A WAVELET CHARACTERIZATION
FOR THE DUAL OF WEIGHTED HARDY SPACES

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Abstract. We define the weighted Carleson measure space $CMO^p_w$ using wavelets, where the weight function $w$ belongs to the Muckenhoupt class. Then we show that $CMO^p_w$ is the dual space of the weighted Hardy space $H^p_w$ by using sequence spaces. As an application, we give a wavelet characterization of $BMO^w$.

1. Introduction

Meyer [4] described the Hardy space $H^1$ and $BMO$ via wavelets. He offered several characterizations of $H^1$ in terms of its decompositions with respect to wavelet bases, and characterized $BMO$ in terms of a Carleson condition on wavelet coefficients. A natural extension is to consider their weighted counterparts. In 2001, Garcia-Cuerva and Martell [2] gave a wavelet characterization of weighted Hardy spaces $H^p_{w}(\mathbb{R})$, $0 < p \leq 1$. In this article, we give a wavelet characterization for the dual of $H^p_{w}(\mathbb{R})$, $0 < p \leq 1$. In order to do this, we define the weighted Carleson measure space $CMO^p_w$ and two sequence spaces $s^p_w$ and $c^p_w$. We first show that $c^p_w$ is the dual of $s^p_w$, and then obtain that $CMO^p_w$ is the dual of $H^p_w$. As a consequence, $CMO^1_w$ is the same as $BMO^w$, and hence we succeed by an approach different from the one in [5] for the wavelet characterization of $BMO^w$.

Let $\psi$ be an orthonormal wavelet; that is, $\psi \in L^2(\mathbb{R})$ such that the system

$$\psi_{j,k}(x) := 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z},$$

is an orthonormal basis for $L^2(\mathbb{R})$. We define the operator $W_\psi$ by

$$W_\psi f = \left\{ \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 |I_{j,k}|^{-1} \chi_{I_{j,k}} \right\}^{1/2}, \quad f \in L^2(\mathbb{R}),$$
where $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)]$. Denoting by $\mathcal{D}$ the set of all dyadic intervals $I_{j,k}$ with $j,k \in \mathbb{Z}$, and letting $\psi_{I_{j,k}} = \psi_{j,k}$, we can also write

$$W_\psi f = \left\{ \sum_{I \in \mathcal{D}} |(f, \psi_I)|^2 |I|^{-1} \chi_I \right\}^{1/2}.$$  

Henceforth, we always use $I$ and $J$ to denote dyadic intervals. In what follows, we shall work exclusively with the one-dimensional case. For $\alpha \geq 1$, we say that $\psi$ belongs to the regularity class $R^\alpha$ if $\psi \in C^{[\alpha]}$ and there exist positive constants $C, r, \varepsilon$ satisfying

\begin{itemize}
  \item[(i)] $\int_R x^n \psi(x) \, dx = 0$ for all $0 \leq n \leq [\alpha] - 1$,
  \item[(ii)] $|\psi(x)| \leq \frac{C}{(1 + |x|)^{1+\varepsilon}}$ for all $x \in \mathbb{R}$,
  \item[(iii)] $|\psi^{(n)}(x)| \leq \frac{C}{(1 + |x|)^{\alpha + \varepsilon}}$ for all $x \in \mathbb{R}$ and $0 \leq n \leq [\alpha]$.
\end{itemize}

Here $[\alpha]$ denotes the greatest integer not greater than $\alpha$.

The weight functions mentioned in this article refer to the Muckenhoupt $A_q$ weights. A weight $w \geq 0$ belongs to the class $A_q$, $1 < q < \infty$, if there is a constant $C > 0$ such that

$$\left( \int_I w(x) \, dx \right) \left( \int_I w(x)^{-1/(q-1)} \, dx \right)^{q-1} \leq C |I|^q$$

for any interval $I \subset \mathbb{R}$.

The class $A_1$ consists of weights $w$ satisfying for some $C > 0$ that

$$\frac{1}{|I|} \int_I w(x) \, dx \leq C \cdot \text{ess inf}_{x \in I} w(x)$$

for any interval $I \subset \mathbb{R}$, and $A_{\infty} := \bigcup_{1 \leq q < \infty} A_q$. For $w \in A_{\infty}$, denote by $q_w := \inf \{ q > 1 : w \in A_q \}$ the critical index of $w$. We use $w(E)$ to denote the weighted measure $\int_E w(x) \, dx$.

Let $\varphi \in S$ satisfy $\int_R \varphi(x) \, dx = 1$. The maximal function $f^*$ is defined by

$$f^*(x) = \sup_{r > 0} |f \ast \varphi_r(x)|,$$

where $\varphi_r(x) = r^{-1} \varphi(x/r)$, $r > 0$. The weighted Hardy spaces $H^p_w$ consist of those tempered distributions $f \in S'$ for which $f^* \in L^p_w$ with $\|f\|_{H^p_w} = \|f^*\|_{L^p_w}$. We refer readers to \cite{1, 3} for the details about $A_q$ and $H^p_w$.

The following theorem was proved by Garcia-Cuerva and Martell \cite{2}.

**Theorem A.** Let $0 < p \leq 1$ and $w \in A_{\infty}$. If $\psi \in R^\alpha$ is an orthonormal wavelet with $\alpha \geq q_w/p$, then there exist two constants $0 < c \leq C < \infty$ such that

$$c \|f\|_{H^p_w} \leq \|W_\psi f\|_{L^p_w} \leq C \|f\|_{H^p_w}.$$  

**Definition.** For $0 < p \leq 1$ and $w \in A_{\infty}$, let $\psi \in R^\alpha$ be an orthonormal wavelet with $\alpha \geq q_w/p$. The weighted Carleson space $CM^p_w$ is the set of all $g \in L^1_{loc}$ satisfying

$$\|g\|_{CM^p_w} := \sup_{J \in \mathcal{D}} \left\{ \frac{1}{w(J)^{1/p - 1}} \sum_{I \subset J} |(g, \psi_I)|^2 \frac{|I|}{w(I)} \right\}^{1/2} < \infty.$$
Remark 1. If \( w \equiv \text{constant} \) and \( p = 1 \), then the above definition reduces to the Carleson condition that characterizes \( BMO \) (cf. [4, p. 154]). Theorem A implies that the wavelet characterization of \( H^p_w \) is independent of the choice of \( \psi \), and hence, by the following Theorem \( 1 \) the definition of \( CMO^p_w \) is independent of the choice of \( \psi \), too.

We now state our main result as follows.

**Theorem 1.** For \( 0 < p \leq 1 \) and \( w \in A_\infty \), let \( \psi \in \mathcal{R}^\alpha \) be an orthonormal wavelet with \( \alpha \geq q_w/p \). The dual of \( H^p_w \) is \( CMO^p_w \) in the following sense.

(a) For each \( g \in CMO^p_w \), there is a linear functional \( \ell_g \), initially defined on \( H^p_w \cap L^2 \), which has a continuous extension to \( H^p_w \) and \( \|\ell_g\| \leq C\|g\|_{CMO^p_w} \).

(b) Conversely, every continuous linear functional \( \ell \) of \( H^p_w \) can be realized as \( \ell = \ell_g \) with some \( g \in CMO^p_w \) and \( \|g\|_{CMO^p_w} \leq C\|\ell\| \).

It is known that the dual space of \( H^1_w \) is

\[
BMO_w = \left\{ f \in L^1_{loc} : \sup_{\text{interval } Q} \frac{1}{w(Q)} \int_Q |f(x) - f_Q| dx < \infty \right\}
\]

for \( w \in A_\infty \), and the dual space of \( H^p_w \), \( 0 < p < 1 \), is

\[
\left\{ \frac{f(x)}{w(x)} \in L^r_{loc}(w(x)dx) : \left( \int_Q \left| \frac{f(x) - P_Q(x)}{w(x)} \right|^r \frac{w(x)dx}{w(Q)} \right)^{1/r} \leq Cw(Q)^{1/p - 1} \right\}
\]

for \( w \in A_r \), \( 1 \leq r < \infty \), where \( f_Q = \frac{1}{|Q|} \int_Q f(x)dx \) and \( P_Q \) is the unique polynomial of degree \( \leq \lfloor q_w/p \rfloor - 1 \) such that \( \int_Q (f(x) - P_Q(x))x^kdx = 0 \) for \( k = 0, 1, \ldots, \lfloor q_w/p \rfloor - 1 \) (see [1]). Thus, we have a wavelet characterization of \( BMO_w \) and a continuous characterization of \( CMO^p_w \) as follows.

**Corollary 2.** Let \( 0 < p \leq 1 \) and \( w \in A_r \), \( 1 \leq r \leq \infty \). Also let \( \psi \in \mathcal{R}^\alpha \) be an orthonormal wavelet with \( \alpha \geq q_w/p \).

(a) For \( p = 1 \) and \( w \in A_\infty \), \( f \in BMO_w \) if and only if its wavelet coefficients \( \langle f, \psi_I \rangle \) satisfy Carleson’s condition:

\[
\sup_{J \in \mathcal{D}} \frac{1}{w(J)} \sum_{I \subset J} |\langle f, \psi_I \rangle|^2 \frac{|I|}{w(I)} \leq C.
\]

(b) For \( 0 < p < 1 \) and \( w \in A_r \), \( 1 \leq r < \infty \), \( f \) satisfies (1) if and only if \( f \in CMO^p_w \).

**Remark 2.** When the wavelet \( \psi \) has compact support, the above characterization of \( BMO_w \) was given by Wu [5]. Here we offer a different but simpler approach.

2. **Sequence spaces**

In this section, we introduce two sequence spaces \( s^p_w \) and \( c^p_w \), \( 0 < p \leq 1 \).

**Definition.** Let \( 0 < p \leq 1 \) and \( w \geq 0 \) be a weight function. The sequence space \( s^p_w \) is defined to be the collection of all complex-valued sequences

\[
s^p_w = \left\{ \{s_I\} : \|\{s_I\}\|_{s^p_w} := \left\| \left( \sum_I |s_I|^2 |I|^{-1} \chi_I \right)^{1/2} \right\|_{L^p_w} < \infty \right\}
\]
Similarly, \( c_w^p \) is defined to be the collection of all complex-valued sequences
\[
\{ t_I : \| \{ t_I \} \|_{c_w^p} := \sup_{J \subseteq D} \left( \frac{1}{w(J)^{1/p}} \sum_{I \subseteq J} |t_I|^2 \frac{|I|^p}{w(I)} \right)^{1/p} \leq \infty \}.
\]

**Theorem 3.** Let \( 0 < p \leq 1 \) and \( w \in A_\infty \). The dual of \( s_w^p \) is \( c_w^p \) in the following sense.

(a) For each \( \{ t_I \} \in c_w^p \), the linear functional \( \{ s_I \} \mapsto \sum_I s_I \cdot t_I \) is continuous on \( s_w^p \).

(b) Conversely, every continuous linear functional on \( s_w^p \) arises as in (a) with a unique element \( \{ t_I \} \) of \( c_w^p \).

Moreover, the norm of \( \{ t_I \} \) as a linear functional on \( s_w^p \) is equivalent to its \( c_w^p \)-norm.

**Proof.** (a) Given \( \{ t_I \} \in c_w^p \), it suffices to show that
\[
\left| \sum_I s_I \cdot t_I \right| \leq C \| \{ s_I \} \|_{s_w^p} \| \{ t_I \} \|_{c_w^p}
\]
for all \( \{ s_I \} \in s_w^p \).

For \( \{ s_I \} \in s_w^p \), write
\[
\Omega_k = \left\{ x \in \mathbb{R} : S(x) := \left( \sum_I |s_I|^2 \frac{|I|}{w(I)} \right)^{1/2} > 2^k \right\}
\]
and
\[
B_k = \left\{ I : w(I \cap \Omega_k) > \frac{1}{2} w(I) \quad \text{and} \quad w(I \cap \Omega_{k+1}) \leq \frac{1}{2} w(I) \right\}.
\]
Then
\[
\left| \sum_I s_I \cdot t_I \right| = \left| \sum_k \sum_{I \in B_k} \sum_{I \subseteq J \subseteq \tilde{B}_k} s_I \cdot t_I \right| \leq \sum_k \sum_{I \in B_k} \sum_{I \subseteq J \subseteq \tilde{B}_k} |s_I||t_I|,
\]
where \( \tilde{I} \)'s are the maximal dyadic intervals in \( B_k \). Applying the inequality \( \| \cdot \|_{L^1} \leq \| \cdot \|_{L^p} \), we get
\[
\sum_k \sum_{I \in B_k} \sum_{I \subseteq J \subseteq \tilde{B}_k} |s_I||t_I| \leq \sum_k \sum_{I \in B_k} \left( \sum_{I \subseteq J \subseteq \tilde{B}_k} |s_I|^2 \frac{w(I)}{|I|} \right)^{1/2} \left( \sum_{I \subseteq J \subseteq \tilde{B}_k} |t_I|^2 \frac{|I|^p}{w(I)} \right)^{1/p}
\]
\[
\leq \left\{ \sum_k \sum_{I \in B_k} \left( \sum_{I \subseteq J \subseteq \tilde{B}_k} |s_I|^2 \frac{w(I)}{|I|} \right)^{p/2} \left( \sum_{I \subseteq J \subseteq \tilde{B}_k} |t_I|^2 \frac{|I|^p}{w(I)} \right)^{p/2} \right\}^{1/p}
\]
\[
\leq \| \{ t_I \} \|_{c_w^p} \left\{ \sum_k \sum_{I \in B_k} w(I)^{1-p/2} \left( \sum_{I \subseteq J \subseteq \tilde{B}_k} |s_I|^2 \frac{w(I)}{|I|} \right)^{p/2} \right\}^{1/p}.
\]

Write
\[
\tilde{\Omega}_k = \{ x \in \mathbb{R} : M_w(\chi_{\Omega_k})(x) > 1/2 \},
\]
where \( M_w \) is the weighted Hardy-Littlewood maximal function defined by
\[
M_w f(x) = \sup_{\text{interval } Q \ni x} \frac{1}{w(Q)} \int_Q |f(x)|w(x)dx.
\]
Then \( I \subset \tilde{\Omega}_k \) for any \( I \in B_k \). Since the \( \tilde{I} \)'s are mutually disjoint dyadic intervals, 
\[
\sum_{\tilde{I} \in B_k} w(\tilde{I}) \leq w(\Omega_k). 
\]
We then apply Hölder’s inequality to obtain
\[
\left\| \sum_{I} s_I \cdot \tilde{t}_I \right\| \leq \| \{ t_I \} \| \left\| \sum_{k} 2^{k/p} w(\tilde{\Omega}_k) \right\|^{1/p}. 
\]
We claim that
\[
\sum_{I \in B_k} |s_I|^2 \frac{w(I)}{|I|} \leq C 2^{k} w(\tilde{\Omega}_k). 
\]
\( M_w \) is of weak type \((1,1)\) with respect to \( w(x)dx \), so \( w(\tilde{\Omega}_k) \leq C w(\Omega_k) \) and the claim gives
\[
\left\| \sum_{I} s_I \cdot \tilde{t}_I \right\| \leq C \| \{ t_I \} \| \| S \|_{L^p_w}. 
\]
To prove the claim, by the definitions of \( S(x) \) and \( B_k \), we have
\[
\int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} S^2(x) w(x) \, dx \leq 2^{2k+2} w(\tilde{\Omega}_k) 
\]
and
\[
\int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} S^2(x) w(x) \, dx \geq \int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} \sum_{I \in B_k} |s_I|^2 |I|^{-1} \chi_{I}(x) w(x) \, dx 
\]
\[
= \sum_{I \in B_k} |s_I|^2 \frac{w(I \cap (\tilde{\Omega}_k \setminus \Omega_{k+1}))}{|I|} 
\]
\[
\geq \frac{1}{2} \sum_{I \in B_k} |s_I|^2 \frac{w(I)}{|I|}. 
\]
(b) Clearly, every \( \ell \in (s_w^p)' \) is of the form
\[
\ell(\{ s_I \}) = \sum_{I} s_I \tilde{t}_I, \quad \{ s_I \} \in s_w^p, 
\]
where \( \{ t_I \} \) is a certain sequence. Fix a dyadic interval \( J \). Let \( S_J = \{ I \in D : I \subset J \} \) and define a measure \( \nu \) on \( S_J \) by
\[
d\nu(I) = \frac{|I|}{w(J)^{\frac{p}{2} - 1}} \quad \text{for} \quad I \in S_J, 
\]
By duality,
\[
\left( \frac{1}{w(J)^{\frac{1}{p}-1}} \sum_{I \subset J} |t_I|^2 \frac{|I|}{w(I)} \right)^{1/2} = \left\| \{ t_I \frac{1}{w(I)^{\frac{1}{p}}} \} \right\|_{L^2(S_J,d\nu)} 
\]
\[
\leq \left\| \ell \right\| \sup_{\| \{ s_I \} \|_{\ell^2(S_J,d\nu)} \leq 1} \left\| \left\{ s_I \frac{1}{w(J)^{\frac{1}{p}-1} w(I)^{\frac{1}{2}}} \right\} \right\|_{s_w^p}. 
\]

For \( \{ s_I \} \in \ell^2(S_J,d\nu) \), Hölder’s inequality yields
\[
\left\| \left\{ s_I \frac{|I|}{w(J)^{\frac{1}{p}-1} w(I)^{\frac{1}{2}}} \right\} \right\|_{s_w^p} = \frac{1}{w(J)^{\frac{1}{p}-1}} \left\{ \int_J \left( \sum_{I \subset J} |s_I|^2 \frac{|I|}{w(I)} \chi_I(x) \right)^{p/2} w(x) \, dx \right\}^{1/p} 
\]
\[
\leq \left\{ \frac{1}{w(J)^{\frac{1}{p}-1}} \int_J \sum_{I \subset J} |s_I|^2 \frac{|I|}{w(I)} \chi_I(x) w(x) \, dx \right\}^{1/2} 
\]
\[
= \left\| \{ s_I \} \right\|_{\ell^2(S_J,d\nu)}, 
\]
and hence
\[
\sup_{\| \{ s_I \} \|_{\ell^2(S_J,d\nu)} \leq 1} \left\| \left\{ s_I \frac{|I|}{w(J)^{\frac{1}{p}-1} w(I)^{\frac{1}{2}}} \right\} \right\|_{s_w^p} \leq 1.
\]
Taking the supremum over \( J \in \mathcal{D} \) in (2), we obtain \( \| \{ t_I \} \|_{s_w^p} \leq \left\| \ell \right\| \). \( \square \)

3. PROOF OF THE MAIN THEOREM

In this section we show that Theorem 1 follows as a consequence of Theorem 3. Let \( \psi \in \mathcal{R}^\alpha, \alpha \geq 1 \), be an orthonormal wavelet. Define a map \( P \) from the family of complex sequences into \( S' \) by
\[
P(\{ s_I \}) = \sum_I s_I \psi_I.
\]

Define another map \( L \) from function space into the family of complex sequences by
\[
L(f) = \{ \langle f, \psi_I \rangle \}
\]
such that all \( \langle f, \psi_I \rangle \)'s are well defined. Figure 1 illustrates the relationship among \( s_w^p, c_w^p, H_w^p, \) and \( CMO_w^p \). Then \( L \circ L \mid_{L^2} \) is the identity on \( L^2 \). For \( 0 < p \leq 1 \) and
\[
\begin{array}{ccc}
S_w^p & \text{dual relation (by Theorem 3)} & c_w^p \\
P & L & P \\
H_w^p & \text{dual relation (by Theorem 1)} & CMO_w^p
\end{array}
\]

Figure 1. Diagram for spaces and maps

\( w \in A_\infty \) with critical index \( q_w \), if \( \alpha \geq q_w/p \), then Theorem A yields
\[
\| \{ L(f) \} \|_{s_w^p} \leq C \| f \|_{H_w^p} \quad \text{for } f \in H_w^p \cap L^2
\]
and
\[
\|P\{s_I\}\|_{H^p_w} \leq C\|\mathcal{W}_w P\{s_I\}\|_{L^p_w} = C\|\{s_I\}\|_{s^p_w} \quad \text{for } \{s_I\} \in s^p_w.
\]

By the definitions of \(c^p_w\) and \(C\text{MO}_w\),
\[
\|\{L(g)\}\|_{c^p_w} = \|g\|_{C\text{MO}_w} \quad \text{for } g \in C\text{MO}_w
\]
and
\[
\|P\{t_I\}\|_{C\text{MO}_w} = \|\{t_I\}\|_{c^p_w} \quad \text{for } \{t_I\} \in c^p_w.
\]

**Proof of Theorem 3**: For \(g \in C\text{MO}_w\), define a linear functional \(\tilde{\ell}_g\) by
\[
\tilde{\ell}_g(f) = \langle L(f), L(g) \rangle \quad \text{for } f \in H^p_w \cap L^2.
\]

By (3), (5), and Theorem 3
\[
|\tilde{\ell}_g(f)| \leq C\|L(f)\|_{s^p_w} \|L(g)\|_{c^p_w} \leq C\|f\|_{H^p_w} \|g\|_{C\text{MO}_w} \quad \text{for } f \in H^p_w \cap L^2.
\]

Since \(H^p_w \cap L^2\) is dense in \(H^p_w\), the map \(\tilde{\ell}_g\) can be extended to a continuous linear functional \(\ell_g\) on \(H^p_w\) satisfying \(\|\ell_g\| \leq C\|g\|_{C\text{MO}_w}\).

Conversely, let \(\ell \in (H^p_w)'\) and set \(\ell_1 = \ell \circ P\) on \(s^p_w\). It follows from (4) that \(\ell_1 \in (s^p_w)'\). By Theorem 3 there exists \(\{t_I\} \in c^p_w\) such that
\[
\ell_1(\{s_I\}) = \sum_I s_I \cdot t_I \quad \text{for } \{s_I\} \in s^p_w,
\]
and
\[
\|\{t_I\}\|_{c^p_w} \approx \|\ell_1\| \leq C\|\ell\|.
\]
For \(f \in H^p_w \cap L^2\), we have
\[
\ell(f) = \ell_1 \circ L(f) = \sum_I \langle f, \psi_I \rangle t_I = \langle L(f), L(g) \rangle,
\]
where \(g = \sum_I t_I \psi_I\). This shows that \(\ell = \ell_g\), and (2) gives
\[
\|g\|_{C\text{MO}_w} = \|\{t_I\}\|_{c^p_w} \leq C\|\ell\|.
\]

Hence, the proof is finished. \(\square\)

**References**

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