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GENERALIZATIONS OF RIGID ANALYTIC PICARD THEOREMS

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Abstract. Berkovich's Picard theorem states that there are no non-constant analytic maps from the affine line to the complement of two points on a non-singular projective curve. The purpose of this article is to find generalizations of this result in higher dimensional varieties.

1. Introduction

Let K be an algebraically closed field complete with respect to a non-Archimedean absolute value of arbitrary characteristic. Berkovich's Picard theorem states that there are no nonconstant analytic maps from K to the complement of two points on a nonsingular projective curve. Some results in this direction for higher dimensional varieties were obtained by Ru [11], An [1], An-Wang-Wong [3]. All of these results were recently generalized by An-Cherry-Wang [2], who proved the following theorem.

Theorem (An-Cherry-Wang)**.** *Let X be a non-singular projective variety over K. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible, effective, ample divisors in general position over X. Let r be the rank of the subgroup of the Neron-Severi group $NS(X)$ generated by the classes $\{[D_i]\}_{i=1}^{\ell}$ in $NS(X)$. Let f be an analytic map from K to the complement of the union of the divisors D_i, $1 \leq i \leq \ell$, on X. Then the image of f is contained in an algebraic subvariety Y of X such that*

$$\dim Y \leq \max\{\dim X + r - \ell, \frac{r}{\ell} \cdot \dim X\}. \tag{1}$$

In particular, f is constant if

$$\ell \geq \max\{\dim X + r, r \cdot \dim X + 1\}. \tag{2}$$

Here, a collection of irreducible divisors D_i in a projective variety X of dimension n are said to be *in general position* if for each $1 \leq k \leq n+1$ and each choice of indices $i_1 < \cdots < i_k$, each irreducible component of

$$D_{i_1} \cap \cdots \cap D_{i_k}$$

has codimension k in X, so in particular is empty when $k = n+1$. When $r = 1$, the inequalities (1) and (2) are both optimal. However, it does not seem to be the case when $r > 1$. In this paper, we will improve the inequality (2) by showing that f is constant if $\ell \geq n+1$. The following is the statement of our main results.

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Main Theorem. *Let X be a normal projective variety over K. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible and effective divisors on X. Let f be an analytic map from K to the complement of the union of the divisors D_i, $1 \le i \le \ell$, on X.*

(A) *If D_i is big for all i and the intersection $\bigcap_{i=1}^{\ell} D_i$ is an empty set, then f is algebraically degenerate, i.e. the image of f is contained in a subvariety of lower dimension in X.*
(B) *Suppose the D_i are ample and in general position over X. Then f is constant if $\ell \ge \dim X + 1$.*

Remark. If the D_i are in general position and $\ell \ge \dim X + 1$, then $\bigcap_{i=1}^{\ell} D_i$ is an empty set.

This paper was inspired by some recent developments such as [6], [8], and [9] in the study of complex analytic curves and integral points in projective varieties. The techniques in these papers are based on the new proof of Siegel's theorem given by Corvaja and Zannier in [5] where they provided a very simple and elegant proof by using Schmidt's subspace theorem. More recently, they used this technique to study integral points on surfaces (see [6]), and Liu and Ru [9] have translated this approach to study complex analytic curves on surfaces. This technique was also generalized by Levin [8] to get results on integral points on higher dimensional varieties, generalizing Siegel's theorem, and, analogously, he also obtained results on complex analytic curves in higher dimensional complex varieties, generalizing Picard's theorem. Although he did not get optimal results, he made some important new conjectures. Our results in this paper provide answers to some non-Archimedean analogs of these conjectures.

Although our theorem is influenced by the above-mentioned papers, the proof is not the same. In the previous cases, one has to filter some chosen vector space of rational functions in order to choose several sets of "good" bases which allows one to construct linear forms with zeros of high multiplicity and apply the Schmidt subspace theorem or Nevanlinna's second main theorem. For the non-Archimedean case, the second main theorem without ramification term is equivalent to the first main theorem (cf. [11]). In other words, the second main theorem in the non-Archimedean case is not a strong tool as in the complex case. Therefore, we will use classical results on the growth modulus of non-Archimedean analytic functions instead of the second main theorem. By doing so, we only need to pick up some "good" rational functions with zeros along some given divisors, and hence there is no need to filter the chosen vector space as in [6], [9], and [8]. This not only allows us to provide a simple proof, but also enables us to obtain optimal bounds.

Although our main theorem answers some of Levin's conjectures in the non-Archimedean setting, we still cannot estimate the degeneration dimension of the map f. In view of the Theorem of An-Cherry-Wang stated above, we make the following conjecture.

Conjecture. *Let X be a non-singular projective variety and $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible effective, ample divisors of X which are in general position. Let f be an analytic map from K to the complement of the union of the divisors D_i, $1 \le i \le \ell$, on X. If $\ell \le \dim X$, then the image of f is contained in an algebraic subvariety Y of X such that* $\dim Y \le \dim X + 1 - \ell$.

2. Preliminaries

We recall the definition of big divisors and some basic properties (cf. [7]).

Definition. Let X be a projective variety of dimension n. A Cartier divisor D is said to be *big* if $h^0(X, \mathcal{O}(kD)) > ck^n$ for some $c > 0$ and some sufficiently large integer k.

Proposition 1. *Let X be a projective variety of dimension n and B a Cartier divisor on X. Then $h^0(X, \mathcal{O}(kB)) \leq O(k^n)$ for every $k > 0$.*

Proof. Let H be a very ample divisor on X such that $H - B$ is linearly equivalent to an effective divisor. Then $h^0(X, \mathcal{O}(kB)) \leq h^0(X, \mathcal{O}(kH))$, and the latter is given by its Hilbert polynomial. □

We now deduce the following:

Lemma 2. *Let B be a big divisor and D be an effective divisor on a normal projective variety X. Let $n = \dim X$. Then there exists a positive constant c such that $h^0(NB - D) \geq cN^n$ for all sufficiently large integer N.*

Proof. From the exact sequence of sheaves

$$0 \to \mathcal{O}_X(NB - D) \to \mathcal{O}_X(NB) \to \mathcal{O}_D(NB) \to 0,$$

we have

$$0 \to H^0(X, \mathcal{O}_X(NB - D)) \to H^0(X, \mathcal{O}_X(NB)) \to H^0(D, \mathcal{O}_D(NB)). \tag{3}$$

Since $\dim D = n-1$, it follows from Proposition 1 that $h^0(D, \mathcal{O}_D(NB)) \leq O(N^{n-1})$. On the other hand, as B is big, $h^0(X, \mathcal{O}_X(NB)) > CN^n$ for some positive constant C and all sufficiently large N. The assertion then follows easily from the exact sequence (3). □

3. Proof of the Main Theorem

We will use the following simple estimate to replace Nevanlinna's second main theorem in the complex analytic setting as in [8] and [9].

Lemma 3. *Let f_1, f_2,...,f_n be non-trivial analytic functions on K. For a fixed positive number r_0, there exists a constant $C > 0$ such that*

$$\sup_{|z|=r} \min_{1 \leq i \leq n} \{|f_i(z)|\} \geq C > 0$$

for all $r \geq r_0$ except a discrete subset of $[r_0, \infty)$.

Proof. Since the f_i are non-trivial, each of them has only finitely many zeros in the disk $|z| \leq r_0$. Therefore, there exists z_0 in the disc $|z| \leq r_0$ such that $f_i(z_0) \neq 0$, for $1 \leq i \leq n$. Then

$$C := \min\{|f_1(z_0)|, ..., |f_n(z_0)|\} > 0.$$

A classical result on the growth modulus of non-Archimedean analytic functions (cf. [10], Chapter 6.1.4) shows that for all $r \geq 0$ except a discrete subset of $[0, \infty)$,

$$\sup_{|z|=r} |f_i(z)| = |f_i(w)|, \qquad \text{for all} \quad |w| = r. \tag{4}$$

Since there are only a finite number of f_i's,

$$\sup_{|z|=r} |f_i(z)| = |f_i(w)|, \qquad \text{for all} \quad |w| = r \text{ and } 1 \le i \le n$$

for all $r \ge r_0$ except a discrete subset of $[r_0, \infty)$. For such r, we can easily deduce that

$$\begin{aligned}\sup_{|z|=r} \min_{1\le i\le n} \{|f_i(z)|\} &= \min_{1\le i\le n} \{\sup_{|z|=r} |f_i(z)|\} \\ &= \min_{1\le i\le n} \{\sup_{|z|\le r} |f_i(z)|\} \ge \min_{1\le i\le n} \{|f_i(z_0)|\} = C > 0. \qquad \square\end{aligned}$$

Proof of the Main Theorem. We will first prove the assertion (A). Let $D = D_1 + D_2 + \cdots + D_\ell$. Fix an integer N sufficiently large, which will be determined later, and consider the following vector space:

$$V_N = \{\phi \in K(X) \mid \operatorname{div}(\phi) + ND \ge 0\},$$

where $K(X)$ is the rational function field of X over K. Let M be the dimension of V_N. For each $1 \le i \le \ell$, the vector space

$$\begin{aligned}L_i &:= \{\phi \in K(X) \mid \operatorname{div}(\phi) + (N+1)D_i - D \ge 0\} \\ &= \{\phi \in K(X) \mid \operatorname{div}(\phi) + ND_i - D_1 - \cdots - D_{i-1} - D_{i+1} - \cdots - D_\ell \ge 0\}\end{aligned}$$

is a subspace of V_N. By Lemma 2, we may choose a sufficiently large integer N such that L_i is non-trivial for each $1 \le i \le \ell$. Then we may take a non-trivial function $\psi_i \in L_i$ for each i. Let $\phi_1, ..., \phi_M$ be a basis of V_N and let

$$\|\phi(P)\| := \max_{1\le j\le M} \{|\phi_j(P)|\}$$

for $P \in X$. We will use the following assertion to complete the proof, and its proof is given at the end.

Claim. There exists a positive constant c such that for all $P \in X$,

$$\min_{1\le i\le \ell} |\psi_i(P)| \le c \|\phi(P)\|^{-\frac{1}{N}}.$$

Let $f : K \to X \setminus \bigcup_{i=1}^{\ell} D_i$ be an analytic map. If $\psi_i(f(z)) = 0$ for all $z \in K$, then the image of f is contained in the support of the zero divisors of ψ_i and hence f is algebraically degenerate. Therefore, we may assume that $\psi_i \circ f$ is not identically zero for each $1 \le i \le \ell$. We can also assume $\phi_i \circ f$ is not constant for at least one i, otherwise f must be algebraically degenerate. It then follows from the claim that

$$\min_{1\le i\le \ell} |\psi_i(f(z))| \le c\,\|\phi(f(z))\|^{-\frac{1}{N}} \tag{5}$$

for some positive constant c independent of z. By (4), for all $r \ge 0$ except a discrete subset of $[0, \infty)$,

$$\sup_{|z|=r} |\psi_i(f(z))| = |\psi_i(f(w))| \quad \text{and} \quad \sup_{|z|=r} |\phi_j(f(z))| = |\phi_j(f(w))|,$$

for all $|w| = r$ and $1 \le i \le \ell$, $1 \le j \le M$. Together with (5), for such r we have

$$\begin{aligned}\sup_{|z|=r} \min_{1\le i\le \ell} |\psi_i(f(z))| &= \min_{1\le i\le \ell} |\psi_i(f(w))| \qquad \text{for } |w| = r \\ &\le c\,\|\phi(f(w))\|^{-\frac{1}{N}} = c\,(\max_{1\le j\le M}\{|\phi_j(f(w))|\})^{-\frac{1}{N}} \\ &= c\,(\max_{1\le j\le M}\{\sup_{|z|=r} |\phi_j(f(z))|\})^{-\frac{1}{N}}. \end{aligned} \tag{6}$$

Since ϕ_i is in V_N, it has only poles along the support of D. Hence $\phi_i \circ f$ is an analytic function. Since $\phi_i \circ f$ is not constant for at least one i, the right hand side of (6) approaches zero as r tends to infinity. However, by Lemma 3 the left hand side of (6) is bounded away from zero by a positive constant independent of r as $r > 1$, which leads to a contradiction to conclude that f must be algebraically degenerate.

We will now prove the claim. Since the intersection of D_1,...,D_ℓ is empty, every point in the support of D is in the intersection of at most $\ell - 1$ distinct D_i's. We can take a finite affinoid covering $\mathcal{U}$ of X such that for each affinoid subdomain U in $\mathcal{U}$, $U \cap D$ is either empty or U intersects D at only D_{i_1},...,D_{i_k}, $k \le \ell - 1$, and moreover there exist regular functions t_{i_j}, $1 \le j \le k$, on U such that $D_{i_j} \cap U = \{t_{i_j} = 0\}$ and $\bigcap_{j=1}^{k} D_{i_j} \cap U = \{t_{i_1} = t_{i_2} = \cdots = t_{i_k} = 0\}$. We note that this can be done by first taking a finite Zariski open covering with required properties and then reducing to the affinoind subdomains by standard construction. Since $\psi_i \in L_i$, it has only a pole of order at most N along D_i and has zeros along D_j for all $j \ne i$. Therefore, each ψ_i is regular on those $U \in \mathcal{U}$ not intersecting the support of D. As a regular function on an affinoid subdomain is bounded, we can find a constant c_1 such that

$$|\psi_i(P)| \le c_1 \tag{7}$$

for all $P \in U$ and $1 \le i \le \ell$. It now remains to consider those $U \in \mathcal{U}$ intersecting at least one of the D_i. For simplicity of notation, we suppose that U only intersects $D_1, D_2, \ldots, D_k$ non-trivially. Then $\cap_{i=1}^{k} D_i \cap U = \{t_1 = t_2 = \cdots = t_k = 0\}$ and $k \le \ell - 1$. Therefore, ψ_{k+1} is regular on U and has zeros along D_i for $1 \le i \le k$. Thus, we can find a rational function ρ_{k+1} on X which is regular on U such that

$$\psi_{k+1} = t_1 \cdots t_k \cdot \rho_{k+1}.$$

Since ρ_{k+1} is bounded above on the affinoid subdomain U, there exists a constant c_2 such that

$$|\psi_{k+1}(P)| \le c_2\,|t_1(P)| \cdots |t_k(P)| \tag{8}$$

for all $P \in U$. On the other hand, since $\mathrm{ord}_{D_i}\phi_j \ge -N$, similar arguments can show that there exists a positive constant c_3 such that for all $P \in U$,

$$\|\phi(P)\| = \max_{1\le j\le M} |\phi_j(P)| \le c_3\,|t_1(P)|^{-N} \cdots |t_k(P)|^{-N}. \tag{9}$$

Equations (8) and (9) imply that there exists a positive constant c_U such that for all $P \in U$,

$$\min_{1\le i\le \ell} |\psi_i(P)| \le |\psi_{k+1}(P)| \le c_U\,\|\phi(P)\|^{-\frac{1}{N}}. \tag{10}$$

As $\mathcal{U}$ is a finite covering, the claim is a consequence of (7) and (10).

We now prove part (B). Suppose now that the D_i are ample and in general position. Then it follows from the previous part of the theorem that the analytic map f is algebraically degenerate if $\ell \geq \dim X + 1$. Suppose that f is not constant. The image of f is then contained in a subvariety Y of X with $\dim Y \geq 1$. We may assume that the map $f : K \to Y$ is algebraically non-degenerate. This analytic map $f : K \to Y$ takes no value on each of the $D_i \cap Y$. Let ℓ_0 be the cardinality of the set

$$\{D_i \cap Y : D_i \not\supset Y\},$$

and note that $D_i \cap Y \neq \emptyset$ for all i because the D_i are assumed ample. Without loss of generality, we may assume that $D_1 \cap Y$, $D_2 \cap Y, \ldots, D_{l_0} \cap Y$ are distinct. Then

$$Y \cap \big(\bigcap_{j=1}^{l_0} D_j \big) = Y \cap \big(\bigcap_{j=1}^{l} D_j \big).$$

The right hand side is an empty set because $l \geq \dim X + 1$ and the D_i are ample and in general position. On the other hand, the dimension on the left hand side is at least $\dim Y - \ell_0$. Therefore,

$$\ell_0 \geq \dim Y + 1. \tag{11}$$

Let $\pi : \tilde{Y} \to Y$ be the normalization of Y. Since the analytic map $f : K \to Y$ is not algebraic degenerate, the image of f is not contained in the indeterminacy locus of the rational map from Y to $\tilde{Y}$, and hence lifts to an analytic map $\tilde{f} : K \to \tilde{Y}$. It is easy to check that the set of divisors $\{\pi^*(D_i \cap Y)\}_{i=1}^{\ell_0}$ are distinct and their intersection is empty and that $\tilde{f}$ takes no value on all the $\pi^*(D_i \cap Y)$. By part (A) of the theorem, we have $l_0 \leq \dim \tilde{Y} = \dim Y$, as the map $\tilde{f}$ is not algebraically degenerate. This yields a contradiction to (11), and hence we can conclude that f must be constant. □

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