $\mathcal{H}$, then the corresponding proper torus-invariant subvariety V_C of $X_\mathcal{H}$ has $\dim V_C = \dim C$. If γ is not in the closure of C, then $U_\gamma \cap V_C = \emptyset$. If γ lies in the closure of C, then $U_\gamma \cap V_C$ is defined by setting the co-ordinates $\{z_i \mid C \subseteq H_{i,-}\}$ and $\{w_i \mid C \subseteq H_{i,+}\}$ equal to zero.

The hypertoric variety $\mathcal{M}_\mathcal{H}$ associated to $\mathcal{H}$ is a $2d$-dimensional complete intersection in $X_\mathcal{H}$ with a canonical orbifold structure. We will describe $\mathcal{M}_\mathcal{H} \cap U_\gamma$ (cf. [21, Proposition 4.3]). If $b_i \notin F$, then we may write $b_i = \sum_{b_j \in F} a_{i,j} b_j$, for unique rational numbers $a_{i,j}$. The ideal I generated by $\langle z_i = \sum_{b_j \in F} a_{i,j} z_j w_j \mid \gamma \in H_{i,-} \smallsetminus H_{i,+} \rangle$ and $\langle w_i = \sum_{b_j \in F} a_{i,j} z_j w_j \mid \gamma \in H_{i,+} \smallsetminus H_{i,-} \rangle$, defines a subvariety $V(I) \subseteq \mathbb{A}^{d+n}$ which is invariant under the action of $N(F)$. The induced subvariety $V(I)/N(F)$ of U_γ is equal to $\mathcal{M}_\mathcal{H} \cap U_\gamma$, and the orbifold structure of $\mathcal{M}_\mathcal{H}$ is locally induced by this quotient. It can be seen from this description that the core of $X_\mathcal{H}$ is contained in $\mathcal{M}_\mathcal{H}$. If $\mathbb{A}^{2d}$ has co-ordinates $\{z_i \mid b_i \in F\}$ and $\{w_i \mid b_i \in F\}$, then the corresponding projection $\pi : \mathbb{A}^{d+n} \to \mathbb{A}^{2d}$ induces a projection $\pi : V(I) \to \mathbb{A}^{2d}$. Since the co-ordinates not projected by π are $\langle z_i \mid \gamma \in H_{i,-} \smallsetminus H_{i,+} \rangle$ and $\langle w_i \mid \gamma \in H_{i,+} \smallsetminus H_{i,-} \rangle$, it follows from the definition of I that π is an isomorphism. The embedding $\iota : N(F) \hookrightarrow (\mathbb{C}^*)^{d+n}$ factors as an embedding $\nu : N(F) \hookrightarrow (\mathbb{C}^*)^{2d}$ followed by the inclusion of $(\mathbb{C}^*)^{2d}$ into $(\mathbb{C}^*)^{d+n}$ by adding 1's in the extra co-ordinates. The embedding ν induces an action of $N(F)$ on $\mathbb{A}^{2d}$ such that the projection π is $N(F)$-equivariant and induces an isomorphism of orbifolds $\mathcal{M}_\mathcal{H} \cap U_\gamma \cong \mathbb{A}^{2d}/N(F)$.

We have seen in the introduction that any G in $M_\mathcal{B}$ determines a hyperplane arrangement $\mathcal{H}^G$ with co-normal vectors $\mathcal{B}_G$. The corresponding Lawrence toric variety X_G and hypertoric variety $\mathcal{M}_G$ may be regarded as subvarieties of $X_\mathcal{H}$ and $\mathcal{M}_\mathcal{H}$ respectively. For g in $N(F)$ corresponding to $v \in \mathrm{Box}(G) \cap N$, one verifies that, as varieties, $X_G \cap U_\gamma = (\mathbb{A}^{d+n})^g/N(F)$ and $\mathcal{M}_G \cap U_\gamma \cong (\mathbb{A}^{2d})^g/N(F)$, where $(\mathbb{A}^{d+n})^g$ and $(\mathbb{A}^{2d})^g$ are the subvarieties of $(\mathbb{A}^{d+n})$ and $(\mathbb{A}^{2d})$ respectively that are invariant under the action of g.

Example 2.1. Consider the notation of Example 1.11 and fix the vertex $\gamma = (-1/2, 0)$ of $\mathcal{H}$ corresponding to $F = \{b_2 b_4\}$ in $M_\mathcal{B}$. The open toric subvariety U_γ of $X_\mathcal{H}$ is the quotient of $\mathbb{A}^6$ with co-ordinates $\{z_1, z_2, z_3, z_4, w_2, w_4\}$ by the action of $N(F) = \mathbb{Z}/2\mathbb{Z}$, acting via multiplication by $(1, -1, 1, -1, -1, -1)$. The open subvariety $\mathcal{M}_\mathcal{H} \cap U_\gamma$ of the hypertoric variety $\mathcal{M}_\mathcal{H}$ is the complete intersection defined by $z_1 = \frac{1}{2} z_2 w_2 + \frac{1}{2} z_4 w_4$ and $z_3 = -z_2 w_2 - z_4 w_4$. The core of $X_\mathcal{H}$ consists of two weighted projective spaces $\mathbb{P}(1, 1, 2)$ that intersect along their unique singular points at $V_\gamma = X_F = \mathcal{M}_F$.

Remark 2.2. Hausel and Sturmfels proved that the embedding of the core of $X_{\mathcal{H}}$ into either $X_{\mathcal{H}}$ or $\mathcal{M}_{\mathcal{H}}$ induces an isomorphism of cohomology rings over $\mathbb{Z}$ [15, Lemma 6.5] and used this to prove Theorem 1.4 and Theorem 1.8.

Orbifold cohomology was introduced by Chen and Ruan in [7] and associates to an orbifold Y a graded $\mathbb{Q}$-algebra $H_{\mathrm{orb}}^*(Y, \mathbb{Q})$. More specifically, one associates to Y an orbifold $I(Y)$, called the *inertia orbifold* of Y, such that, if $Y = X/G$ for some smooth variety X and finite abelian group G, then $I(Y) = \coprod_{g \in G} X^g/G$, where X^g is the subvariety of X fixed by g. If $Y_1, \dots, Y_r$ denote the connected components of $I(Y)$, then, for any $i \in \mathbb{Q}$, Chen and Ruan defined the i'th orbifold cohomology group of Y by

$$H_{\mathrm{orb}}^i(Y, \mathbb{Q}) = \bigoplus_{j=1}^r H^{i-2\,\mathrm{age}(Y_j)}(Y_j, \mathbb{Q}),$$

where $\mathrm{age}(Y_j)$ is the *age* of Y_j. We have $I(U_\gamma) = \coprod_{G \subseteq F} \coprod_{v \in \mathrm{Box}(G) \cap N} (\mathbb{A}^{d+n})^g/N(F)$, $I(\mathcal{M}_{\mathcal{H}} \cap U_\gamma) = \coprod_{G \subseteq F} \coprod_{v \in \mathrm{Box}(G) \cap N} (\mathbb{A}^{2d})^g/N(F)$, and, in both cases, the age of the connected component corresponding to a pair (v, G) is equal to $\dim G$. This implies that we have isomorphisms of varieties, $I(X_{\mathcal{H}}) \cong \coprod_{G \in M} \coprod_{v \in \mathrm{Box}(G) \cap N} X_G$ and $I(\mathcal{M}_{\mathcal{H}}) \cong \coprod_{G \in M} \coprod_{v \in \mathrm{Box}(G) \cap N} \mathcal{M}_G$, as computed by Jiang and Tseng in [20]. We conclude that Theorem 1.4 and the above description of inertia orbifolds implies Theorem 1.5, which computes the orbifold cohomology of $X_{\mathcal{H}}$ and $\mathcal{M}_{\mathcal{H}}$.

3. Ehrhart theory for Lawrence polytopes

The goal of this section is to give a combinatorial proof of Theorem 1.6. We will first show that

$$(1) \qquad \delta_{\mathcal{B}}(t) = \sum_{F \in M} \#\left(\mathrm{Box}(F) \cap N\right) t^{\dim F} h_F^{bd}(t).$$

The proof should be compared with the combinatorial proof of Theorem 1.2 (see [32]) and the proof of [26, Theorem 1.1]. Recall that σ denotes the cone over $P_{\mathcal{B}}$ in $N_{\mathbb{R}} \times \mathbb{R}^n$ and that $\Sigma_{\mathcal{H}}$ is a fan refining σ, such that the cones σ_C in $\Sigma_{\mathcal{H}}$ that do not lie in the boundary of σ are in bijection with the bounded cells C of $\mathcal{H}$. The cone σ_C has codimension equal to $\dim C$, and its rays correspond to the primitive integer vectors $\{(b_i, e_i) \mid C \subseteq H_{i,-}\}$ and $\{(0, e_i) \mid C \subseteq H_{i,+}\}$. If F is an element of $M_{\mathcal{B}}$ and $w = \sum_{b_i \in F} a_i b_i \in \mathrm{Box}(F) \cap N$, for some $0 < a_i < 1$, then let $l(w) = \sum_{b_i \in F}(a_i b_i, e_i) \in \sigma \cap (N \times \mathbb{Z}^n)$. If v is a lattice point in the interior of σ and C is the largest bounded cell in $\mathcal{H}$ such that $v \in \sigma_C$, then we have a unique expression

$$v = l(w) + \sum_{b_i \notin F, C \subseteq H_{i,-}} (b_i, e_i) + \sum_{b_i \notin F, C \subseteq H_{i,+}} (0, e_i) + \sum_{C \subseteq H_{i,-}} \alpha_i(b_i, e_i) + \sum_{C \subseteq H_{i,+}} \beta_i(0, e_i),$$

where $w \in \mathrm{Box}(F) \cap N$, for some F in $M_{\mathcal{B}}$ such that C is a bounded cell in $\mathcal{H}^F$, and α_i, β_i are non-negative integers. Conversely, given $w \in \mathrm{Box}(F) \cap N$, a bounded cell C in $\mathcal{H}^F$ and non-negative integers $\{\alpha_i \mid C \subseteq H_{i,-}\}$ and $\{\beta_i \mid C \subseteq H_{i,+}\}$, the lattice point $v = l(w) + \sum_{b_i \notin F, C \subseteq H_{i,-}} (b_i, e_i) + \sum_{b_i \notin F, C \subseteq H_{i,+}} (0, e_i) + \sum_{C \subseteq H_{i,-}} \alpha_i(b_i, e_i) + \sum_{C \subseteq H_{i,+}} \beta_i(0, e_i)$ lies in the interior of σ and C is the largest bounded cell in $\mathcal{H}$ such that $v \in \sigma_C$. Recall from the proof of Theorem 1.3 that $\psi : N \times \mathbb{Z}^n \to \mathbb{Z}$ is given by $\psi(\lambda + \sum_{i=1}^n \mu_i e_i) = \sum_{i=1}^n \mu_i$, for $\lambda \in N$ and integers μ_i, and that

$\psi^{-1}(m) \cap \sigma = mP_{\mathcal{B}}$ and $\psi(l(w)) = \dim F$. It is a standard result of Ehrhart theory (see, for example, [4]) that

$$\sum_{m \geq 1} \# \left(\text{Int}(mP_{\mathcal{B}}) \cap (N \times \mathbb{Z}^n) \right) t^m = \frac{t^{d+n} \delta_{\mathcal{B}}(t^{-1})}{(1-t)^{d+n}}.$$

Using the above facts and setting $\mathcal{H}_{\text{bd}}^F$ to be the collection of bounded cells in $\mathcal{H}^F$, we calculate

$$t^{d+n} \delta_{\mathcal{B}}(t^{-1}) = (1-t)^{d+n} \sum_{v \in \text{Int}(\sigma) \cap (N \times \mathbb{Z}^n)} t^{\psi(v)}$$

$$= (1-t)^{d+n} \sum_{F \in M} \# \left(\text{Box}(F) \cap N \right) t^{\dim F} \sum_{C \in \mathcal{H}_{\text{bd}}^F} \frac{t^{d+n-\dim C - 2\dim F}}{(1-t)^{d+n-\dim C}}$$

$$= t^{d+n} \sum_{F \in M} \# \left(\text{Box}(F) \cap N \right) t^{-\dim F} \sum_{C \in \mathcal{H}_{\text{bd}}^F} (t^{-1} - 1)^{\dim C},$$

and (1) follows immediately as desired. We present below the remainder of the proof of Theorem 1.6.

Proof. Observe that the constant coefficient in $h_F(t)$ is 1 and the coefficient of t in $h_{\{0\}}(t)$ is $n - d$. The elements of $M_{\mathcal{B}}$ of dimension 1 correspond to the elements b_i in $\mathcal{B}$ and $\text{Box}(\{b_i\}) \cap N$ consists of $a_i - 1$ lattice points. We deduce that $\delta_1 = n - d + \sum_{i=1}^n (a_i - 1) = \sum_{i=1}^n a_i - d$. The leading term of $h_F^{bd}(t)$ is $R_F^{bd} t^{\text{codim } F}$ and hence $\delta_d = \sum_{F \in M} \# \left(\text{Box}(F) \cap N \right) R_F^{bd}$ and $\delta_{d+1} = \cdots = \delta_{d+n-1} = 0$. It is a standard fact of Ehrhart theory (see, for example, [4]) that the normalised volume of P_B is equal to $\frac{1}{(d+n-1)!} \delta_{\mathcal{B}}(1)$. The last statement follows from the observation that $h_F(1) = h_F^{\text{bd}}(1) = V_F$. $\square$

Remark 3.1. If $P_1, \ldots, P_n$ are lattice polytopes in $N_{\mathbb{R}}$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$, then the *Cayley sum* $P = P_1 * \cdots * P_n$ is the convex hull of $P_1 \times \{e_1\}, \ldots, P_n \times \{e_n\}$ in $N \times \mathbb{R}^n$. If the affine span of the union of the P_i is $N_{\mathbb{R}}$, then P is a $(d + n - 1)$-dimensional lattice polytope. Setting P_i to be the convex hull of the origin and b_i, we recover the Lawrence polytope $P_{\mathcal{B}}$. The degree s of a lattice polytope Q is the degree of its Ehrhart δ-polynomial, and it is a standard fact that $(\dim Q + 1 - s)Q$ is the smallest dilate of Q that contains a lattice point in its relative interior (see, for example, [4]). Cayley sums provide examples of lattice polytopes with small degree relative to their dimension since the degree of P is at most d [2, Proposition]. In [2], Batyrev and Nill used Lawrence polytopes in the classification of lattice polytopes of degree 1.

Remark 3.2. It is a standard result of Ehrhart theory (see, for example, [4]) that $\delta_1 = f_{\mathcal{B}}(1) - \dim P_{\mathcal{B}} - 1$ and hence Theorem 1.6 implies that $P_{\mathcal{B}}$ contains $\sum_{i=1}^n (a_i + 1)$ lattice points. More specifically, the lattice points in $P_{\mathcal{B}}$ are $\{(\lambda_i v_i, e_i) \mid 0 \leq \lambda_i \leq a_i, 1 \leq i \leq n\}$. We noted in Remark 3.1 that $mP_{\mathcal{B}}$ contains no interior lattice points for $1 \leq m \leq n - 1$. It is a standard fact that $\delta_d = \#(\text{Int}(nP_{\mathcal{B}}) \cap (N \times \mathbb{Z}^n))$ (see, for example, [4]), and hence Theorem 1.6 implies that nP contains $\sum_{F \in M} \# \left(\text{Box}(F) \cap N \right) R_F^{bd}$ interior lattice points. More specifically, if w is a lattice point in $\text{Box}(F)$ and R is a bounded region in $\mathcal{H}^F$, then the corresponding lattice point in the relative interior of $nP_{\mathcal{B}}$ is $l(w) + \sum_{b_i \notin F, R \subseteq H_{i,-}} (b_i, e_i) + \sum_{b_i \notin F, R \subseteq H_{i,+}} (0, e_i) \in \sigma_R$.

Remark 3.3. We present an alternative combinatorial proof of the formula

$$\delta_{\mathcal{B}}(t) = \sum_{F \in M} \# \left(\mathrm{Box}(F) \cap N \right) t^{\dim F} h_F(t).$$

If v is a lattice point in σ, then v lies in some maximal cone σ_γ of $\Sigma_\mathcal{H}$, and we may write $v = \sum_{\gamma \in H_{i,-}} \alpha_i (b_i, e_i) + \sum_{\gamma \in H_{i,+}} \beta_i (0, e_i)$, for unique non-negative integers α_i and β_i. The element G of $M_\mathcal{B}$ generated by $\{ b_i \mid \alpha_i, \beta_i \neq 0 \}$ and the sets $I_- = \{ i \mid \alpha_i \neq 0, b_i \notin \mathrm{span}\, G \}$ and $I_+ = \{ i \mid \beta_i \neq 0, b_i \notin \mathrm{span}\, G \}$ are independent of the choice of maximal cone σ_γ. Note that $b_i \in \mathrm{span}\, G$ if and only if $M_{G,\mathbb{R}} = \bigcap_{b_i \in G} H_i$ is contained in either $H_{i,-}$ or $H_{i,+}$. If we consider the projection $\phi_G : N \to N/(N \cap \mathrm{span}\, G)$ and the hyperplane arrangement $\{ \phi_G(H_i) \mid i \in I = I_- \cup I_+ \}$, then the intersection of the half-spaces $\{ \phi_G(H_{i,-}) \mid i \in I_- \}$ and $\{ \phi_G(H_{i,+}) \mid i \in I_+ \}$ is a region in the hyperplane arrangement and, moreover, every region in the hyperplane arrangement has this form. We deduce that v has a unique expression

$$v = l(w) + \sum_{b_i \in G \setminus F} ((b_i, e_i) + (0, e_i)) + \sum_{M_{G,\mathbb{R}} \subseteq H_{i,-}} \mu_i (b_i, e_i) + \sum_{M_{G,\mathbb{R}} \subseteq H_{i,+}} \mu_i' (0, e_i)$$
$$+ \sum_{i \in I_-} \nu_i (b_i, e_i) + \sum_{i \in I_+} \nu_i' (0, e_i),$$

where $w \in \mathrm{Box}(F) \cap N$, for some $F \subseteq G$, μ_i, μ_i' are non-negative integers and ν, ν' are positive integers. Conversely, consider elements $F \subseteq G$ in $M_\mathcal{B}$, a subset I in $\{ i \mid b_i \notin \mathrm{span}\, G \}$ and a region R in the hyperplane arrangement $\{ \phi_G(H_i) \mid i \in I \}$. If we set $I_- = \{ i \mid R \subseteq \phi_G(H_{i,-}) \}$ and $I_+ = \{ i \mid R \subseteq \phi_G(H_{i,+}) \}$, and consider some $w \in \mathrm{Box}(F) \cap N$, some non-negative integers μ_i, μ_i' and some positive integers ν, ν', then the right hand side of the above expression gives a lattice point v in σ. By a theorem of Zaslavsky [34], the number of regions $r_{G,I}$ in the hyperplane arrangement $\{ \phi_G(H_i) \mid i \in I \}$ is equal to the number of elements G' in $M_\mathcal{B}$ satisfying $G \subseteq G' \subseteq G \cup \{ b_i \mid i \in I \}$. Setting $I_G = \{ i \mid b_i \notin \mathrm{span}\, G \}$, we compute

$$\delta_\mathcal{B}(t) = (1 - t)^{d+n} \sum_{v \in \sigma \cap (Ns \times \mathbb{Z}^n)} t^{\psi(v)}$$

$$= (1-t)^{d+n} \sum_{F \in M} \# \left(\mathrm{Box}(F) \cap N \right) t^{\dim F} \sum_{F \subseteq G} \frac{t^{2(\dim G - \dim F)}}{(1-t)^{\dim G + n - |I_G|}} \sum_{I \subseteq I_G} r_{G,I} \frac{t^{|I|}}{(1-t)^{|I|}}$$

$$= \sum_{F \in M} \# (\mathrm{Box}(F) \cap N) \, t^{-\dim F} \sum_{F \subseteq G} t^{2\dim G} (1 - t)^{d - \dim G + |I_G|} \sum_{G \subseteq G'} \frac{t^{\dim G' - \dim G}}{(1 - t)^{|I_G|}}$$

$$= \sum_{F \in M} \# \left(\mathrm{Box}(F) \cap N \right) t^{\dim F} \sum_{F \subseteq G'} t^{\dim G' - \dim F} (1 - t)^{d - \dim G'}$$

$$\times \sum_{F \subseteq G \subseteq G'} t^{\dim G - \dim F} (1 - t)^{\dim G' - \dim G}$$

$$= \sum_{F \in M} \# \left(\mathrm{Box}(F) \cap N \right) t^{\dim F} h_F(t).$$

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