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# CHERN SUBRINGS

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Abstract. Let $p$ be an odd prime. We show that for a simply connected semisimple complex linear algebraic group, if its integral homology has $p$-torsion, the Chern classes do not generate the Chow ring of its classifying space.

## 1. Introduction

Let $p$ be an odd prime. Let $h^*(-)$ be one of the mod $p$ cohomology $H\mathbb{Z}/p$, the cohomology $H\mathbb{Z}_{(p)}$ with coefficient $\mathbb{Z}_{(p)}$ and the Brown-Peterson cohomology $BP$ with $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$. Let $G$ be a compact connected Lie group and $G(\mathbb{C})$ its complexification; that is, $G(\mathbb{C})$ is a complex linear algebraic group which is homotopy equivalent to the compact connected Lie group $G$. Considering a finite dimensional complex representation $\rho : G \to GL_m(\mathbb{C})$, we have Chern classes $c_i(\rho)$ in the cohomology $h^*(BG)$ of classifying space and the Chern subring $Ch_h(G) \subset h^*(BG)$ , a subalgebra over $h_*$ generated by Chern classes, where $\rho$ ranges over all finite dimensional representations. If $G$ is one of the classical groups $SU(n)$, $\mathrm{Spin}(n)$ and $Sp(n)$, the cohomology $h^*(BG)$ is generated by Chern classes and $h^*(BG) = Ch_h(G)$ for an arbitrary odd prime $p$.

The case of the Brown-Peterson cohomology is particularly interesting in conjunction with the study of Chow rings of classifying spaces of complex linear algebraic groups defined by Totaro. In [To], Totaro considered the classifying space of the linear algebraic group $G(\mathbb{C})$ as a limit of algebraic varieties, defined the Chow ring for it and showed that the cycle map factors through the Brown-Peterson cohomology,

$$CH^*(BG(\mathbb{C}))_{(p)} \to BP^*(BG) \otimes_{BP_*} \mathbb{Z}_{(p)} \to H^{even}(BG; \mathbb{Z}_{(p)}),$$

where $H^{even}(BG; \mathbb{Z}_{(p)})$ is the direct sum of $H^{2i}(BG; \mathbb{Z}_{(p)})$ $(i \geq 0)$. He also conjectured that the left homomorphism $CH^*(BG(\mathbb{C}))_{(p)} \to BP^*(BG) \otimes_{BP_*} \mathbb{Z}_{(p)}$ is an isomorphism. We may consider a Chern subring for the Chow ring $CH^*(BG(\mathbb{C}))$ as in the case of the above $Ch_h(G)$.

In [Ka-Ya] and [Vi], the Chow ring $CH^*(BPGL_p(\mathbb{C}))_{(p)}$ of the complex linear algebraic group $PGL_p(\mathbb{C})$, which is the complexification of the projective unitary

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group $PU(p)$, and related cohomology theories were computed, and it was shown that

$$CH^*(BPGL_p(\mathbb{C}))_{(p)} = BP^*(BPU(p)) \otimes_{BP_*} \mathbb{Z}_{(p)} = H^{even}(BPU(p); \mathbb{Z}_{(p)})$$

through the cycle map above. In [Ka-Ya, Proposition 5.7], we showed similar results for $(p, G) = (3, F_4)$, $(5, E_8)$. For $p = 3$, the computation of the Brown-Peterson cohomology was done by Kono and Yagita in [Ko-Ya], and Kono and Yagita showed that $x_8^a$ is not in the Chern subring unless $a$ is divisible by 2. In [Ta], Targa showed that $x_{2p+2}^a$ in $CH^*(BPGL_p(\mathbb{C}))_{(p)}$, where $a \leq p-2$, is not in the Chern subring for an arbitrary odd prime $p$.

In this paper, we prove the following and generalize the above computation of Kono, Yagita and Targa. Let $Q_i$ be the Milnor operations of degree $2p^i - 1$ which acts on the mod $p$ cohomology of a space.

**Theorem 1.1.** *For $(p, G) = (p, PU(p))$, let $x = Q_0Q_1x_2$ where $x_2$ is the generator of $H^2(BG; \mathbb{Z}/p) = \mathbb{Z}/p$. For $(p, G) = (3, F_4)$, $(3, E_6)$, $(3, E_7)$, $(3, E_8)$, $(5, E_8)$, let $x = Q_1Q_2x_4$, where $x_4$ is the generator of $H^4(BG; \mathbb{Z}/p) = \mathbb{Z}/p$. Then, $x^a$ is not in the Chern subring $Ch_{H\mathbb{Z}/p}(G)$ unless $a$ is divisible by $p-1$.*

This theorem implies that if $x$ comes from the Chow ring through the cycle map, then the Chow ring is not generated by Chern classes. Recall that the motivic cohomology $H^{*,*'}(BG(\mathbb{C}), \mathbb{Z}/p)$ contains $CH^*(BG(\mathbb{C}))/p$ as

$$CH^*(BG(\mathbb{C}))/p = H^{2*,*}(BG(\mathbb{C}), \mathbb{Z}/p).$$

Moreover, the motivic cohomology has the action of Milnor operations $Q_i$ where the degree of $Q_i$ is $(2p^i - 1, p^i - 1)$. We refer the reader to Voevodsky's paper [Vo] for motivic cohomology operations. If there exists an element $x_{4,3}$ in $H^{4,3}(BG(\mathbb{C}), \mathbb{Z}/p)$ corresponding to $x_4$ in $H^4(BG; \mathbb{Z}/p)$, then $x = Q_1Q_2(x_{4,3})$ is in the Chow ring

$$CH^{p^2+p+1}(BG(\mathbb{C}))/p = H^{2p^2+2p+2, p^2+p+1}(BG(\mathbb{C}), \mathbb{Z}/p)$$

and through the cycle map it maps to $x$ in Theorem 1.1. In [Ya, Lemma 9.6], Yagita proved that if $px_4 \in H^4(BG; \mathbb{Z}_{(p)})$ is a Chern class of some representation, then the element $x_{4,3}$ above exists. In [Sc-Ya], Schuster and Yagita showed that for $(p, G) = (3, F_4)$, $3x_4$ is the Chern class of the complexification of the irreducible representation of $F_4$. In this paper, by computing the Chern class of the adjoint representation of $E_8$, we prove the following proposition.

**Proposition 1.2.** *For $(p, G) = (3, F_4)$, $(3, E_6)$, $(3, E_7)$, $(3, E_8)$ and $(5, E_8)$, there exist a complex representation $\alpha$ of $G$ and $\gamma \in \mathbb{Z}_{(p)}^\times$ such that the element $\gamma p x_4 \in H^4(BG; \mathbb{Z}_{(p)})$ is a Chern class $c_2(\alpha)$.*

Thus, we have the following result on Chern subrings of Chow rings.

**Theorem 1.3.** *For $(p, G) = (p, PU(p))$, $(3, F_4)$, $(3, E_6)$, $(3, E_7)$, $(3, E_8)$ and $(5, E_8)$, the Chow ring $CH^*(BG(\mathbb{C}))_{(p)}$ is not generated by Chern classes.*

Our results seem to support Totaro's conjecture since not only elements in $Ch_{BP}(G)$ but also some elements that are not in the Chern subrings come from Chow rings $CH^*(BG(\mathbb{C}))$ for the above $(p, G)$.

In §2, we consider Chern classes of elementary abelian $p$-groups. In §3, we prove Theorem 1.1. In §4, we prove Proposition 1.2. Throughout the rest of this paper, we write $H^*(-)$ for $H^*(-; \mathbb{Z}/p)$ whenever the coefficient $\mathbb{Z}/p$ is clear from the context.

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## 2. Chern classes of elementary abelian $p$-groups

In this section, we investigate the total Chern class of a finite dimensional complex representation $\rho : A_n \to GL_m(\mathbb{C})$ of an elementary abelian $p$-group $A_n$ of rank $n$.

First, we recall the cohomology of $BA_n$. The mod $p$ cohomology of an elementary abelian $p$-group is a polynomial tensor exterior algebra

$$\mathbb{Z}/p[t_1, \dots, t_n] \otimes \Lambda(dt_1, \dots, dt_n).$$

The elements $dt_1, \dots, dt_n \in H^1(BA_n)$ correspond to the dual of the basis of $\pi_1(BA_n) = H_1(BA_n)$. The elements $t_1, \dots, t_n$ are obtained from $dt_1, \dots, dt_n$ by applying the Milnor operation $Q_0$. For the mod $p$ cohomology of a space, there exists an action of Milnor operations $Q_0, Q_1, Q_2, \dots$ and reduced power operations $\wp^0 = 1, \wp^1, \wp^2, \dots$. The action of Milnor operations on the mod $p$ cohomology of an elementary abelian $p$-group is given by

$$Q_i(dt_k) = t_k^{p^i}, \quad Q_i t_k = 0, \quad Q_i(x \cdot y) = Q_i(x) \cdot y + (-1)^{\deg x} x \cdot Q_i(y).$$

The action of reduced power operations is given by

$$\wp^i dt_k = 0, \quad \wp^i t_k = \begin{cases} t_k^p & (i = 1) \\ 0 & (i \geq 2), \end{cases} \quad \wp^j (x \cdot y) = \sum_{i=0}^{j} \wp^{i-j} x \cdot \wp^j y.$$

Second, we recall the invariant theory of finite general linear groups and special linear groups. The action of Milnor operations commutes with the action of the general linear group $GL_n(\mathbb{Z}/p)$ since the action of the general linear group on the mod $p$ cohomology comes from the one on the elementary abelian $p$-group $A_n$. For the sake of notational simplicity, we write $V_n$ for the subspace spanned by $t_1, \dots, t_n$,

$$V_n = \mathbb{Z}/p\{t_1, \dots, t_n\}.$$

Let

$$SM_n = H^*(BA_n)^{SL_n(\mathbb{Z}/p)}, \quad M_n = H^*(BA_n)^{GL_n(\mathbb{Z}/p)},$$
$$SD_n = \mathbb{Z}/p[t_1, \dots, t_n]^{SL_n(\mathbb{Z}/p)}, \quad D_n = \mathbb{Z}/p[t_1, \dots, t_n]^{GL_n(\mathbb{Z}/p)},$$

respectively. Kameko and Mimura [Ka-Mi] gave a simpler description for $SM_n$, $M_n$ using Milnor operations. For the Dickson invariants $SD_n$, $D_n$ and the Mùi invariants $SM_n, M_n$ above, we refer the reader to [Ka-Mi] and its references. Let us define $c_{n,i}$ for $n = 1, \dots, n-1$ as follows: Consider the polynomial

$$f_n(X) = \prod_{v \in V_n} (X + v)$$

in $\mathbb{Z}/p[t_1, \dots, t_n][X]$. We define $(-1)^{n-i} c_{n,i}$ to be the coefficient of $X^{p^{n-i}}$ in $f_n(X)$. We define $e_n$ by $e_n = Q_0 \cdots Q_{n-1}(dt_1 \cdots dt_n)$. Then, we have the following. For a ring $R$ and for a finite set $\{a_1, \dots, a_r\}$, we denote by $R\{a_1, \dots, a_r\}$ a free $R$-module with the basis $\{a_1, \dots, a_r\}$.

**Proposition 2.1.** *The following hold:*

(1) $c_{n,0} = e_n^{p-1}$.
(2) $f_n(X) = X^{p^n} - c_{n,n-1} X^{p^{n-1}} + \cdots + (-1)^n c_{n,0} X$.
(3) *$SD_n$ is a polynomial algebra $\mathbb{Z}/p[e_n, c_{n,n-1}, \dots, c_{n,1}]$.*
(4) *$D_n$ is also a polynomial algebra $\mathbb{Z}/p[c_{n,n-1}, \dots, c_{n,1}, c_{n,0}]$.*

(5) $M_n$ *is a free* $D_n$*-module*

$$D_n\{1, e_n^{p-2}dt_1 \dots dt_n, e_n^{p-2}Q_{i_1} \dots Q_{i_r}(dt_1 \cdots dt_n)\} \quad \textit{and}$$

(6) $SM_n$ *is a free* $SD_n$*-module*

$$SD_n\{1, dt_1 \dots dt_n, Q_{i_1} \dots Q_{i_r}(dt_1 \cdots dt_n)\},$$

*where* $0 \leq i_1 < \cdots < i_r \leq n-1$, $1 \leq r \leq n-1$.

Third, we consider Chern classes. Any finite dimensional complex representation of an abelian group is a direct sum of 1-dimensional complex representations. (See Serre's book [Se, Theorem 9].) Therefore, the total Chern class $c(\rho)$ is a product of $c(\lambda)$'s where $c(\lambda) = 1 + v$, $v \in V_n$. Thus, the Chern classes are in $\mathbb{Z}/p[t_1, \dots, t_n]$ instead of $H^*(BA_n)$. Let us consider the total Chern class $c(reg)$ of the regular representation $reg : A_n \to GL_{p^n}(\mathbb{C})$. It is clear that $c(reg) \in M_n$.

**Proposition 2.2.** *It follows that*

$$c(reg) = \prod_{v \in V_n \backslash \{0\}} (1+v) = 1 - c_{n,n-1} + \cdots + (-1)^n c_{n,0} \in D_n.$$

For a group $W$ acting on $V_n \backslash \{0\}$, we say that the action of $W$ is transitive on $V_n \backslash \{0\}$ if and only if for each $u, v$ in $V_n \backslash \{0\}$, there exists $w \in W$ such that $wu = v$. We investigate the total Chern class $c(\rho)$ when the image of the induced homomorphism $B\rho^* : H^*(BGL_m(\mathbb{C})) \to \mathbb{Z}/p[t_1, \dots, t_n]$ is invariant under a certain group action.

**Lemma 2.3.** *Let $\rho : A_n \to GL_m(\mathbb{C})$ be a complex representation of an elementary abelian $p$-group $A_n$ of rank $n$. Suppose that a subgroup $W$ of $GL_n(\mathbb{Z}/p)$ acts on $A_n$ in the obvious manner. Suppose that the total Chern class $c(\rho)$ is in $\mathbb{Z}/p[t_1, \dots, t_n]^W$ and suppose that the action of $W$ on $V_n \backslash \{0\}$ is transitive. Then, $c(\rho) = c(reg)^a$ for some $a \geq 0$.*

*Proof.* Suppose that

$$c(\rho) = \prod_{v \in V_n \backslash \{0\}} (1+v)^{\mu(v)}.$$

The non-negative integer $\mu(v)$ is the divisibility of $c(\rho)$ by $1 + v$. In other words, $c(\rho)$ is divisible by $(1+v)^{\mu(v)}$ but not divisible by $(1+v)^{\mu(v)+1}$. In order to prove the lemma, it suffices to show that $\mu(v)$ is a constant function of $v \in V_n \backslash \{0\}$. Suppose that $\mu(u) < \mu(v)$ for some $u, v \in V_n \backslash \{0\}$. Let $w \in W$ be an element such that $wv = u$ . Then, since $w$ acts trivially on $c(\rho)$, we have

$$c(\rho) = wc(\rho) = \prod_{v' \in V_n \backslash \{0\}} (w(1+v'))^{\mu(v')} = \left( \prod_{v' \in V_n \backslash \{0, v\}} (1 + wv')^{\mu(v')} \right) (1+u)^{\mu(v)}.$$

This implies that $\mu(u) \geq \mu(v)$, a contradiction. Hence, we have the desired result. ∎

By Proposition 2.2 and Lemma 2.3, we have the following result:

**Proposition 2.4.** *Let $G$ be a compact connected Lie group and let $A_n$ be an elementary abelian $p$-subgroup of $G$. Suppose that the Weyl group of $A_n$, that is, the quotient of the normalizer of $A_n$ in $G$ by the centralizer of $A_n$ in $G$, acts transitively on $V_n \backslash \{0\}$. Then, $B\eta^*(Ch_{H\mathbb{Z}/p}(G)) \subset D_n$, where $\eta : A_n \to G$ is the inclusion of $A_n$ into $G$.*

We end this section by recalling the following fact:

**Proposition 2.5.** *The action of $SL_n(\mathbb{Z}/p)$ on $V_n\backslash\{0\}$ is transitive for $n \geq 2$.*

Thus, in order to prove Theorem 1.1, it suffices to show that there exists an elementary abelian $p$-subgroup $A_n$ whose Weyl group is $SL_n(\mathbb{Z}/p)$ and that $B\eta^*(x) \notin D_n$. This is what we do in the next section.

## 3. Chern subrings

In this section, we prove Theorem 1.1 by examining the cohomology of a non-toral elementary abelian $p$-subgroup of $G$. There exist non-toral elementary abelian $p$-subgroups in a compact connected Lie group if the integral homology of the Lie group has $p$-torsion. These non-toral elementary abelian $p$-subgroups and their Weyl groups are known for $(p, G) = (p, PU(n)), (3, F_4), (3, E_6), (3, E_7), (3, E_8)$, and $(5, E_8)$. We refer the reader to Andersen et al. [A-G-M-V] and its references. In this paper, we use the following results for $(p, G) = (p, PU(p)), (3, F_4)$ and $(5, E_8)$ only:

**Proposition 3.1.** *The following hold:*
(1) *For $(p, G) = (p, PU(p))$, there exists a non-toral elementary abelian $p$-subgroup $A_2$ of rank 2 such that its Weyl group in $G$ is the special linear group $SL_2(\mathbb{Z}/p)$.*
(2) *For $(p, G) = (3, F_4), (5, E_8)$, there exists a non-toral elementary $p$-subgroup $A_3$ of rank 3 such that its Weyl group in $G$ is the special linear group $SL_3(\mathbb{Z}/p)$.*

Let $\eta : A_n \to G$ be the inclusion of a non-toral elementary abelian $p$-subgroup in $G$. In [Ka-Ya], we computed the image of the induced homomorphism

$$B\eta^* : H^*(BG) \to SM_n$$

for $(p, G) = (p, PU(p))$, $n = 2$ and for $(p, G) = (3, F_4)$, $(3, E_6)$, $(3, E_7)$, $(5, E_8)$, $n = 3$. Since we wish to include the case $(p, G) = (3, E_8)$ in Theorem 1.1, instead of making use of the computation of the image of $B\eta^*$, we use the following result, which is also used in the computation of the image of $B\eta^*$:

**Proposition 3.2.** *The following hold:*
(1) *The induced homomorphism*

$$H^2(BPU(p)) \to SM_2^2 = \mathbb{Z}/p\{dt_1 dt_2\}$$

*is an isomorphism.*
(2) *For $(p, G) = (3, F_4)$ and $(5, E_8)$, the induced homomorphism*

$$H^4(BG) \to SM_3^4 = \mathbb{Z}/p\{Q_0(dt_1 dt_2 dt_3)\}$$

*is an isomorphism.*

Now, we prove Theorem 1.1 for $(p, G) = (3, E_8)$. As we mentioned at the end of the previous section, it suffices to show that $B\eta^*(x) \notin D_3$. There is a sequence of inclusions

$$F_4 \to E_6 \to E_7 \to E_8,$$

and the induced homomorphisms

$$H^4(BF_4) \leftarrow H^4(BE_6) \leftarrow H^4(BE_7) \leftarrow H^4(BE_8) = \mathbb{Z}/p$$

are isomorphisms. Recall that we denote the generator of $H^4(BE_8; \mathbb{Z}/3)$ by $x_4$. We define $x \in H^{26}(BE_8; \mathbb{Z}/3)$ by $x = Q_1 Q_2(x_4)$. Since the induced homomorphism maps $x_4$ to $Q_0(dt_1dt_2dt_3)$ by Proposition 3.2, it maps $x$ to $e_3 = Q_0Q_1Q_2(dt_1dt_2dt_3)$ in $SD_3$. Since $D_3$ is a polynomial algebra generated by $e_3^{p-1}, c_{3,1}$ and $c_{3,2}$ and since $SD_3$ is a polynomial algebra generated by $e_3, c_{3,1}, c_{3,2}$, it is clear that $e_3^a$ is in $D_3$ if and only if $a$ is divisible by $p-1$. Thus, we have Theorem 1.1 for $(p, G) = (3, E_8)$. Theorem 1.1 for the other $(p, G)$'s can be proved in the same manner.

## 4. Proof of Proposition 1.2

In this section, we prove Proposition 1.2 by computing the second Chern class of the adjoint representation of the exceptional Lie group $\alpha : E_8 \to SO(248)$. Similar computations were done in [Sc-Ya] for the irreducible representation $F_4 \to SO(26)$.

Since the induced homomorphisms

$$H^4(BF_4; \mathbb{Z}_{(3)}) \leftarrow H^4(BE_6; \mathbb{Z}_{(3)}) \leftarrow H^4(BE_7; \mathbb{Z}_{(3)}) \leftarrow H^4(BE_8; \mathbb{Z}_{(3)}) = \mathbb{Z}_{(3)}$$

are isomorphisms, if $3x_4$ in $H^4(BE_8; \mathbb{Z}_{(3)})$ is a Chern class, so it is in $H^4(BG; \mathbb{Z}_{(3)})$ for $G = F_4, E_6, E_7$. So, it suffices to show the proposition for $G = E_8$.

Let $\alpha : E_8 \to SO(248)$ be the adjoint representation of $E_8$. By the construction of the exceptional Lie group $E_8$ in [Ad], there exists a homomorphism $\beta : \mathrm{Spin}(16) \to E_8$ such that the induced representation $\alpha \circ \beta$ is the direct sum of $\lambda_{16}^2 : \mathrm{Spin}(16) \to SO(120)$ and $\Delta_{16}^+ : \mathrm{Spin}(16) \to SO(128)$. See [Ad, Corollary 7.3] and [Mi-Ni, p. 143]. Let $T^8$ be the maximal torus of $\mathrm{Spin}(16)$. Let $T^1$ be the first factor of $T^8$ and $\eta : T^1 \to \mathrm{Spin}(16)$ the inclusion of $T^1$ into $\mathrm{Spin}(16)$. Denote by $R(G)$ the complex representation ring of $G$. The complexification of $\lambda_{16}^2$ corresponds to the second elementary symmetric function of $z_1^2 + z_1^{-2}, \dots, z_8^2 + z_8^{-2}$ in $R(T^8)$ and the complexification of $\Delta_{16}^+$ corresponds to $\displaystyle\sum_{\varepsilon_1 \cdots \varepsilon_8 = 1} z_1^{\varepsilon_1} \dots z_8^{\varepsilon_8}$ in $R(T^8)$, where $\varepsilon_r = \pm 1$ for $r = 1, \dots, 8$.

So, the restriction of the complexification of $\lambda_{16}^2$ to $T^1$ corresponds to

$$2^2 \begin{pmatrix} 7 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 7 \\ 1 \end{pmatrix} (z_1^2 + z_1^{-2}) = 84 + 14(z_1^2 + z_1^{-2}) \quad \text{in } R(T^1).$$

The restriction of the complexification of $\Delta_{16}^+$ to $T^1$ corresponds to

$$2^6(z_1 + z_1^{-1}) = 64(z_1 + z_1^{-1}) \quad \text{in } R(T^1).$$

Therefore, the total Chern class of the complexification of $\alpha \circ \beta \circ \eta$ is

$$\{(1+2u)(1-2u)\}^{14}\{(1+u)(1-u)\}^{64} = 1 - 120u^2 + \cdots \in \mathbb{Z}[u] = H^*(BT^1; \mathbb{Z}),$$

where $u$ is the generator of $H^2(BT^1; \mathbb{Z}) = \mathbb{Z}$. Since $120 = 2^3 \cdot 3 \cdot 5$, the Chern class $c_2(\alpha)$ represents $\gamma p x_4$ for $p = 3, 5$ in $H^4(BE_8; \mathbb{Z}_{(p)})$, where $\gamma$ is a unit in $\mathbb{Z}_{(p)}$ and $x_4$ is the generator of $H^4(BE_8; \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}$. This completes the proof of Proposition 1.2.

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