ON SHARP EMBEDDINGS OF BESOV AND TRIEBEL-LIZORKIN SPACES IN THE SUBCRITICAL CASE

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ABSTRACT. We discuss the growth envelopes of Fourier-analytically defined Besov and Triebel-Lizorkin spaces $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$ in the limiting case $s = \sigma_p := n \max\left(\frac{1}{p} - 1, 0\right)$. These results may also be reformulated as optimal embeddings into the scale of Lorentz spaces $L_{p,q}(\mathbb{R}^n)$. We close several open problems outlined already in [H. Triebel, The structure of functions, Birkhäuser, Basel, 2001] and explicitly stated in [D. D. Haroske, Envelopes and sharp embeddings of function spaces, Chapman & Hall/CRC, Boca Raton, FL, 2007].

1. INTRODUCTION AND MAIN RESULTS

In this paper we prove sharp embedding theorems for Besov and Triebel-Lizorkin spaces $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$ in some limiting cases of the range guaranteeing that these spaces consist of locally integrable functions. As proven in [12, Theorem 3.3.2],

\begin{equation}
B_{p,q}^s(\mathbb{R}^n) \hookrightarrow L_{1}^{\text{loc}}(\mathbb{R}^n) \iff \begin{cases}
\text{either } s > \sigma_p := n \max\left(\frac{1}{p} - 1, 0\right), \\
\text{or } s = \sigma_p, 1 < p \leq \infty, 0 < q \leq \min(p, 2), \\
\text{or } s = \sigma_p, 0 < p \leq 1, 0 < q \leq 1
\end{cases}
\end{equation}

and

\begin{equation}
F_{p,q}^s(\mathbb{R}^n) \hookrightarrow L_{1}^{\text{loc}}(\mathbb{R}^n) \iff \begin{cases}
\text{either } s > \sigma_p, \\
\text{or } s = \sigma_p, 1 \leq p < \infty, 0 < q \leq 2, \\
\text{or } s = \sigma_p, 0 < p < 1, 0 < q \leq \infty
\end{cases}
\end{equation}

The embeddings can be measured quantitatively by the growth envelope function of $X$ as defined by D. D. Haroske and H. Triebel (see [5], [6], [16] and the references given there) by

$$
\mathcal{E}_G^X(t) := \sup_{||f||_X \leq 1} f^*(t), \quad 0 < t < 1,
$$

where $f^*$ denotes the non-increasing rearrangement of $f$.

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In the case where $\mathcal{E}_G(t) \approx t^{-\alpha}$ for $0 < t < 1$ and some $\alpha > 0$ the growth envelope index $u_X$ is given as the infimum of all numbers $v$, $0 < v \leq \infty$, such that

$$
(3) \quad \left( \int_0^t \left[ \frac{f^*(t)}{\mathcal{E}_G(t)} \right]^v \frac{dt}{t} \right)^{1/v} \leq c ||f||
$$

(with the usual modification for $v = \infty$) holds for some $c > 0$ and all $f \in X$. The pair $\mathcal{E}_G(X) = (\mathcal{E}_G^X, u_X)$ is called the growth envelope for the function space $X$.

In the case $s < s$, the growth envelopes of $A^s_{p,q}(\mathbb{R}^n)$ are known; cf. [16 Theorem 15.2] and [6 Theorem 8.1]. If $s = \sigma_p$ and [11 or 24] is fulfilled in the $B$ or $F$ case, respectively, then the growth function is given by $t^{-\min(s,n)}$, but the known information about the growth index $u$ is not complete; cf. [16 Remarks 12.5, 15.1] and [6 Props. 8.12, 8.14 and Remark 8.15].

The growth index of $B^0_{p,q}(\mathbb{R}^n)$ satisfies

$$
(4) \quad \begin{cases}
q \leq u \leq p & \text{if } 1 \leq p < \infty \text{ and } 0 < q \leq \min(p,2), \\
q \leq u \leq 1 & \text{if } 0 < p < 1 \text{ and } 0 < q \leq 1.
\end{cases}
$$

The growth index of $F^\sigma_{p,q}(\mathbb{R}^n)$ satisfies $p \leq u \leq 1$ if $0 < p < 1$ and $0 < q \leq \infty$ and is equal to $p$ if $1 \leq p < \infty$ and $0 < q \leq 2$.

The growth envelopes of $B^0_{p,q}$ defined on the torus $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ with $1 \leq q \leq 2$ were identified recently by Seeger and Trebels in [10] and are equivalent to $|t|^{1/q'}$ for $0 < t \leq 1/2$. We fill the remaining gaps for the range $p < \infty$.

**Theorem 1.1.** (i) Let $1 \leq p < \infty$ and $0 < q \leq \min(p,2)$. Then $$\mathcal{E}_G(B^0_{p,q}) = (t^{-\frac{1}{p}}, p).$$

(ii) Let $0 < p < 1$ and $0 < q \leq 1$. Then $$\mathcal{E}_G(B^\sigma_{p,q}) = (t, q).$$

(iii) Let $0 < p < 1$ and $0 < q \leq \infty$. Then $$\mathcal{E}_G(F^\sigma_{p,q}) = (t^{-1}, p).$$

These results are closely related to optimal embeddings into the scale of Lorentz spaces. In this context, we prove the following.

**Theorem 1.2.** (i) Let $1 \leq p < \infty$ and $0 < q \leq \min(p, 2)$. Then $$B^0_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n).$$

(ii) Let $0 < p < 1$ and $0 < q \leq 1$. Then

$$B^\sigma_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,q}(\mathbb{R}^n).$$

(iii) Let $0 < p < 1$ and $0 < q \leq \infty$. Then

$$F^\sigma_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,p}(\mathbb{R}^n)$$

and all these embeddings are optimal with respect to the second fine parameter of the scale of the Lorentz spaces.

**Remark 1.3.** (i) Let us observe that [5] improves [12 Theorem 3.2.1] and [11 Theorem 2.2.3], where the embedding $B^\sigma_{p,q}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n)$ is proved for all $0 < p < 1$ and $0 < q \leq 1$. 

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spaces do not coincide for \( s = 0 \), an effect observed in detail recently by Schneider [9].

We denote the Lebesgue and Lorentz spaces by \( L_p(\mathbb{R}^n) \) and \( L_{p,q}(\mathbb{R}^n) \), respectively. The reader may consult [13, Chapter 5, Section 3] or [1, Chapter 4, Section 4]. We shall use the following well-known property of Lorentz spaces \( L_{1,q} \). Its proof follows immediately from Hardy’s lemma (cf. [1, Chapter 2, Proposition 3.6]).

**Lemma 1.4.** Let \( 0 < q < 1 \). Then the \( \| \cdot |L_{1,q}(\mathbb{R}^n)\| q \) is the \( q \)-norm; it means that

\[
\| f_1 + f_2 |L_{1,q}(\mathbb{R}^n)\| q \leq \| f_1 |L_{1,q}(\mathbb{R}^n)\| q + \| f_2 |L_{1,q}(\mathbb{R}^n)\| q
\]

holds for all \( f_1, f_2 \in L_{1,q}(\mathbb{R}^n) \).

We work with Fourier analytically defined Besov and Triebel-Lizorkin spaces \( B^s_{p,q}(\mathbb{R}^n) \) and \( F^s_{p,q}(\mathbb{R}^n) \) as studied for example in [8], [14], [15] and [17]. We shall also use the sequence spaces \( b^s_{p,q} \) associated to \( B^s_{p,q}(\mathbb{R}^n) \) in a way described in [17, Chapters 2 and 3]. This approach goes back to [3] and [4].

All the unimportant constants are denoted by the letter \( c \), whose meaning may differ from one occurrence to another. If \( \{a_n\}_{n=1}^\infty \) and \( \{b_n\}_{n=1}^\infty \) are two sequences of positive real numbers, we write \( a_n \lesssim b_n \) if, and only if, there is a positive real number \( c > 0 \) such that \( a_n \leq c b_n, n \in \mathbb{N} \). Furthermore, \( a_n \approx b_n \) means that \( a_n \lesssim b_n \) and simultaneously \( b_n \lesssim a_n \).

2. Proofs of the main results

2.1. Proof of Theorem 1.1 (i). In view of (4), it is enough to prove that for \( 1 \leq p < \infty \) and \( 0 < q \leq \min(p,2) \) the index \( u \) associated to \( B^0_{p,q}(\mathbb{R}^n) \) is greater than or equal to \( p \).

We assume to the contrary that (3) is fulfilled for some \( 0 < v < p, \epsilon > 0, c > 0 \) and all \( f \in B^0_{p,q}(\mathbb{R}^n) \). Let \( \psi \) be a non-vanishing \( C^\infty \) function in \( \mathbb{R}^n \) supported in \([0,1]^n\) with \( \int_{\mathbb{R}^n} \psi(x)dx = 0 \).

Let \( J \in \mathbb{N} \) be such that \( 2^{-Jn} < \epsilon \) and consider the function

\[
f_j = \sum_{m=1}^{2^{(j-J)n}} \lambda_{jm} \psi(2^j(x-(m,0,\ldots,0))), \quad j > J,
\]

where

\[
\lambda_{jm} = \frac{1}{m^p \log^\frac{2}{p}(m+1)}, \quad m = 1,\ldots,2^{(j-J)n}.
\]

Then (3) represents an atomic decomposition of \( f \) in the space \( B^0_{p,q}(\mathbb{R}^n) \) according to [17, Chapter 1.5], and we obtain (recall that \( v < p \))

\[
\| f_j B^0_{p,q}(\mathbb{R}^n)\| \lesssim 2^{-\frac{j}{p}} \left( \sum_{m=1}^{2^{(j-J)n}} \lambda_{jm}^p \right)^{1/p} \lesssim 2^{-\frac{j}{p}} \left( \sum_{m=1}^{\infty} m^{-1} \log(m+1) \right)^{-\frac{1}{p}} \lesssim 2^{-\frac{j}{p}}.
\]
On the other hand,
\[
\left( \int_0^\epsilon \left[ f_j^*(t) \right]^v \frac{dt}{t} \right)^{1/v} \geq \left( \int_0^{2^{-j_n}} f_j^*(t)^{v/p} \frac{dt}{t} \right)^{1/v} \\
\geq \left( \sum_{m=1}^{2^{(j-j_n)}} \lambda_j^{v_m} \int_{2^{-j_n}(m-1)}^{2^{-j_n}m} t^{v/p-1} \frac{dt}{t} \right)^{1/v} \geq \left( \sum_{m=1}^{2^{(j-j_n)}} \lambda_j^{v_m} 2^{-jn \nu/v} m^{v/p-1} \right)^{1/v} \\
= 2^{-j_n} \left( \sum_{m=1}^{\infty} \frac{1}{m \log(m+1)} \right)^{1/v}.
\]

As the last series is divergent for \( j \to \infty \), this is in contradiction with (7), and (5) cannot hold for all \( f_j, j > J \).

Remark 2.1. Observe that Theorem 1.2 (i) is a direct consequence of Theorem 1.1 (i). The embeddings \( B_{1,q}^0(\mathbb{R}^n) \hookrightarrow B_{1,1}^0(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n) \) if \( p = 1 \) and \( B_{p,q}^0(\mathbb{R}^n) \hookrightarrow F_{p,q}^0(\mathbb{R}^n) = L_p(\mathbb{R}^n) \) if \( 1 < p < \infty \) show that \( B_{p,q}^0(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \). Theorem 1.1 (i) implies that if \( B_{p,q}^0(\mathbb{R}^n) \hookrightarrow L_{p,s}(\mathbb{R}^n) \) for some \( 0 < s < \infty \), then \( p \leq v \). This proves the optimality of Theorem 1.2 (i) in the frame of the scale of Lorentz spaces.

2.2. Proof of Theorem 1.1 (ii) and Theorem 1.2 (ii). Let \( 0 < p < 1, 0 < q \leq 1 \) and \( s = \sigma_p = \left( \frac{1}{p} - 1 \right) \). We first prove Theorem 1.2 (ii); i.e. we show that

\[
B_{p,q}^\mathbb{R}^n \hookrightarrow L_{1,q}(\mathbb{R}^n),
\]

or, equivalently, that

\[
\left( \int_0^\infty \left[ tf^*(t) \right]^q \frac{dt}{t} \right)^{1/q} \leq c \| f \|_{B_{p,q}^\mathbb{R}^n}, \quad f \in B_{p,q}^\mathbb{R}^n.
\]

Let

\[
f = \sum_{j=0}^\infty f_j = \sum_{j=0}^\infty \sum_{m \in \mathbb{Z}^n} \lambda_j a_{jm}
\]

be the optimal atomic decomposition of an \( f \in B_{p,q}^\mathbb{R}^n \), again in the sense of [17, Chapter 1.5]. Then

\[
\| f \|_{B_{p,q}^\mathbb{R}^n} \approx \left( \sum_{j=0}^\infty 2^{-jn} \left( \sum_{m \in \mathbb{Z}^n} |\lambda_j|^p \right)^{q/p} \right)^{1/q}
\]

and by Lemma 1.4

\[
\| f \|_{L_{1,q}(\mathbb{R}^n)} = \left( \sum_{j=0}^\infty \| f_j \|_{L_{1,q}(\mathbb{R}^n)} \right)^{1/q} \leq \left( \sum_{j=0}^\infty \| f_j \|_{L_{1,q}(\mathbb{R}^n)} \right)^{1/q}.
\]

We shall need only one property of the atoms \( a_{jm} \), namely, that their support is contained in the cube \( Q_{jm} \), a cube centred at the point \( 2^{-j}m \) with sides parallel to the coordinate axes and side length \( \alpha 2^{-j} \), where \( \alpha > 1 \) is fixed and independent.
of $f$. We denote by $\tilde{\chi}_{jm}(x)$ the characteristic functions of $\tilde{Q}_{jm}$ and by $\chi_{jl}$ the characteristic function of the interval $(l2^{-jn}, (l+1)2^{-jn})$. Hence

$$f_j(x) \leq c \sum_{m \in \mathbb{Z}} |\lambda_{jm}| \tilde{\chi}_{jm}(x), \quad x \in \mathbb{R}^n$$

and

$$||f_j||_{L^1,q}(\mathbb{R}^n) \leq \left( \int_0^\infty \sum_{l=0}^\infty (\lambda_j t)^q \chi_{jl}(t) t^{q-1} dt \right)^{1/q} \leq \left( \sum_{l=0}^\infty (\lambda_j)^q \int_{2^{-jn}}^{2^{-jn+1}} t^{q-1} dt \right)^{1/q} \leq 2^{-jn} \left( \sum_{l=0}^\infty (\lambda_j)^q \right)^{1/q} \leq 2^{-jn} ||\lambda_j||_{\ell_p}.$$

(10)

The last inequality follows by $(l+1)^{q-1} \leq 1$ and $\ell_p \hookrightarrow \ell_q$ if $p \leq q$. If $p > q$, the same follows by Hölder’s inequality with respect to the indices $\alpha = \frac{p}{q}$ and $\alpha' = \frac{p}{p-q}$:

$$\left( \sum_{l=0}^\infty (\lambda_j)^q (l+1)^{q-1} \right)^{1/q} \leq \left( \sum_{l=0}^\infty (\lambda_j)^q \right)^{1/q} \leq c ||\lambda_j||_{\ell_p}.$$ 

Here, we used that for $0 < q < p < 1$ the exponent $\frac{(q-1)p}{p-q} = -1 + \frac{(p-1)q}{p-q}$ is strictly smaller than $-1$.

The proof now follows by (8), (9) and (10):

$$||f||_{L^1,q}(\mathbb{R}^n) \leq \left( \sum_{j=0}^\infty ||f_j||_{L^1,q}(\mathbb{R}^n) ||q \right)^{1/q} \leq c \left( \sum_{j=0}^\infty 2^{-jnq} ||\lambda_j||_{\ell_p} \right)^{1/q} \leq c ||f||_{B^\alpha_{p,q}(\mathbb{R}^n)}.$$

Remark 2.2. We actually proved that (3) holds for $X = B^\alpha_{p,q-n} \cap \mathbb{R}^n$, $v = q$ and $\epsilon = \infty$. This, together with (4), implies immediately Theorem 1.1 (ii).

2.3. Proof of Theorem 1.1 (iii) and Theorem 1.2 (iii). Let $0 < p < 1$ and $0 < q \leq \infty$. By the Jawerth embedding (cf. [19] or [18]) and Theorem 1.1 (ii) we get for any $0 < p < \tilde{p} < 1$,

$$F^\alpha_{p,q}(\mathbb{R}^n) \hookrightarrow B^\alpha_{p,p}(\mathbb{R}^n) \hookrightarrow L^1, \mathbb{R}^n.$$
References


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