A WAVELET CHARACTERIZATION
FOR THE DUAL OF WEIGHTED HARDY SPACES

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Abstract. We define the weighted Carleson measure space $CMO^p_w$ using
wavelets, where the weight function $w$ belongs to the Muckenhoupt class. Then
we show that $CMO^p_w$ is the dual space of the weighted Hardy space $H^p_w$ by
using sequence spaces. As an application, we give a wavelet characterization
of $BMO_w$.

1. Introduction

Meyer [4] described the Hardy space $H^1$ and $BMO$ via wavelets. He offered several
characterizations of $H^1$ in terms of its decompositions with respect to wavelet
bases, and characterized $BMO$ in terms of a Carleson condition on wavelet coef-
ficients. A natural extension is to consider their weighted counterparts. In 2001,
Garcia-Cuerva and Martell [2] gave a wavelet characterization of weighted Hardy
spaces $H^p_w(\mathbb{R})$, $0 < p \leq 1$. In this article, we give a wavelet characterization for the
dual of $H^p_w(\mathbb{R})$, $0 < p \leq 1$. In order to do this, we define the weighted Carleson
measure space $CMO^p_w$ and two sequence spaces $s^p_w$ and $c^p_w$. We first show that $c^p_w$
is the dual of $s^p_w$, and then obtain that $CMO^p_w$ is the dual of $H^p_w$. As a consequence,
$CMO^1_w$ is the same as $BMO_w$, and hence we succeed by an approach different from
the one in [5] for the wavelet characterization of $BMO_w$.

Let $\psi$ be an orthonormal wavelet; that is, $\psi \in L^2(\mathbb{R})$ such that the system

$$
\psi_{j,k}(x) := 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z},
$$

is an orthonormal basis for $L^2(\mathbb{R})$. We define the operator $W_\psi$ by

$$
W_\psi f = \left\{ \sum_{j,k \in \mathbb{Z}} |(f, \psi_{j,k})|^2 |I_{j,k}|^{-1/2} \right\}^{1/2}, \quad f \in L^2(\mathbb{R}),
$$
where $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)]$. Denoting by $\mathcal{D}$ the set of all dyadic intervals $I_{j,k}$ with $j, k \in \mathbb{Z}$, and letting $\psi_{I_{j,k}} = \psi_{j,k}$, we can also write

$$W_{\psi}f = \left\{ \sum_{I \in \mathcal{D}} |(f, \psi_{I})|^2 |I|^{-1} \chi_{I} \right\}^{1/2}.$$ 

Henceforth, we always use $I$ and $J$ to denote dyadic intervals. In what follows, we shall work exclusively with the one-dimensional case. For $\alpha \geq 1$, we say that $\psi$ belongs to the regularity class $\mathcal{R}^\alpha$ if $\psi \in C^{(\alpha)}$ and there exist positive constants $C, r, \varepsilon$ satisfying $C, r, \varepsilon$

\begin{align*}
(\text{i}) & \quad \int_{\mathbb{R}} x^n \psi(x) \, dx = 0 \quad \text{for all } 0 \leq n \leq [\alpha] - 1, \\
(\text{ii}) & \quad |\psi(x)| \leq \frac{C}{(1 + |x|)^{1+[\alpha]+\varepsilon}} \quad \text{for all } x \in \mathbb{R}, \\
(\text{iii}) & \quad |\psi^{(n)}(x)| \leq \frac{C}{(1 + |x|)^{n+\varepsilon}} \quad \text{for all } x \in \mathbb{R} \text{ and } 0 \leq n \leq [\alpha].
\end{align*}

Here $[\alpha]$ denotes the greatest integer not greater than $\alpha$.

The weight functions mentioned in this article refer to the Muckenhoupt $A_q$ weights. A weight $w \geq 0$ belongs to the class $A_q$, $1 < q < \infty$, if there is a constant $C > 0$ such that

$$\left( \int_I w(x) \, dx \right) \left( \int_I w(x)^{-1/(q-1)} \, dx \right)^{q-1} \leq C |I|^q \quad \text{for any interval } I \subset \mathbb{R}.$$ 

The class $A_1$ consists of weights $w$ satisfying for some $C > 0$

$$\frac{1}{|I|} \int_I w(x) \, dx \leq C \cdot \text{ess} \inf_{x \in I} w(x) \quad \text{for any interval } I \subset \mathbb{R},$$

and $A_{\infty} := \bigcup_{1 \leq q < \infty} A_q$. For $w \in A_{\infty}$, denote by $q_w := \inf \{ q > 1 : w \in A_q \}$ the critical index of $w$. We use $w(E)$ to denote the weighted measure $\int_E w(x) \, dx$.

Let $\varphi \in \mathcal{S}$ satisfy $\int_{\mathbb{R}} \varphi(x) \, dx = 1$. The maximal function $f^*$ is defined by

$$f^*(x) = \sup_{r > 0} |f * \varphi_r(x)|,$$

where $\varphi_r(x) = r^{-1} \varphi(x/r)$, $r > 0$. The weighted Hardy spaces $H^p_w$ consist of those tempered distributions $f \in \mathcal{S}'$ for which $f^* \in L^p_w$ with $\|f\|_{H^p_w} = \|f^*\|_{L^p_w}$. We refer readers to [13] for the details about $A_q$ and $H^p_w$.

The following theorem was proved by Garcia-Cuerva and Martell [2].

**Theorem A.** Let $0 < p \leq 1$ and $w \in A_{\infty}$. If $\psi \in \mathcal{R}^\alpha$ is an orthonormal wavelet with $\alpha \geq q_w/p$, then there exist constants $0 < c \leq C < \infty$ such that

$$c \|f\|_{H^p_w} \leq \|W_{\psi}f\|_{L^p_w} \leq C \|f\|_{H^p_w}.$$ 

**Definition.** For $0 < p \leq 1$ and $w \in A_{\infty}$, let $\psi \in \mathcal{R}^\alpha$ be an orthonormal wavelet with $\alpha \geq q_w/p$. The weighted Carleson space $CMO_w^p$ is the set of all $g \in L^1_{\text{loc}}$ satisfying

$$\|g\|_{CMO_w^p} := \sup_{J \in \mathcal{D}} \left\{ \frac{1}{w(J)^{\frac{1}{p}-1}} \sum_{I \subset J} |(g, \psi_{I})|^2 \frac{|I|}{w(I)} \right\}^{1/2} < \infty.$$
Remark 1. If \( w \equiv \text{constant} \) and \( p = 1 \), then the above definition reduces to the Carleson condition that characterizes \( BMO \) (cf. [4, p. 154]). Theorem A implies that the wavelet characterization of \( H^p_w \) is independent of the choice of \( \psi \), and hence, by the following Theorem 1, the definition of \( CMO^p_w \) is independent of the choice of \( \psi \), too.

We now state our main result as follows.

**Theorem 1.** For \( 0 < p \leq 1 \) and \( w \in A_\infty \), let \( \psi \in \mathcal{R}^n \) be an orthonormal wavelet with \( \alpha \geq q_w/p \). The dual of \( H^p_w \) is \( CMO^p_w \) in the following sense.

(a) For each \( g \in CMO^p_w \), there is a linear functional \( \ell_g \), initially defined on \( H^p_w \cap L^2 \), which has a continuous extension to \( H^p_w \) and \( \| \ell_g \| \leq C \| g \|_{CMO^p_w} \).

(b) Conversely, every continuous linear functional \( \ell \) of \( H^p_w \) can be realized as \( \ell = \ell_g \) with some \( g \in CMO^p_w \) and \( \| g \|_{CMO^p_w} \leq C \| \ell \| \).

It is known that the dual space of \( H^1_w \) is

\[
BMO_w = \left\{ f \in L^1_{\text{loc}} : \sup_{\text{interval } Q} \frac{1}{w(Q)} \int_Q |f(x) - f_Q| \, dx < \infty \right\}
\]

for \( w \in A_\infty \), and the dual space of \( H^p_w \), \( 0 < p < 1 \), is

\[
\left\{ \frac{f(x)}{w(x)} \in L^1_{\text{loc}}(w(x) \, dx) : \left( \int_Q \left| \frac{f(x) - P_Q(x)}{w(x)} \right|^{r'} \frac{w(x) \, dx}{w(Q)} \right)^{1/r'} \leq C w(Q)^{1/p - 1} \right\}
\]

for \( w \in A_r \), \( 1 \leq r < \infty \), where \( f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx \) and \( P_Q \) is the unique polynomial of degree \( \leq [q_w/p] - 1 \) such that \( \int_Q (f(x) - P_Q(x)) \, x^k \, dx = 0 \) for \( k = 0, 1, \ldots, [q_w/p] - 1 \) (see [1]). Thus, we have a wavelet characterization of \( BMO_w \) and a continuous characterization of \( CMO^p_w \) as follows.

**Corollary 2.** Let \( 0 < p \leq 1 \) and \( w \in A_r \), \( 1 \leq r \leq \infty \). Also let \( \psi \in \mathcal{R}^n \) be an orthonormal wavelet with \( \alpha \geq q_w/p \).

(a) For \( p = 1 \) and \( w \in A_\infty \), \( f \in BMO_w \) if and only if its wavelet coefficients \( \langle f, \psi_l \rangle \) satisfy Carleson’s condition:

\[
\sup_{J \in \mathcal{D}} \frac{1}{w(J)} \sum_{l \subseteq J} |\langle f, \psi_l \rangle|^2 \frac{|I|}{w(I)} \leq C.
\]

(b) For \( 0 < p < 1 \) and \( w \in A_r \), \( 1 \leq r < \infty \), \( f \) satisfies (1) if and only if \( f \in CMO^p_w \).

**Remark 2.** When the wavelet \( \psi \) has compact support, the above characterization of \( BMO_w \) was given by Wu [5]. Here we offer a different but simpler approach.

2. Sequence spaces

In this section, we introduce two sequence spaces \( s^p_w \) and \( c^p_w \), \( 0 < p \leq 1 \).

**Definition.** Let \( 0 < p \leq 1 \) and \( \geq 0 \) be a weight function. The sequence space \( s^p_w \) is defined to be the collection of all complex-valued sequences

\[
s^p_w = \left\{ \{s_I\} : \|\{s_I\}\|_{s^p_w} := \left( \sum_I |s_I|^2 |I|^{-1} \chi_I \right)^{1/2} \in L^p_w \right\}.
\]
Similarly, \( c_w^p \) is defined to be the collection of all complex-valued sequences
\[
c_w^p = \left\{ \{ t_I \} : \| \{ t_I \} \|_{c_w^p} := \sup_{J \in D} \left( \frac{1}{w(J)^{p-1}} \sum_{I \subset J} |t_I|^2 \frac{|I|}{w(I)} \right)^{1/2} < \infty \right\}.
\]

**Theorem 3.** Let \( 0 < p \leq 1 \) and \( w \in A_\infty \). The dual of \( s_w^p \) is \( c_w^p \) in the following sense.

(a) For each \( \{ t_I \} \in c_w^p \), the linear functional \( \{ s_I \} \mapsto \sum_I s_I \cdot \overline{t_I} \) is continuous on \( s_w^p \).

(b) Conversely, every continuous linear functional on \( s_w^p \) arises as in (a) with a unique element \( \{ t_I \} \) of \( c_w^p \).

Moreover, the norm of \( \{ t_I \} \) as a linear functional on \( s_w^p \) is equivalent to its \( c_w^p \)-norm.

**Proof.** (a) Given \( \{ t_I \} \in c_w^p \), it suffices to show that
\[
\left| \sum_I s_I \cdot \overline{t_I} \right| \leq C \| \{ s_I \} \|_{s_w^p} \| \{ t_I \} \|_{c_w^p} \quad \text{for all } \{ s_I \} \in s_w^p.
\]

For \( \{ s_I \} \in s_w^p \), write
\[
\Omega_k = \left\{ x \in \mathbb{R} : S(x) := \left( \sum_I |s_I|^2 |I|^{-1} \chi_I(x) \right)^{1/2} > 2^k \right\}
\]
and
\[
B_k = \left\{ I : w(I \cap \Omega_k) > \frac{1}{2} w(I) \quad \text{and} \quad w(I \cap \Omega_{k+1}) \leq \frac{1}{2} w(I) \right\}.
\]
Then
\[
\left| \sum_I s_I \cdot \overline{t_I} \right| = \left| \sum_k \sum_{I \in B_k} \sum_{I' \subset I \in \tilde{B}_k} s_I \cdot \overline{t_I} \right| \leq \sum_k \sum_{I \in B_k} \sum_{I' \subset I \in \tilde{B}_k} |s_I| |t_I|,
\]
where \( \tilde{I} \)'s are the maximal dyadic intervals in \( B_k \). Applying the inequality \( \| \cdot \|_{l^p} \leq \| \cdot \|_{c_w^p} \), we get
\[
\sum_k \sum_{I \in B_k} \sum_{I' \subset I \in \tilde{B}_k} |s_I| |t_I| \leq \sum_k \sum_{I \in B_k} \left( \sum_{I' \subset I \in \tilde{B}_k} |s_I|^2 \frac{w(I)}{|I|} \right)^{1/2} \left( \sum_{I' \subset I \in \tilde{B}_k} |t_I|^2 \frac{|I|}{w(I)} \right)^{1/2}
\]
\[
\leq \left\{ \sum_k \sum_{I \in B_k} \left( \sum_{I' \subset I \in \tilde{B}_k} |s_I|^2 \frac{w(I)}{|I|} \right)^{p/2} \left( \sum_{I' \subset I \in \tilde{B}_k} |t_I|^2 \frac{|I|}{w(I)} \right)^{p/2} \right\}^{1/p}
\]
\[
\leq \| \{ t_I \} \|_{c_w^p} \left\{ \sum_k \sum_{I \in B_k} w(I)^{1-2/p} \left( \sum_{I' \subset I \in \tilde{B}_k} |s_I|^2 \frac{w(I)}{|I|} \right)^{p/2} \right\}^{1/p}.
\]

Write
\[
\tilde{\Omega}_k = \{ x \in \mathbb{R} : M_w(\chi_{\tilde{\Omega}_k})(x) > 1/2 \},
\]
where \( M_w \) is the weighted Hardy-Littlewood maximal function defined by
\[
M_w f(x) = \sup_{\text{interval } Q \ni x} \frac{1}{w(\tilde{Q})} \int_Q |f(x)| w(x) dx.
\]
Then \( I \subset \tilde{\Omega}_k \) for any \( I \in B_k \). Since the \( \tilde{I} \)'s are mutually disjoint dyadic intervals, \( \sum_{I \in B_k} w(\tilde{I}) \leq w(\tilde{\Omega}_k) \). We then apply Hölder's inequality to obtain

\[
\left| \sum s_I \cdot t_I \right| \leq \| \{ t_I \} \|_c \left\{ \sum k w(\tilde{\Omega}_k)^{1-p/2} \left( \sum_{I \in B_k} |s_I|^2 w(I) \right)^{p/2} \right\}^{1/p}.
\]

We claim that

\[
\sum_{I \in B_k} |s_I|^2 \frac{w(I)}{|I|} \leq C2^{2k}w(\tilde{\Omega}_k).
\]

\( M_w \) is of weak type \((1,1)\) with respect to \( w(x)dx \), so \( w(\tilde{\Omega}_k) \leq Cw(\Omega_k) \) and the claim gives

\[
\left| \sum s_I \cdot t_I \right| \leq C\| \{ t_I \} \|_c \| S \|_{L_p^w} \leq C\| \{ t_I \} \|_c \| \{ s_I \} \|_{s_p^w}.
\]

To prove the claim, by the definitions of \( S(x) \) and \( B_k \), we have

\[
\int_{\tilde{\Omega}_k \setminus \tilde{\Omega}_{k+1}} S^2(x)w(x) \, dx \leq 2^{2k+2}w(\tilde{\Omega}_k)
\]

and

\[
\int_{\tilde{\Omega}_k \setminus \tilde{\Omega}_{k+1}} S^2(x)w(x) \, dx \geq \int_{\tilde{\Omega}_k \setminus \tilde{\Omega}_{k+1}} \sum_{I \in B_k} |s_I|^2 |I|^{-1} \chi_I(x)w(x) \, dx
\]

\[
= \sum_{I \in B_k} |s_I|^2 \frac{w(I \cap (\tilde{\Omega}_k \setminus \tilde{\Omega}_{k+1}))}{|I|}
\]

\[
\geq \frac{1}{2} \sum_{I \in B_k} |s_I|^2 \frac{w(I)}{|I|}.
\]

(b) Clearly, every \( \ell \in (s_{p}^w)' \) is of the form

\[
\ell(\{ s_I \}) = \sum_I s_I t_I, \quad \{ s_I \} \in s_{p}^w,
\]

where \( \{ t_I \} \) is a certain sequence. Fix a dyadic interval \( J \). Let \( S_J = \{ I \in \mathcal{D} : I \subset J \} \) and define a measure \( \nu \) on \( S_J \) by

\[
d\nu(I) = \frac{|I|}{w(J)^{p-1}} \quad \text{for} \ I \in S_J.
\]
By duality, 
\[
\left( \frac{1}{w(J)^{\frac{1}{p}}} \sum_{i \in J} |t_I|^2 \frac{|I|}{w(I)} \right)^{1/2} = \left\| \left\{ \frac{1}{w(I)^{\frac{1}{p}}} \frac{|I|}{w(I)^{\frac{1}{2}}} \right\} \right\|_{\ell^p(S_J, d\nu)}
\]
(2) 
\[
\leq \sup_{\{s_I\} \in \ell^2(S_J, d\nu)} \left\| s_I \frac{|I|}{w(J)^{\frac{1}{p}} \cdot w(I)^{\frac{1}{2}}} \right\|_{s^p_w}.
\]
For \( \{s_I\} \in \ell^2(S_J, d\nu) \), Hölder’s inequality yields 
\[
\left\| s_I \frac{|I|}{w(J)^{\frac{1}{p}} \cdot w(I)^{\frac{1}{2}}} \right\|_{s^p_w} = \frac{1}{w(J)^{\frac{1}{p}} \cdot w(I)^{\frac{1}{2}}} \left\{ \int_J \left( \sum_{I \in J} |s_I|^2 \frac{|I|}{w(I)^{\frac{1}{2}}} \chi_I(x) \right)^{p/2} w(x) \, dx \right\}^{1/p}
\]
\[
\leq \left\{ \frac{1}{w(J)^{\frac{1}{p}} \cdot w(I)^{\frac{1}{2}}} \int_J \sum_{I \in J} |s_I|^2 \frac{|I|}{w(I)^{\frac{1}{2}}} \chi_I(x) w(x) \, dx \right\}^{1/2}
\]
and hence
\[
\sup_{\{s_I\} \in \ell^2(S_J, d\nu)} \left\| s_I \frac{|I|}{w(J)^{\frac{1}{p}} \cdot w(I)^{\frac{1}{2}}} \right\|_{s^p_w} \leq 1.
\]
Taking the supremum over \( J \in \mathcal{D} \) in (2), we obtain \( \| \{t_I\} \|_{c^p_w} \leq \| \ell \|. \)

3. Proof of the main theorem

In this section we show that Theorem 1 follows as a consequence of Theorem 3. Let \( \psi \in \mathcal{R}^\alpha \), \( \alpha \geq 1 \), be an orthonormal wavelet. Define a map \( P \) from the family of complex sequences into \( \mathcal{S}' \) by
\[
P(\{s_I\}) = \sum_I s_I \psi_I.
\]
Define another map \( L \) from function space into the family of complex sequences by
\[
L(f) = \{ \langle f, \psi_I \rangle \}
\]
such that all \( \{f, \psi_I\} \)'s are well defined. Figure 1 illustrates the relationship among \( s^p_w \), \( c^p_w \), \( H^p_w \), and \( CMO^p_w \). Then \( P \circ L \mid _{L^2} \) is the identity on \( L^2 \). For \( 0 < p \leq 1 \) and

\[
\text{dual relation (by Theorem 3)}
\]

\[
\begin{array}{ccc}
P & L & P \\ \\
S^p_w & \downarrow & L \\
\text{dual relation (by Theorem 1)} & & \\
H^p_w & \downarrow & CMO^p_w
\end{array}
\]

**Figure 1.** Diagram for spaces and maps

\( w \in A^\infty \) with critical index \( q_w \), if \( \alpha \geq q_w / p \), then Theorem A yields
\[
(3) \quad \| \{L(f)\} \|_{s^p_w} \leq C \|f\|_{H^p_w} \quad \text{for } f \in H^p_w \cap L^2
\]
and
\[ \|P(\{s_I\})\|_{H^p_w} \leq C \|W_\varepsilon P(\{s_I\})\|_{L^p_w} = C \|\{s_I\}\|_{s^p_w} \quad \text{for } \{s_I\} \in s^p_w. \]

By the definitions of $c^p_w$ and $CMO^p_w$,
\[ \|\{L(g)\}\|_{c^p_w} = \|g\|_{CMO^p_w} \quad \text{for } g \in CMO^p_w \]
and
\[ \|P(\{t_I\})\|_{CMO^p_w} = \|\{t_I\}\|_{c^p_w} \quad \text{for } \{t_I\} \in c^p_w. \]

**Proof of Theorem 1.** For $g \in CMO^p_w$, define a linear functional $\hat{\ell}_g$ by
\[ \hat{\ell}_g(f) = \langle L(f), L(g) \rangle \quad \text{for } f \in H^p_w \cap L^2. \]

By (3), (5), and Theorem 3
\[ |\hat{\ell}_g(f)| \leq C \|L(f)\|_{c^p_w} \|L(g)\|_{c^p_w} \leq C \|f\|_{H^p_w} \|g\|_{CMO^p_w} \quad \text{for } f \in H^p_w \cap L^2. \]

Since $H^p_w \cap L^2$ is dense in $H^p_w$, the map $\hat{\ell}_g$ can be extended to a continuous linear functional $\ell_g$ on $H^p_w$ satisfying $\|\ell_g\| \leq C \|g\|_{CMO^p_w}$.

Conversely, let $\ell \in (H^p_w)'$ and set $\ell_1 = \ell \circ P$ on $s^p_w$. It follows from (4) that $\ell_1 \in (s^p_w)'$. By Theorem 3 there exists $\{t_I\} \in c^p_w$ such that
\[ \ell_1(\{s_I\}) = \sum_I s_I \cdot t_I \quad \text{for } \{s_I\} \in s^p_w, \]
and
\[ \|\{t_I\}\|_{c^p_w} \approx \|\ell_1\| \leq C \|\ell\|. \]

For $f \in H^p_w \cap L^2$, we have
\[ \ell(f) = \ell_1 \circ L(f) = \sum_I \langle f, \psi_I \rangle t_I = \langle L(f), L(g) \rangle, \]
where $g = \sum_I t_I \psi_I$. This shows that $\ell = \ell_g$, and (3) gives
\[ \|g\|_{CMO^p_w} = \|\{t_I\}\|_{c^p_w} \leq C \|\ell\|. \]

Hence, the proof is finished. \qed

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