

## STABILITY IN $L^1$ OF CIRCULAR VORTEX PATCHES

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ABSTRACT. The motion of incompressible and ideal fluids is studied in the plane. The stability in  $L^1$  of circular vortex patches is established among the class of all bounded vortex patches of equal strength.

For planar incompressible and ideal fluid flow, the theory of Yudovich [9] establishes global well-posedness of the initial value problem with initial vorticity in  $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ . Because vorticity is transported in 2d, it remains constant along particle trajectories. If  $\Phi_t$  is the flow map, then the vorticity is given by  $\omega(t, \Phi_t(y)) = \omega(0, y)$ , for all  $t > 0$  and  $y \in \mathbb{R}^2$ . When the initial vorticity is a patch of unit strength, represented by the indicator (characteristic) function  $I_{\Omega_0}$  of a bounded open set  $\Omega_0 \subset \mathbb{R}^2$ , the resulting vorticity is  $I_{\Omega_t}$ , with  $\Omega_t = \Phi_t(\Omega_0)$ . In the special case when  $\Omega_0$  is equal to a ball  $B$ , the patch is stationary,  $\Phi_t(B) = B$ , for all  $t > 0$ . Theorem 3, our main result, gives the stability in  $L^1(\mathbb{R}^2)$  of any circular patch within the class of all bounded vortex patches of equal strength. No restriction is placed on the  $L^1$  distance of the perturbation to the ball, and the flow region is not limited to a bounded domain, but rather is the entire space  $\mathbb{R}^2$ .

Wan and Pulvirenti [8] were the first to study stability of vortex patches in  $L^1$ . They considered the case where the flow was contained in a bounded region, although for the stability of circular patches they mention that this assumption can be removed. Their key estimate, **(J)**, shows that the total angular momenta of the patches can be used to control the  $L^1$  difference between an arbitrary patch and a circular patch of the same total mass. They allow the strengths of the patches to differ, in which case the two patches are assumed to be close in  $L^1$ . Our generalization of their inequality, given in Lemma 2, estimates the  $L^1$  distance of an arbitrary patch to a circular patch when both patches have equal strength. Stability in  $L^1$ , given in Theorem 3, follows immediately. Weaker stability results were given by Saffman [7] and Dritschel [4]. Dritschel controls the measure of the symmetric difference of two patches through a convenient integral, and this idea is incorporated into our argument in Lemma 1.

Stability in  $L^1$  does not imply that the boundaries of the two patches remain close in any metric. Indeed, numerical simulations give strong evidence of fingering

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and filamentation; see [1, 3]. Spreading of vorticity may also occur. The best upper bound for the growth rate of the patch diameter is  $\mathcal{O}(t \log t)^{1/4}$  given in [5]; see also [6]. Nevertheless, in spite of the fact that the patch geometry may be complicated, smoothness of smooth patch boundaries persists for all time; see [2].

For any bounded open set  $A \subset \mathbb{R}^2$ , denote its mass, momentum, and angular momentum by

$$|A| = \int_A dx, \quad M(A) = \int_A x \, dx, \quad \text{and} \quad i(A) = \int_A |x|^2 dx,$$

respectively. Our arguments depend heavily upon the fact that these three quantities are conserved in time when  $A = \Omega_t$  is a patch moving with the flow.

**Lemma 1.** *If  $A \subset \mathbb{R}^2$  is any bounded open set, then*

$$i(A) - \frac{|A|^2}{2\pi} - \frac{|M(A)|^2}{|A|} \geq 0.$$

*Equality holds if and only if the set  $A$  is a ball.*

*Proof.* For any ball  $B_r(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| < r\}$ , introduce the quantity

$$(1) \quad Q = Q(A; B_r(x_0)) = \int_{A \Delta B_r(x_0)} \left| |x - x_0|^2 - r^2 \right| dx,$$

in which  $A \Delta B_r(x_0) = (A \setminus B_r(x_0)) \cup (B_r(x_0) \setminus A)$  denotes the symmetric difference. Note that  $Q \geq 0$  and  $Q = 0$  if and only if  $A = B_r(x_0)$ .

The quantity  $Q$  can also be written as

$$Q = \int_A (|x - x_0|^2 - r^2) dx + \int_{B_r(x_0)} (r^2 - |x - x_0|^2) dx,$$

since the portions of these two integrals over the set  $A \cap B_r(x_0)$  cancel each other out.

Now, we can expand the first integral in  $Q$  and compute the second to obtain

$$Q = i(A) - 2x_0 \cdot M(A) + (|x_0|^2 - r^2)|A| + \frac{\pi}{2}r^4.$$

A rearrangement of terms gives

$$(2) \quad Q = i(A) - \frac{|A|^2}{2\pi} - \frac{|M(A)|^2}{|A|} + \frac{1}{2\pi} (\pi r^2 - |A|)^2 + |A| \left| x_0 - \frac{M(A)}{|A|} \right|^2.$$

This last expression is minimized by choosing  $B_r(x_0)$  with the same mass and center of mass as  $A$ :

$$|B_r(x_0)| = \pi r^2 = |A| \quad \text{and} \quad x_0 = \frac{M(A)}{|A|}.$$

With this choice, the lemma now follows. □

**Lemma 2.** *If  $B = B_r(0)$ , then for any bounded open set  $A$ ,*

$$\|I_A - I_B\|_{L^1}^2 \leq 4\pi Q(A; B),$$

*in which  $Q(A; B)$  is defined by (1). Equality holds if and only if*

$$(3) \quad A = B_a(0) \cup [B_b(0) \setminus B_r(0)],$$

*with  $a < r < b$  and  $r^2 - a^2 = b^2 - r^2$ .*

*Proof.* Using the identity (2) and then Lemma 1, we have for any bounded open set  $A'$ ,

$$(4) \quad (|A'| - |B|)^2 = (|A'| - \pi r^2)^2 \leq 2\pi Q(A'; B),$$

with equality if and only if  $A'$  is a ball centered at the origin.

Next, we note that

$$\begin{aligned} \|I_A - I_B\|_{L^1}^2 &= |A\Delta B|^2 \\ &= (|A \setminus B| + |B \setminus A|)^2 \\ &\leq 2|A \setminus B|^2 + 2|B \setminus A|^2 \\ &= 2(|A \cup B| - |B|)^2 + 2(|A \cap B| - |B|)^2, \end{aligned}$$

with equality if and only if  $|A \setminus B| = |B \setminus A|$ .

Application of (4) with  $A' = A \cup B$  and  $A' = A \cap B$  yields

$$\begin{aligned} 2(|A \cup B| - |B|)^2 + 2(|A \cap B| - |B|)^2 \\ \leq 4\pi [Q(A \cup B; B) + Q(A \cap B; B)] = 4\pi Q(A; B), \end{aligned}$$

with equality if and only if  $A \cup B$  and  $A \cap B$  are balls centered at the origin. This establishes the desired inequality.

The argument also shows that equality holds if and only if  $A \cup B = B_b(0)$ ,  $A \cap B = B_a(0)$ , with  $a < r < b$ , and

$$|B_b(0) \setminus B| = |A \setminus B| = |B \setminus A| = |B \setminus B_a(0)|,$$

which gives (3). □

**Theorem 3.** *Let  $B = B_r(0)$ . Then for any bounded open set  $\Omega_0 \subset \mathbb{R}^2$ , we have that*

$$\|I_{\Omega_t} - I_B\|_{L^1}^2 \leq 4\pi \sup_{\Omega_0 \Delta B} ||x|^2 - r^2| \|I_{\Omega_0} - I_B\|_{L^1},$$

for all  $t > 0$ .

*Proof.* The identity (2) shows that the quantity  $Q(\Omega_t; B)$  depends only on conserved quantities, and it is therefore also conserved. In other words, we have  $Q(\Omega_t; B) = Q(\Omega_0; B)$ , for all  $t > 0$ . Thus, the result follows from Lemma 2 and the fact that

$$Q(\Omega_0; B) \leq \sup_{\Omega_0 \Delta B} ||x|^2 - r^2| |\Omega_0 \Delta B| = \sup_{\Omega_0 \Delta B} ||x|^2 - r^2| \|I_{\Omega_0} - I_B\|_{L^1}.$$

□

## REFERENCES

- [1] Buttkle, T.F. *A fast adaptive vortex method for patches of constant vorticity in two dimensions.* J. Comput. Phys. **89** (1990), no. 1, 161–186. MR1063151 (91g:76072)
- [2] Chemin, J.-Y. *Sur le mouvement des particules d'un fluide parfait incompressible bidimensionnel.* Invent. Math. **103** (1991), 599–629. MR1091620 (91m:35187)
- [3] Deem, G.S. and N.J. Zabusky. *Vortex waves: stationary “V states”, interactions, recurrence, and breaking.* Phys. Rev. Lett. **41** (1978), no. 13, 859–862.
- [4] Dritschel, D.G. *Nonlinear stability bounds for inviscid, two-dimensional, parallel or circular flows with monotonic vorticity, and the analogous three-dimensional quasi-geostrophic flows.* J. Fluid Mech. **191** (1988), 575–581.
- [5] Iftimie, D., T.C. Sideris, and P. Gamblin. *On the evolution of compactly supported planar vorticity.* Comm. Partial Differential Equations **24** (1999), no. 9–10, 1709–1730. MR1708106 (2000d:76030)

- [6] Marchioro, C. *Bounds on the growth of the support of a vortex patch*. Comm. Math. Phys. **164** (1994), 507–524. MR1291243 (95f:76012)
- [7] Saffman, P.G. *Vortex dynamics*. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York, 1992. MR1217252 (94c:76015)
- [8] Wan, Y.H. and M. Pulvirenti. *Nonlinear stability of circular vortex patches*. Comm. Math. Phys. **99** (1985), no. 3, 435–450. MR795112 (86i:76028)
- [9] Yudovič, V.I. *Non-stationary flow of an ideal incompressible liquid*. Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), no. 6, 1032–1066. MR0158189 (28:1415)

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