EVERY GRAPH HAS AN EMBEDDING IN $S^3$
CONTAINING NO NON-HYPERBOLIC KNOT

ERICA FLAPAN AND HUGH HOWARDS

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Abstract. In contrast with knots, whose properties depend only on their extrinsic topology in $S^3$, there is a rich interplay between the intrinsic structure of a graph and the extrinsic topology of all embeddings of the graph in $S^3$. For example, it was shown by Conway and Gordon that every embedding of the complete graph $K_7$ in $S^3$ contains a non-trivial knot. Later it was shown that for every $m \in \mathbb{N}$ there is a complete graph $K_n$ such that every embedding of $K_n$ in $S^3$ contains a knot $Q$ whose minimal crossing number is at least $m$. Thus there are arbitrarily complicated knots in every embedding of a sufficiently large complete graph in $S^3$. We prove the contrasting result that every graph has an embedding in $S^3$ such that every non-trivial knot in that embedding is hyperbolic. Our theorem implies that every graph has an embedding in $S^3$ which contains no composite or satellite knots.

In contrast with knots, whose properties depend only on their extrinsic topology in $S^3$, there is a rich interplay between the intrinsic structure of a graph and the extrinsic topology of all embeddings of the graph in $S^3$. For example, it was shown in [2] that every embedding of the complete graph $K_7$ in $S^3$ contains a non-trivial knot. Later in [3] it was shown that for every $m \in \mathbb{N}$, there is a complete graph $K_n$ such that every embedding of $K_n$ in $S^3$ contains a knot $Q$ (i.e., $Q$ is a subgraph of $K_n$) such that $|a_2(Q)| \geq m$, where $a_2$ is the second coefficient of the Conway polynomial of $Q$. More recently, in [4] it was shown that for every $m \in \mathbb{N}$, there is a complete graph $K_n$ such that every embedding of $K_n$ in $S^3$ contains a knot $Q$ whose minimal crossing number is at least $m$. Thus there are arbitrarily complicated knots (as measured by $a_2$ and the minimal crossing number) in every embedding of a sufficiently large complete graph in $S^3$.

In light of these results, it is natural to ask whether there is a graph such that every embedding of that graph in $S^3$ contains a composite knot. Or more generally, is there a graph such that every embedding of the graph in $S^3$ contains a satellite knot? Certainly, $K_7$ is not an example of such a graph since Conway and Gordon [2] exhibit an embedding of $K_7$ containing only the trefoil knot. In this paper we answer this question in the negative. In particular, we prove that every graph has an embedding in $S^3$ such that every non-trivial knot in that embedding is hyperbolic. Our theorem implies that every graph has an embedding in $S^3$ which contains no composite or satellite knots. By contrast, for any particular embedding of a graph
we can add local knots within every edge to get an embedding such that every knot in that embedding is composite.

Let \( G \) be a graph. There is an odd number \( n \) such that \( G \) is a minor of \( K_n \). We will show that for every odd number \( n \), there is an embedding of \( K_n \) in \( S^3 \) such that every non-trivial knot in that embedding of \( K_n \) is hyperbolic. It follows that there is an embedding of \( G \) in \( S^3 \) which contains no non-trivial non-hyperbolic knots.

Let \( n \) be a fixed odd number. We begin by constructing a preliminary embedding of \( K_n \) in \( S^3 \) as follows. Let \( h \) be a rotation of \( S^3 \) of order \( n \) with fixed point set \( \alpha \). Let \( V \) denote the complement of an open regular neighborhood of the fixed point set \( \alpha \). Let \( v_1, \ldots, v_n \) be points in \( V \) such that for each \( i \), \( h(v_i) = v_{i+1} \) (throughout the paper we shall consider our subscripts mod \( n \)). These \( v_i \) will be the vertices of the preliminary embedding of \( K_n \).

**Definition 1.** By a solid annulus we shall mean a 3-manifold with boundary which can be parametrized as \( D \times I \) where \( D \) is a disk. We use the term the annulus boundary of a solid annulus \( D \times I \) to refer to the annulus \( \partial D \times I \). The ends of \( D \times I \) are the disks \( D \times \{0\} \) and \( D \times \{1\} \). If \( A \) is an arc in a solid annulus \( W \) with one endpoint in each end of \( W \) and \( A \) co-bounds a disk in \( W \) together with an arc in \( \partial W \), then we say that \( A \) is a longitudinal arc of \( W \).

As follows, we embed the edges of \( K_n \) as simple closed curves in the quotient space \( S^3/h = S^3 \). Observe that since \( V \) is a solid torus, \( V' = V/h \) is also a solid torus. Let \( D' \) denote a meridional disk for \( V' \) which does not contain the point \( v = v_1/h \). Let \( W' \) denote the solid annulus \( \text{cl}(V' - D') \) with ends \( D'_+ \) and \( D'_- \). Since \( n \) is odd, we can choose unknotted simple closed curves \( S_1, \ldots, S_{n-1} \) in the solid torus \( V' \) such that each \( S_i \) contains \( v \) and has winding number \( n + i \) in \( V' \), the \( S_i \) are pairwise disjoint except at \( v \), and for each \( i \), \( W' \cap S_i \) is a collection of \( n + i \) untangled longitudinal arcs (see Figure 1).

![Figure 1](image-url)  
**Figure 1.** For each \( i \), \( W' \cap S_i \) is a collection of \( n + i \) untangled longitudinal arcs.

We define as follows two additional simple closed curves \( J' \) and \( C' \) in \( V' \) whose intersections with \( W' \) are illustrated in Figure 1. First, choose a simple closed curve \( J' \) in \( V' \) whose intersection with \( W' \) is a longitudinal arc which is disjoint from and untangled with \( S_1 \cup \cdots \cup S_{n-1} \). Next we let \( C' \) be the unknotted simple closed
curve in \( W' - (S_1 \cup \cdots \cup S_{n-1} \cup J') \) whose projection is illustrated in Figure 1. In particular, \( C \) contains one half twist between \( J' \) and the set of arcs of \( S_1 \cup \cdots \cup S_{n-1} \) which do not contain \( v \), another half twist between those arcs of \( S_1 \cup \cdots \cup S_{n-1} \) and the set of arcs containing \( v \), and \( r \) full twists between each of the individual arcs of \( S_i \) and \( S_{i+1} \) containing \( v \). We will determine the value of \( r \) later.

Each of the \( \frac{n(n-1)}{2} \) simple closed curves \( S_1, \ldots, S_{n-1} \) lifts to a simple closed curve consisting of \( n \) consecutive edges of \( K_n \). The vertices \( v_1, \ldots, v_n \) together with these \( \frac{n(n-1)}{2} \) edges give us a preliminary embedding \( \Gamma_1 \) of \( K_n \) in \( S^3 \).

Lift the meridional disk \( D' \) of the solid torus \( V' \) to \( n \) disjoint meridional disks \( D_1, \ldots, D_n \) of the solid torus \( V \). Lift the simple closed curve \( C' \) to \( n \) disjoint simple closed curves \( C_1, \ldots, C_n \), and lift the simple closed curve \( J' \) to \( n \) consecutive arcs \( J_1, \ldots, J_n \) whose union is a simple closed curve \( J \). The closures of the components of \( V - (D_1 \cup \cdots \cup D_n) \) are solid annuli, which we denote by \( W_1, \ldots, W_n \). The subscripts of all of the lifts are chosen consistently so that for each \( i \), \( v_i \in W_i \), \( C_i \cup J_i \subseteq W_i \), and \( D_i \) and \( D_{i+1} \) are the ends of the solid annulus \( W_i \). For each \( i \), the pair \( (W_i - (C_i \cup J_i), (W_i - (C_i \cup J_i)) \cap \Gamma_1) \) is homeomorphic to \( (W' - (C' \cup J'), (W' - (C' \cup J')) \cap (S_1 \cup \cdots \cup S_{n-1})) \). For each \( i \), the solid annulus \( W' \) contains \( n + i - 1 \) arcs of \( S_i \) which are disjoint from \( v \). Hence each edge of the embedded graph \( \Gamma_1 \) meets each solid annulus \( W_i \) in at least one arc not containing \( v_i \).

Let \( \kappa \) be a simple closed curve in \( \Gamma_1 \). For each \( i \), we let \( k_i \) denote the set of those arcs of \( \kappa \cap W_i \) which do not contain \( v_i \), and we let \( e_i \) denote either the single arc of \( \kappa \cap W_i \) which does contain \( v_i \) or the empty set if \( v_i \) is not on \( \kappa \). Observe that since \( \kappa \) is a simple closed curve, it contains at least three edges of \( \Gamma_1 \); and as we observed above, each edge of \( \kappa \) contains at least one arc of \( k_i \). Thus for each \( i \), \( k_i \) contains at least three arcs. Either \( e_i \) is empty, the endpoints of \( e_i \) are in the same end of the solid annulus \( W_i \), or the endpoints of \( e_i \) are in different ends of \( W_i \). We illustrate these three possibilities for \((W_i, C_i \cup J_i \cup k_i \cup e_i)\) in Figure 2 as forms a), b) and c) respectively. The number of full twists represented by the labels \( t, u, x, \) or \( z \) in Figure 2 is some multiple of \( r \) depending on the particular simple closed curve \( \kappa \).

![Figure 2](image_url)

**Figure 2.** The forms of \((W_i, C_i \cup J_i \cup k_i \cup e_i)\).

With each of the forms of \((W_i, C_i \cup J_i \cup k_i \cup e_i)\) illustrated in Figure 2 we will associate an additional arc and an additional collection of simple closed curves as follows (illustrated in Figure 3). Let the arc \( B_i \) be the core of a solid annulus neighborhood of the union of the arcs \( k_i \) in \( W_i \) such that \( B_i \) is disjoint from \( J_i, C_i \), ...
and $e_i$. Let the simple closed curve $Q$ be obtained from $C_i$ by removing the full twists $z, x, t,$ and $u$. Let $Z, X, T,$ and $U$ be unknotted simple closed curves which wrap around $Q$ in place of $z, x, t,$ and $u$ as illustrated in Figure 3.

**Figure 3.** The forms of $W_i$ with associated simple closed curves and the arc $B_i$.

For each $i$, let $M_i$ denote an unknotted solid torus in $S^3$ obtained by gluing together two identical copies of $W_i$ along $D_i$ and $D_{i+1}$, making sure that the endpoints of the arcs of $J_i, B_i,$ and $e_i$ match up with their counterparts in the second copy to give simple closed curves $j, b,$ and $E$, respectively, in $M_i$. Thus $M_i$ has a 180° rotational symmetry around a horizontal line which goes through the center of the figure and the endpoints of both copies of $J_i, B_i,$ and $e_i$. Recall that in form a), $e_i$ is the empty set, and hence so is $E$. Let $Q_1$ and $Q_2$, $X_1$ and $X_2$, $Z_1$ and $Z_2$, $T_1$ and $T_2$, and $U_1$ and $U_2$ denote the doubles of the unknotted simple closed curves $Q, X, Z, T,$ and $U$ respectively.

Let $Y$ denote the core of the solid torus $c(S^3 - M_i)$. We associate to Form a) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y$. We associate to Form b) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E \cup X_1 \cup X_2 \cup Z_1 \cup Z_2$. We associate to Form c) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E \cup T_1 \cup T_2 \cup U_1 \cup U_2$. Figure 3 illustrates the three forms of the link $L$.

The software program SnapPea\(^1\) can be used to determine whether or not a given knot or link in $S^3$ is hyperbolic, and if it is, SnapPea estimates the hyperbolic volume of the complement. We used SnapPea to verify that each of the three forms of the link $L$ illustrated in Figure 3 is hyperbolic.

A 3-manifold is unchanged by doing Dehn surgery on an unknot if the boundary slope of the surgery is the reciprocal of an integer (though such surgery may change a knot or link in the manifold). According to Thurston’s Hyperbolic Dehn Surgery Theorem \(^1\)\(^5\), all but finitely many Dehn fillings of a hyperbolic link complement yield a hyperbolic manifold. Thus there is some $r \in \mathbb{N}$ such that for any $m \geq r$, if we do Dehn filling with slope $\frac{1}{m}$ along the components $X_1$, $X_2$, $Z_1$, $Z_2$ of the link $L$ in form b) or along the components $T_1$, $T_2$, $U_1$, $U_2$ of the link $L$ in form c), then we obtain a hyperbolic link $\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup Y \cup E$, where the simple closed curves $\overline{Q}_1$ and $\overline{Q}_2$ are obtained by adding $m$ full twists to $Q_1$ and $Q_2$ in place of each of the surgered curves.

\(^1\)Available at http://www.geometrygames.org/SnapPea/index.html.
We fix the value of \( r \) according to the above paragraph, and this is the value of \( r \) that we use in Figure 1. Recall that the number of twists \( x \), \( z \), \( u \), and \( t \) in the simple closed curves \( C_i \) in Figure 2 are each a multiple of \( r \). Thus the particular simple closed curves \( C_i \) are determined by our choice of \( r \) together with our choice of the simple closed curve \( \kappa \). Now we do Dehn fillings along \( X_1 \) and \( X_2 \) with slope \( \frac{1}{z} \), along \( Z_1 \) and \( Z_2 \) with slope \( \frac{1}{y} \), along \( U_1 \) and \( U_2 \) with slope \( \frac{1}{u} \), and along \( T_1 \) and \( T_2 \) with slope \( \frac{1}{t} \). Since \( x \), \( z \), \( u \), and \( t \) are each greater than or equal to \( r \), the link \( Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E \) that we obtain will be hyperbolic. In Form a), \( E \) is the empty set, and the link \( Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E \) was already seen to be hyperbolic from using SnapPea. In this case, we do no surgery and let \( Q_1 = Q_1 \) and \( Q_2 = Q_2 \).

It follows that each form of \( M_j - (Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E) \) is a hyperbolic 3-manifold. Observe that \( M_j - (Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E) \) is the double of \( W_i - (C_i \cup J_i \cup B_i \cup e_i) \).

Now that we have fixed \( C_i \), we let \( N(C_i) \), \( N(J_i) \), \( N(B_i) \), and \( N(e_i) \) be pairwise disjoint regular neighborhoods of \( C_i \), \( J_i \), \( B_i \), and \( e_i \) respectively in the interior of each of the forms of the solid annulus \( W_i \) (illustrated in Figure 2). We choose \( N(B_i) \) such that it contains the union of the arcs \( k_i \). Note that in Form a) \( e_i \) is the empty set and hence so is \( N(e_i) \). Let \( N(k_i) \) denote a collection of pairwise disjoint regular neighborhoods, each containing an arc \( k_i \), such that \( N(k_i) \subseteq N(B_i) \). Let

![Form a) of L](image1)

![Form b) of L](image2)

![Form c) of L](image3)

**Figure 4.** The possible forms of the link \( L \).
Definition 2. Let $X$ be a 3-manifold. A sphere in $X$ is said to be essential if it does not bound a ball in $X$. A properly embedded disk $D$ in $X$ is said to be essential if $\partial D$ does not bound a disk in $\partial X$. A properly embedded annulus is said to be essential if it is incompressible and not boundary parallel. A torus in $X$ is said to be essential if it is incompressible and not boundary parallel.

Lemma 1. For each $i$, $V'_i$ contains no essential torus, sphere, or disk whose boundary is in $D_i \cup D_{i+1}$. Also, any incompressible annulus in $V'_i$ whose boundary is in $D_i \cup D_{i+1}$ either is boundary parallel or can be expressed as $\sigma \times I$ (possibly after a change in parameterization of $\Delta$), where $\sigma$ is a non-trivial simple closed curve in $D_i \cap \Delta$.

Proof. Since $k_i$ contains at least three disjoint arcs, $F$ is a disk with at least three holes. Let $\beta$ denote the double of $\Delta$ along $\Delta \cap (D_i \cup D_{i+1})$. Then $\beta = F \times S^1$. Now it follows from Waldhausen [7] that $\beta$ contains no essential sphere or properly embedded disk and that any incompressible torus in $\beta$ can be expressed as $\sigma \times S^1$ (after a possible change in parameterization of $\beta$) where $\sigma$ is a non-trivial simple closed curve. We see as follows that $\nu \cup \beta$ contains no essential sphere and that any essential torus in $\nu \cup \beta$ can be expressed (after a possible change in parameterization of $\beta$) as $\sigma \times S^1$, where $\sigma$ is a non-trivial simple closed curve in $D_i \cap \Delta$. Let $\tau$ be an essential sphere or torus in $\nu \cup \beta$, and let $\gamma$ denote the torus $\nu \cap \beta$. By doing an isotopy as necessary, we can assume that $\tau$ intersects $\gamma$ in a minimal number of disjoint simple closed curves. Suppose there is a curve of intersection which bounds a disk in the essential surface $\tau$. Let $c$ be an innermost curve of intersection on $\tau$ which bounds a disk $\delta$ in $\tau$. Then $\delta$ is a properly embedded disk in either $\gamma$ or $\beta$. Since neither $\nu$ nor $\beta$ contains a properly embedded essential disk or an essential sphere, there is an isotopy of $\tau$ which removes $c$ from the collection of curves of intersection. Thus by the minimality of the number of curves in $\tau \cap \gamma$, we can assume that none of the curves in $\tau \cap \gamma$ bounds a disk in $\tau$.

Suppose that $\tau$ is an essential sphere in $\nu \cup \beta$. Since none of the curves in $\tau \cap \gamma$ bounds a disk in $\tau$, $\tau$ must be contained entirely in either $\nu$ or $\beta$. However, we saw above that neither $\nu$ nor $\beta$ contains any essential sphere. Thus $\tau$ cannot be an essential sphere and hence must be an essential torus. Since $\tau \cap \gamma$ is minimal, if $\tau \cap \nu$ is non-empty, then the components of $\tau$ in $\nu$ are all incompressible annuli. However, we saw above that $\nu$ contains no essential annuli. Thus $\tau \cap \nu$ is empty. Since $\nu$ contains no essential torus, the essential torus $\tau$ must be contained in $\beta$. Hence $\tau$ can be expressed (after a possible change in parameterization of $\beta$) as $\sigma \times S^1$, where $\sigma$ is a non-trivial simple closed curve in $D_i \cap \Delta$. 

$V_i = \text{cl}([W_i - (N(C_1) \cup N(J_i) \cup N(B_i) \cup N(e_i))], \text{let } \Delta = \text{cl}(N(B_i) - N(k_i))$, and let $V'_i = V_i \cup \Delta$. Since $N(B_i)$ is a solid annulus, it has a product structure $D^2 \times I$. Without loss of generality, we assume that each of the components of $N(k_i)$ respects the product structure of $N(B_i)$. Thus $\Delta = F \times I$ where $F$ is a disk with holes.


Now we consider essential surfaces in $V'_i$. Suppose that $V'_i$ contains an essential sphere $S$. Since $\nu \cap \beta$ contains no essential sphere, $S$ bounds a ball $B$ in $\nu \cap \beta$. Now the ball $B$ cannot contain any of the boundary components of $\nu \cap \beta$. Thus $B$ cannot contain either $D_i$ or $D_{i+1}$. Since $S$ is disjoint from $D_i \cup D_{i+1}$, it follows that $B$ must be disjoint from $D_i \cup D_{i+1}$. Thus $B$ is contained in $V'_i$. Hence $V'_i$ cannot contain an essential sphere.

We see as follows that $V'_i$ cannot contain an essential disk whose boundary is in $D_i \cup D_{i+1}$. Let $\epsilon$ be a disk in $V'_i$ whose boundary is in $D_i \cup D_{i+1}$. Let $\epsilon'$ denote the double of $\epsilon$ in $\nu \cup \beta$. Then $\epsilon'$ is a sphere which meets $D_i \cup D_{i+1}$ in the simple closed curve $\partial \epsilon$. Since $\nu \cup \beta$ contains no essential sphere, $\epsilon'$ bounds a ball $B$ in $\nu \cup \beta$. It follows that $B$ cannot contain any of the boundary components of $\nu \cup \beta$. Thus $B$ cannot contain any of the boundary components of $D_i \cup D_{i+1}$. Therefore, $D_i \cup D_{i+1}$ intersects the ball $B$ in a disk bounded by $\partial \epsilon$. Hence the simple closed curve $\partial \epsilon$ bounds a disk in $(D_i \cup D_{i+1}) \cap V'_i$, and therefore the disk $\epsilon$ was not essential in $V'_i$. Thus, $V'_i$ contains no essential disk whose boundary is in $D_i \cup D_{i+1}$.

Now suppose that $V'_i$ contains an essential torus $T$. Suppose that $T$ is not essential in $\nu \cup \beta$. Then either $T$ is boundary parallel or $T$ is compressible in $\nu \cup \beta$. However, $T$ cannot be boundary parallel in $\nu \cup \beta$ since $T \subseteq V'_i$. Thus $T$ must be compressible in $\nu \cup \beta$. Let $\delta$ be a compression disk for $T$ in $\nu \cup \beta$. Since $V'_i$ contains no essential sphere or essential disk whose boundary is in $D_i \cup D_{i+1}$, we can use an innermost disk argument to push $\delta$ off of $D_i \cup D_{i+1}$. Hence $T$ is compressible in $V'_i$, contrary to our initial assumption. Thus $T$ must be essential in $\nu \cup \beta$. It follows that $T$ has the form $\sigma \times S^1$, where $\sigma \subseteq D_i \cap \Delta$. However, since $\nu \cup \beta$ is the double of $V'_i$, the intersection of $\sigma \times S^1$ with $V'_i$ is an annulus $\sigma \times I$. In particular, $V'_i$ cannot contain $\sigma \times S^1$. Therefore, $V'_i$ cannot contain an essential torus.

Suppose that $V''_i$ contains an incompressible annulus $\alpha$ whose boundary is in $D_i \cup D_{i+1}$. Let $\tau$ denote the double of $\alpha$ in $\nu \cup \beta$. Then $\tau$ is a torus. If $\tau$ is essential in $\nu \cup \beta$, then we saw above that $\tau$ can be expressed as $\sigma \times S^1$ (after a possible change in parameterization of $\beta$) where $\sigma$ is a non-trivial simple closed curve in $D_i \cap \Delta$. In this case, $\alpha$ can be expressed as $\sigma \times I$.

On the other hand, if $\tau$ is inessential in $\nu \cup \beta$, then either $\tau$ is parallel to a component of $\partial(\nu \cup \beta)$, or $\tau$ is compressible in $\nu \cup \beta$. If $\tau$ is parallel to a boundary component of $\nu \cup \beta$, then $\alpha$ is parallel to the annulus boundary component of $W_i$, $N(J_i)$, $N(e_i)$, $N(B_i)$, or one of the boundary components of $N(k_i)$.

Thus we suppose that the torus $\tau$ is compressible in $\nu \cup \beta$. In this case, it follows from an innermost loop–outermost arc argument that either the annulus $\alpha$ is compressible in $V'_i$ or $\alpha$ is $\partial$-compressible in $V'_i$. Since we assumed $\alpha$ was incompressible in $V'_i$, $\alpha$ must be $\partial$-compressible in $V'_i$. Now according to a lemma of Waldhausen \cite{Waldhausen}, if a 3-manifold contains no essential sphere or properly embedded essential disk, then any annulus which is incompressible but boundary compressible must be boundary parallel. We saw above that $V''_i$ contains no essential sphere or essential disk whose boundary is in $D_i \cup D_{i+1}$. Since the boundary of the incompressible annulus $\alpha$ is contained in $D_i \cup D_{i+1}$, it follows from Waldhausen’s lemma that $\alpha$ is boundary parallel in $V'_i$.

It follows from Lemma \cite{Waldhausen} that for any $i$, any incompressible annulus in $V'_i$ whose boundary is in $D_i \cup D_{i+1}$ either is parallel to an annulus in $D_i$ or $D_{i+1}$ or co-bounds a solid annulus in the solid annulus $W_i$ with ends in $D_i \cup D_{i+1}$. Recall that $\kappa$ is a simple closed curve in $\Gamma_1$ such that $\kappa \cap W_i = k_i \cup e_i$. Also $J = J_1 \cup \cdots \cup J_n$. Let $N(\kappa)$
and \(N(J)\) be regular neighborhoods of the simple closed curves \(\kappa\) and \(J\) respectively, such that for each \(i\), \(N(\kappa) \cap W_i = N(k_i) \cup N(e_i)\) and \(N(J) \cap W_i = N(J_i)\). Recall that \(V = W_1 \cup \cdots \cup W_n\). Thus \(\text{cl}(V - (N(C_1) \cup \cdots \cup N(C_n) \cup N(J) \cup N(\kappa))) = V_1' \cup \cdots \cup V_n'\).

**Proposition 1.** \(H = \text{cl}(V - (N(C_1) \cup \cdots \cup N(C_n) \cup N(J) \cup N(\kappa)))\) contains no essential sphere or torus.

*Proof.* Suppose that \(S\) is an essential sphere in \(H\). Without loss of generality, \(S\) intersects the collection of disks \(D_1 \cup \cdots \cup D_n\) transversely in a minimal number of simple closed curves. By Lemma 1 for each \(i\), \(V_i'\) contains no essential sphere or essential disk whose boundary is in \(D_i \cup D_{i+1}\). Thus the sphere \(S\) cannot be entirely contained in one \(V_i'\). Let \(c\) be an innermost curve of intersection on \(S\). Then \(c\) bounds a disk \(\delta\) in some \(V_i'\). However, since the number of curves of intersection is minimal, \(\delta\) must be essential, contrary to Lemma 1. Hence \(H\) contains no essential sphere.

Suppose \(T\) is an incompressible torus in \(H\). We show as follows that \(T\) is parallel to some boundary component of \(H\). Without loss of generality, the torus \(T\) intersects the collection of disks \(D_1 \cup \cdots \cup D_n\) transversely in a minimal number of simple closed curves. By Lemma 1 for each \(i\), \(V_i'\) contains no essential torus, essential sphere, or essential disk whose boundary is in \(D_i \cup D_{i+1}\). Thus the torus \(T\) cannot be entirely contained in one \(V_i'\). Also, by the minimality of the number of curves of intersection, we can assume that if \(V_i' \cap T\) is non-empty, then it consists of a collection of incompressible annuli in \(V_i'\) whose boundary components are in \(D_i \cup D_{i+1}\). Furthermore, by Lemma 1, each such annulus either is boundary parallel or is contained in \(N(B_i)\) and can be expressed (after a possible change in parameterization of \(N(B_i)\)) as \(\sigma_i \times I\) for some non-trivial simple closed curve \(\sigma_i\) in \(D_i \cap \Delta\). If some annulus component of \(V_i' \cap T\) is parallel to an annulus in \(D_i \cup D_{i+1}\), then we could remove that component by an isotopy of \(T\). Thus we can assume that each annulus in \(V_i' \cap T\) is parallel to the annulus boundary component of one of the solid annuli \(W_i, N(J_i)\), or \(N(e_i)\), or can be expressed as \(\sigma_i \times I\). In any of these cases the annulus co-bounds a solid annulus in \(W_i\) with ends in \(D_i \cup D_{i+1}\).

Consider some \(i\) such that \(V_i' \cap T\) is non-empty. Hence it contains an incompressible annulus \(A_i\) which has one of the above forms. By the connectivity of the torus \(T\), either there is an incompressible annulus \(A_{i+1} \subset V_{i+1}' \cap T\) such that \(A_i\) and \(A_{i+1}\) share a boundary component, or there is an incompressible annulus \(A_{i-1} \subset V_{i-1}' \cap T\) such that \(A_i\) and \(A_{i-1}\) share a boundary component, or both. We will assume, without loss of generality, that there is an incompressible annulus \(A_{i+1} \subset V_{i+1}' \cap T\) such that \(A_i\) and \(A_{i+1}\) share a boundary component. Now it follows that \(A_i\) co-bounds a solid annulus \(F_i\) in \(W_i\) with ends in \(D_{i+1} \cup D_{i+2}\) and that \(A_{i+1}\) co-bounds a solid annulus \(F_{i+1}\) in \(W_{i+1}\) together with two disks in \(D_{i+1} \cup D_{i+2}\). Hence the solid annuli \(F_i\) and \(F_{i+1}\) meet in one or two disks in \(D_{i+1}\).

We consider several cases where \(A_i\) is parallel to some boundary component of \(V_i'\). Suppose that \(A_i\) is parallel to the annulus boundary component of the solid annulus \(N(J_i)\). Then the solid annulus \(F_i\) contains \(N(J_i)\) and is disjoint from the arcs \(k_i\) and \(e_i\). Now the arcs \(J_i\) and \(J_{i+1}\) share an endpoint contained in \(F_i \cap F_{i+1}\), and there is no endpoint of any arc of \(k_i\) or \(e_i\) in \(F_i \cap F_{i+1}\). It follows that the solid annulus \(F_{i+1}\) contains the arc \(J_{i+1}\) and contains no arcs of \(k_{i+1}\). Hence, by Lemma 1 the incompressible annulus \(A_{i+1}\) must be parallel to \(\partial N(J_{i+1})\). Continuing from one \(V_i'\) to the next, we see that in this case \(T\) is parallel to \(\partial N(J)\).
Suppose that $A_i$ is parallel to the annulus boundary component of the solid annulus $\partial N(e_i)$ or one of the solid annuli in $\partial N(k_i)$. Using an argument similar to that in the above paragraph, we see that $A_{i+1}$ is parallel to the annulus boundary component of the solid annulus $\partial N(e_{i+1})$ or one of the solid annuli in $\partial N(k_{i+1})$. Continuing as above, we see that in this case $T$ is parallel to $\partial N(\kappa)$.

Suppose that the annulus $A_i$ is parallel to the annulus boundary component of the solid annulus $W_i$. Then the solid annulus $F_i$ contains all of the arcs of $J_i$, $k_i$, and $e_i$. It follows as above that the solid annulus $F_{i+1}$ contains the arc $J_{i+1}$ and some arcs of $k_{i+1} \cup e_{i+1}$. Thus by Lemma 1, $A_{i+1}$ must be parallel to the annulus boundary component of the solid annulus $W_{i+1}$. Continuing in this way, we see that in this case $T$ is parallel to $\partial N(V)$.

Thus we now assume that no component of any $V_i' \cap T$ is parallel to an annulus boundary component of $V_i'$. Hence if any $V_i' \cap T$ is non-empty, then by Lemma 1 it consists of disjoint incompressible annuli in $N(B_i)$ which can each be expressed (after a possible re-parametrization of $N(B_i)$) as $\sigma \times I$ for some non-trivial simple closed curve $\sigma \subseteq D_i \cap \Delta$. Choose $i$ such that $V_i' \cap T$ is non-empty. Since $N(B_i)$ is a solid annulus, there is an innermost incompressible annulus $A_i$ of $N(B_i) \cap T$. Now $A_i$ bounds a solid annulus $F_i$ in $N(B_i)$, and $F_i$ contains more than one arc of $k_i$. Since $A_i$ is innermost in $N(B_i)$, $\text{int}(F_i)$ is disjoint from $T$. Now there is an incompressible annulus $A_{i+1}$ in $V_{i+1}' \cap T$ such that $A_i$ and $A_{i+1}$ meet in a circle in $D_{i+1}$. Furthermore, this circle bounds a disk in $D_{i+1}$ which is disjoint from $T$ and, by our assumption, is contained in $N(B_i)$. Thus by Lemma 1 the incompressible annulus $A_{i+1}$ has the form $\sigma_{i+1} \times I$ for some non-trivial simple closed curve $\sigma_{i+1} \subseteq D_{i+1} \cap \Delta$. Thus $A_{i+1}$ bounds a solid annulus $F_{i+1}$ in $N(B_{i+1})$, and $\text{int}(F_{i+1})$ is also disjoint from $T$. We continue in this way considering consecutive annuli to conclude that for every $j$, every component $A_j$ of $T \cap V_j'$ is an incompressible annulus which bounds a solid annulus $F_j$ whose interior is disjoint from $T$.

Recall that $V = W_1 \cup \cdots \cup W_n$ is a solid torus. Let $Q$ denote the component of $V - T$ which is disjoint from $\partial V$. Then $Q$ is the union of the solid annuli $F_j$. Since some $F_i$ contains some arcs of $k_i$, the simple closed curve $\kappa$ must be contained in $Q$.

Recall that the simple closed curve $\kappa$ contains at least three vertices of the embedded graph $\Gamma_\kappa$. Also each vertex of $\kappa$ is contained in some arc $e_j$. Since each such $e_j$ satisfies $e_j \subseteq \kappa \subseteq Q$, some component $F_j$ of $Q \cap W_j$ contains the arc $e_j$. By our assumption, for any $V_i' \cap T$ which is non-empty, $V_i' \cap T$ consists of disjoint incompressible annuli in $N(B_i)$. In particular, $V_j \cap T \subseteq N(B_j)$. Now the annulus boundary of $F_j$ is contained in $N(B_j)$ and hence $F_j \subseteq N(B_j)$. But this is impossible since $e_j \subseteq F_j$ and $e_j$ is disjoint from $N(B_j)$. Hence our assumption that no component of any $V_i' \cap T$ is parallel to an annulus boundary component of $V_i'$ is wrong. Thus, as we saw in the previous cases, $T$ must be parallel to a boundary component of $H$. Therefore $H$ contains no essential annulus. □

Recall that the value of $r$, the simple closed curves, and the manifold $H$ all depend on the particular choice of simple closed curve $\kappa$. In the following theorem we do not fix a particular $\kappa$, so none of the above are fixed.

**Theorem 1.** Every graph can be embedded in $S^3$ in such a way that every non-trivial knot in the embedded graph is hyperbolic.
Proof. Let $G$ be a graph, and let $n \geq 3$ be an odd number such that $G$ is a minor of the complete graph on $n$ vertices, $K_n$. Let $\Gamma_1$ be the embedding of $K_n$ given in our preliminary construction. Then $\Gamma_1$ contains at most finitely many simple closed curves, $\kappa_1, \ldots, \kappa_m$. For each $\kappa_j$, we use Thurston’s Hyperbolic Dehn Surgery Theorem [1] to choose an $r_j$ in the same manner that we chose $r$ after we fixed a particular simple closed curve $\kappa$. Now let $R = \max\{r_1, \ldots, r_m\}$, and let $R$ be the value of $r$ in Figure 1. This determines the simple closed curves $C_1, \ldots, C_n$.

Let $P = \text{cl}(V - (N(C_1) \cup \cdots \cup N(C_n) \cup N(J)))$ where $V$ and $J$ are given in our preliminary construction. Then the embedded graph is such that $\Gamma_1 \subseteq P$. For each $j = 1, \ldots, m$, let $H_j = \text{cl}(P - N(\kappa_j))$. It follows from Proposition 1 that each $H_j$ contains no essential sphere or torus. Since each $H_j$ has more than three boundary components, no $H_j$ can be Seifert fibered. Hence by Thurston’s Hyperbolization Theorem [6], every $H_j$ is a hyperbolic manifold.

We will glue solid tori $Y_1, \ldots, Y_{n+2}$ to $P$ along its $n + 2$ boundary components $\partial V$, $\partial N(C_1)$, $\ldots$, $\partial N(C_n)$, and $\partial N(J)$ to obtain a closed manifold $\overline{P}$ as follows. For each $j$, any gluing of solid tori along the boundary components of $P$ defines a Dehn filling of $H_j = \text{cl}(P - N(\kappa_j))$ along all of its boundary components except $\partial N(\kappa_j)$. Since each $H_j$ is hyperbolic, by Thurston’s Hyperbolic Dehn Surgery Theorem [1], all but finitely many such Dehn fillings of $H_j$ result in a hyperbolic 3-manifold. Furthermore, since $P$ is obtained by removing solid tori from $S^3$, for any integer $q$, if we attach the solid tori $Y_1, \ldots, Y_{n+2}$ to $P$ with slope $\frac{1}{q}$, then $\overline{P} = S^3$. In this case each $H_j \cup Y_1 \cup \cdots \cup Y_{n+2}$ is the complement of a knot in $S^3$. There are only finitely many $H_j$’s, and for each $j$, only finitely many slopes $\frac{1}{q}$ are excluded by Thurston’s Hyperbolic Dehn Surgery Theorem. Thus there is some integer $q$ such that if we glue the solid tori $Y_1, \ldots, Y_{n+2}$ to any of the $H_j$ along $\partial N(C_1)$, $\ldots$, $\partial N(C_n)$, $\partial N(J)$, and $\partial V$ with slope $\frac{1}{q}$, then we obtain the complement of a hyperbolic knot in $S^3$.

Let $\Gamma_2$ denote the re-embedding of $\Gamma_1$ obtained as a result of gluing the solid tori $Y_1, \ldots, Y_{n+2}$ to the boundary components of $P$ with slope $\frac{1}{q}$. Now $\Gamma_2$ is an embedding of $K_n$ in $S^3$ such that every non-trivial knot in $\Gamma_2$ is hyperbolic. Now there is a minor $G'$ of the embedded graph $\Gamma_2$, which is an embedding of our original graph $G$, such that every non-trivial knot in $G'$ is hyperbolic. \qed

References


Department of Mathematics, Pomona College, 610 North College Avenue, Claremont, California 91711-6348

Department of Mathematics, Wake Forest University, P.O. Box 7388, Winston-Salem, North Carolina 27109-7388