THE REVERSE ULTRA LOG-CONCAVITY
OF THE BOROS-MOLL POLYNOMIALS

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Abstract. We prove the reverse ultra log-concavity of the Boros-Moll polynomials. We further establish an inequality which implies the log-concavity of the sequence \( \{d_i(m)\} \) for any \( m \geq 2 \), where \( d_i(m) \) are the coefficients of the Boros-Moll polynomials \( P_m(a) \). This inequality also leads to the fact that in the asymptotic sense, the Boros-Moll sequences are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We propose two conjectures on the log-concavity and reverse ultra log-concavity of the sequence \( \{d_{i-1}(m)d_{i+1}(m)/d_i(m)^2\} \) for \( m \geq 2 \).

1. Introduction

This paper is concerned with the reverse ultra log-concavity of the Boros-Moll polynomials. A sequence \( \{a_k\}_{k \geq 0} \) of real numbers is said to be log-concave if \( a_k^2 \geq a_{k+1}a_{k-1} \) holds for all \( k \geq 1 \). A polynomial is said to be log-concave if the sequence of its coefficients is log-concave; see Brenti [4] and Stanley [11]. Furthermore, a sequence \( \{a_k\}_{0 \leq k \leq n} \) is called ultra log-concave if \( \{a_k/(n\choose k)\} \) is log-concave; see Liggett [8]. This condition can be restated as

\[
k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \geq 0.
\]

It is well known that if a polynomial has only real zeros, then its coefficients form an ultra log-concave sequence. As noticed by Liggett [8], if a sequence \( \{a_k\}_{0 \leq k \leq n} \) is ultra log-concave, then the sequence \( \{ka_k\}_{0 \leq k \leq n} \) is log-concave.

A sequence is said to be reverse ultra log-concave if it satisfies the reverse relation of (1.1), that is,

\[
k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \leq 0.
\]

For example, it is easy to verify that for \( n \geq 2 \), the Bessel polynomial [6]

\[y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k! (n-k)!} x^k\]

is log-concave and reverse ultra log-concave.
The Boros-Moll polynomials, denoted by $P_m(a)$, arise in the following evaluation of a quartic integral:

$$
\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} \, dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a),
$$

where

$$
P_m(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m - 2k}{m - k} \binom{m + k}{k} (a + 1)^k;
$$

see [1, 2, 3, 9]. Write

$$
P_m(a) = \sum_{i=0}^{m} d_i(m) a^i.
$$

The sequence $\{d_i(m)\}_{0 \leq i \leq m}$ is called a Boros-Moll sequence. The expression (1.3) gives the following formula for the coefficients $d_i(m)$:

$$
d_i(m) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m - 2k}{m - k} \binom{m + k}{k} \binom{k}{i}.
$$

Clearly, the coefficients $d_i(m)$ are positive. Moll conjectured that the sequence $\{d_i(m)\}_i$ is log-concave for $m \geq 2$, that is, $d_i(m)^2 \geq d_{i-1}(m)d_{i+1}(m)$ (1 \leq i \leq m - 1). This conjecture has been proved by Kauers and Paule [7].

Despite the log-concavity of $\{d_i(m)\}_i$, we find that inverse ultra log-concavity holds.

**Theorem 1.1.** For $m \geq 2$ and $1 \leq i \leq m - 1$, we have

$$
\left( \frac{d_{i-1}(m)}{d_{i+1}(m)} \right) \cdot \left( \frac{d_{i+1}(m)}{d_i(m)} \right) > \left( \frac{d_i(m)}{d_{i-1}(m)} \right)^2
$$

or, equivalently,

$$
\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m - i + 1)(i + 1)}{(m - i)i}.
$$

On the other hand, it can be shown that the coefficients $d_i(m)$ satisfy an inequality stronger than log-concavity. To be more specific, we will give a lower bound on $d_i(m)^2 / (d_{i-1}(m)d_{i+1}(m))$ which is very close to the upper bound in (1.5).

**Theorem 1.2.** For $m \geq 2$ and $1 \leq i \leq m - 1$, we have

$$
\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} > \frac{(m - i + 1)(i + 1)(m + i)}{(m - i)i(m + i + 1)}.
$$

This paper is organized as follows. We establish an upper bound for $d_i(m+1)/d_i(m)$ in Section 2 which leads to the reverse ultra log-concavity of $\{d_i(m)\}$. In Section 3 we give the proof of Theorem 1.2. We conclude this paper with two conjectures concerning the log-concavity and reverse ultra log-concavity of the sequence $\{d_{i-1}(m)d_{i+1}(m)/d_i^2(m)\}$ for $m \geq 2$. 
2. AN UPPER BOUND FOR $d_i(m+1)/d_i(m)$

In this section, we establish an upper bound for the ratio $d_i(m+1)/d_i(m)$ that will lead to the reverse ultra log-concavity of the sequence of $\{d_i(m)\}$. For $m \geq 1$ and $0 \leq i \leq m$, set

$$T(m, i) = \frac{4m^2 + 7m + 3 + i\sqrt{4m + 4i^2 + 1} - 2i}{2(m - i + 1)(m + 1)}.
$$

**Theorem 2.1.** For all $m \geq 2$, $1 \leq i \leq m - 1$, we have

$$\frac{d_i(m+1)}{d_i(m)} < T(m, i),
$$

and for $m \geq 1$, we have

$$\frac{d_0(m+1)}{d_0(m)} = T(m, 0), \quad \frac{d_m(m+1)}{d_m(m)} = T(m, m).$$

The following lemma will be needed in the proof of Theorem 2.1.

**Lemma 2.2.** For $m \geq 2$ and $1 \leq i \leq m - 1$,

$$T(m, i) < F(m, i),$$

where

$$F(m, i) = \frac{(m + i + 1)(4m + 3)(4m + 5)}{2(2m + 1)(4m^2 - 2i^2 + 9m + 5 + i\sqrt{4m + 4i^2 + 5})}.
$$

**Proof.** Let $A = \sqrt{4m + 4i^2 + 1}$ and $B = \sqrt{4m + 4i^2 + 5}$. It is easy to check that

$$F(m, i) - T(m, i) = \frac{i(X - Y)}{2(2m + 1)(m - i + 1)(4m^2 + 9m + 5 - 2i^2 - iB)},
$$

where

$$X = i - 4i^3 + iAB,$$

$$Y = (5 + 4m^2 + 9m - 2i^2)A - (3 + 4m^2 + 7m - 2i^2)B.$$  

Since $(4m^2 + 9m + 5 - 2i^2)^2 - (iB)^2 = (4m + 5)^2(m + i + 1)(m - i + 1) > 0$, it remains to show that the numerator of $X-Y$ is also positive. We claim that $X > 0$ and $X^2 > Y^2$.

Since $m > i$, we have $A > 2i + 1$ and $B > 2i + 1$. Moreover, since $i \geq 1$, we find that

$$X = (i - 4i^3 + iAB) \geq i - 4i^3 + i(2i + 1)^2 = 4i^2 + 2i > 0.$$  

It is routine to check that $X^2 - Y^2 = G(m, i) - H(m, i)$, where

$$G(m, i) = (32m^4 - 32m^2+128m^3 - 64mi^2 + 190m^2 - 30i^2 + 124m + 30)AB,$$

$$H(m, i) = 128m^5 + 608m^4 + 1128m^3 + 1014m^2 + 436m + 128m^4i^2 + 384m^3i^2 + 408m^2i^2 - 128m^2i^4 + 200mi^2 - 256mi^4 - 120i^4 + 50i^2 + 70.$$  

Since $i < m$, it is easily seen that $G(m, i) > 0$ and $H(m, i) > 0$. To prove that $G(m, i) > H(m, i)$, it suffices to show that $G(m, i) > H(m, i)$ for $1 \leq i \leq m - 1$.

$$G(m, i)^2 - H(m, i)^2 = 16(4m + 5)^2(16m^2 + 12i^2 - 1)(m + i + 1)^2(m - i + 1)^2 > 0.$$  

This yields $X^2 > Y^2$. Since $X > 0$, we see that $X > Y$, and hence (2.3) holds for $1 \leq i \leq m - 1$.  

$\Box$
Proof of Theorem 2.1 It is easy to check (2.3). To prove (2.2), we proceed by induction on \( m \). For \( m = 2 \) and \( i = 1 \), we have \( d_1(3)/d_1(2) = 43/15 < T(2, 1) = (31 + \sqrt{13})/12 \). We now assume that (2.2) is true for \( m \), that is,

\[
d_i(m + 1) < T(m, i)d_i(m), \quad 1 \leq i \leq m - 1.
\]

It will be shown that

\[
d_i(m + 2) < T(m + 1, i)d_i(m + 1), \quad 1 \leq i \leq m - 1.
\]

Using the recurrence relation (3.3), we may write (2.7) in the following form:

\[
-4i^2 + 8m^2 + 24m + 19 \left( \frac{d_i(m + 1)}{2(m - i + 2)(m + 2)} \right) - \frac{(m + i + 1)(4m + 3)(4m + 5)}{4(m + 1)(m + 2)(m - i + 2)} d_i(m)
\]

\[
< T(m + 1, i)d_i(m + 1).
\]

Since \( m > i \), we have \( 4m + 4i^2 + 5 < 12m + 4m^2 + 9 \). It follows that

\[
R(m, i) = \frac{-4i^2 + 8m^2 + 24m + 19}{2(m - i + 2)(m + 2)} - T(m + 1, i)
\]

\[
= \frac{4m^2 + 9m + 5 - 2i^2 - i\sqrt{4m + 4i^2 + 5}}{2(m - i + 2)(m + 2)}
\]

\[
\geq \frac{4m^2 + 9m + 5 - 2i^2 - i(2m + 3)}{2(m - i + 2)(m + 2)} > 0.
\]

Therefore, (2.8) is equivalent to the inequality

\[
(2.9) \quad \frac{d_i(m + 1)}{d_i(m)} < F(m, i),
\]

which is a consequence of (2.6) and Lemma 2.2.

It remains to consider the case \( i = m \). We aim to show that

\[
(2.10) \quad \frac{d_m(m + 2)}{d_m(m + 1)} < T(m + 1, m).
\]

By an easy computation, we find that

\[
\frac{d_m(m + 2)}{d_m(m + 1)} = \frac{(m + 1)(4m^2 + 18m + 21)}{2(2m + 3)(m + 2)},
\]

\[
T(m + 1, m) = \frac{2m^2 + 15m + 14 + m\sqrt{4m^2 + 4m + 5}}{4(m + 2)}.
\]

Thus (2.10) can be rewritten as

\[
(2.11) \quad (2m^2 + 3m)\sqrt{4m^2 + 4m + 5} > 4m^3 + 8m^2 + 5m.
\]

Denote by \( U \) and \( V \) the left hand side and the right hand side of (2.11), respectively. Then \( U^2 - V^2 = 4m^2(4m + 5) > 0 \), and so (2.11) is verified. This completes the proof. \( \square \)
3. The reverse ultra log-concavity

In this section, we give the proof of Theorem 1.1. Our approach can be described as follows. Let \( f(x) = ax^2 + bx + c \) be a quadratic function with \( a > 0 \). Suppose that the equation \( f(x) = 0 \) has two distinct real zeros \( x_1 \) and \( x_2 \), where \( x_1 < x_2 \). Then \( f(x) > 0 \) if \( x > x_2 \) or \( x < x_1 \), and \( f(x) < 0 \) if \( x_1 < x < x_2 \). The key step is to transform the inequality (1.5), that is,

\[
\frac{d_i(m)^2}{d_{i-1}(m) d_{i+1}(m)} < \frac{(m - i + 1)(i + 1)}{(m - i)i},
\]

into a quadratic inequality in the ratio \( d_i(m + 1)/d_i(m) \).

We will need the following recurrence relations for the coefficients \( d_i(m) \). For \( m \geq 1 \) and \( 0 \leq i \leq m + 1 \),

\[
\begin{align*}
(3.1) \quad & 2(m + 1) d_i(m + 1) = 2(m + i)d_{i-1}(m) + (4m + 2i + 3)d_i(m) \\
(3.2) \quad & 2(m + 1)(m + 1 - i)d_i(m + 1) = (4m - 2i + 3)(m + i + 1)d_i(m) - 2i(i + 1)d_{i+1}(m) \\
(3.3) \quad & 4(m + 2 - i)(m + 1)(m + 2)d_i(m + 2) = 2(m + 1)(-4i^2 + 8m^2 + 24m + 19)d_i(m + 1) - (m + i + 1)(4m + 3)(4m + 5)d_i(m).
\end{align*}
\]

These recurrence relations were derived by Kauers and Paule [7]. The relation (3.3) was also derived independently by Moll [10]. Based on these recurrence relations, Kauers and Paule [7] derived the following lower bound for \( d_i(m + 1)/d_i(m) \) in their proof of the log-concavity of the Boros-Moll polynomials:

\[
\frac{d_i(m + 1)}{d_i(m)} \geq Q(m, i), \quad 0 \leq i \leq m,
\]

where

\[
Q(m, i) = \frac{4m^2 + 7m + i + 3}{2(m + 1 - i)(m + 1)}.
\]

Note that Chen and Xia [5] have shown that inequality (3.4) becomes strict, that is,

\[
\frac{d_i(m + 1)}{d_i(m)} > Q(m, i),
\]

for \( m \geq 2 \) and \( 1 \leq i \leq m - 1 \).

Now we are ready to prove the reverse ultra log-concavity of \( \{d_i(m)\} \).

Proof of Theorem 1.1. Applying (3.1) and (3.2), we may reformulate (1.5) in the form

\[
\begin{align*}
(3.7) \quad & 4(m - i + 1)^2 (m + 1)^2 \left( \frac{d_i(m + 1)}{d_i(m)} \right)^2 \\
& - 4(m - i + 1)(m + 1)(4m^2 - 2i^2 + 7m + 3) \left( \frac{d_i(m + 1)}{d_i(m)} \right) \\
& - (32mi^2 - 56m^3 - 73m^2 - 42m + 13i^2 - 9 - 16m^4 + 16i^2 m^2) < 0.
\end{align*}
\]
For $1 \leq i \leq m - 1$, the discriminant of the above quadratic function in $d_i(m+1)/d_i(m)$ is
\[
\Delta = 16i^2(m + 1)^2(4i^2 + 4m + 1)(m - i + 1)^2 > 0.
\]
We see that the quadratic function on the left hand side of (3.7) has two real roots:
\[
x_1 = \frac{4m^2 - 2i^2 + 7m + 3 - i\sqrt{4m + 4i^2 + 1}}{2(m - i + 1)(m + 1)},
\]
\[
x_2 = \frac{4m^2 - 2i^2 + 7m + 3 + i\sqrt{4m + 4i^2 + 1}}{2(m - i + 1)(m + 1)}.
\]
Clearly, $Q(m, i) > x_1$. In view of (3.4), we deduce that $d_i(m+1)/d_i(m) \geq Q(m, i) > x_1$. Observe that $x_2$ coincides with the upper bound $T(m, i)$ in Theorem 2.1. Thus we have $d_i(m+1)/d_i(m) < x_2$. So we have shown that for $1 \leq i \leq m - 1$,
\[
x_1 < \frac{d_i(m+1)}{d_i(m)} < x_2,
\]
which implies (3.7). This completes the proof of Theorem 1.1. \hfill \square

4. A lower bound for $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$

In this section, we give the proof of Theorem 1.2, providing a lower bound on $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$. As will be seen, this lower bound is very close to the upper bound in (1.6) for the reverse ultra log-concavity. So in the asymptotic sense, we may say that the Boros-Moll polynomials are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We conclude this paper with two conjectures.

Proof of Theorem 1.2. Utilizing the recurrence relations (3.1) and (3.2), the inequality (1.6) can be restated as
\[
4(m + 1)^2(m - i + 1)^2\left(\frac{d_i(m+1)}{d_i(m)}\right)^2 - 4(m - i + 1)(m + 1)(4m^2 + 7m - 2i^2 + 3)\frac{d_i(m+1)}{d_i(m)} + (4m^2 + 7m + 3)(-4i + 3 + 4m)(m + i + 1) > 0.
\]
For $1 \leq i \leq m - 1$, the discriminant of the above quadratic function in $d_i(m+1)/d_i(m)$ is
\[
\delta = 16i^2(2i + 1)^2(m + 1)^2(m - i + 1)^2 > 0.
\]
Hence the above quadratic function has two real roots,
\[
x_1 = \frac{4m^2 + 7m - 4i^2 - i + 3}{2(m + 1)(m - i + 1)},
\]
\[
x_2 = \frac{4m^2 + 7m + i + 3}{2(m + 1)(m - i + 1)}.
\]
As $x_2 = Q(m, i)$, it follows from (3.6) that $d_i(m+1)/d_i(m) > x_2$. So we arrive at (1.6). This completes the proof. \hfill \square
Notice that for \(1 \leq i \leq m - 1\),
\[
\frac{(m - i + 1)(i + 1)(m + i)}{(m - i)i(m + i + 1)} > \frac{i + 1}{i}.
\]
As a consequence of Theorem 1.2 we obtain the log-concavity of the sequence \(\{i!d_i(m)\}\).

**Corollary 4.1.** For \(m \geq 2\) and \(1 \leq i \leq m - 1\),
\[
\frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} > \frac{i + 1}{i}
\]
or, equivalently, the sequence \(\{i!d_i(m)\}\) is log-concave.

**Corollary 4.2.** For \(1 \leq i \leq m - 1\), let
\[
c_i(m) = \frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} \quad \text{and} \quad u_i(m) = \left(1 + \frac{1}{i}\right)\left(1 + \frac{1}{m-i}\right).
\]
Then for any \(i \geq 1\),
\[
\lim_{m \to \infty} \frac{c_i(m)}{u_i(m)} = 1.
\]

**Proof.** By Theorems 1.1 and 1.2 we find that
\[
\frac{m + i}{m + i + 1} < \frac{c_i(m)}{u_i(m)} < 1,
\]
which implies \(\lim_{m \to \infty} \frac{c_i(m)}{u_i(m)} = 1\). \(\square\)

We remark that even when \(m\) is small, \(c_i(m)\) is quite close to \(u_i(m)\) for any \(1 \leq i \leq m - 1\). Numerical evidence indicates that \(c_i(m)/u_i(m)\) is increasing for any given \(m\). For example, when \(m = 8\), the values of \(c_i(m)/u_i(m)\) for \(1 \leq i \leq 7\) are given below:

0.956593, 0.969751, 0.978293, 0.983956, 0.987811, 0.990507, 0.992445.

We propose the following two conjectures on the log-concavity and reverse ultra log-concavity of the sequence \(\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}\).

**Conjecture 4.3.** For \(m \geq 2\), the sequence \(\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2 \leq i \leq m-2}\) is log-concave.

**Conjecture 4.4.** For \(m \geq 2\), the sequence \(\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2 \leq i \leq m-2}\) is reverse ultra log-concave.

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