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GROUPS WHERE ALL THE IRREDUCIBLE CHARACTERS ARE SUPER-MONOMIAL

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Abstract. Isaacs has defined a character to be super-monomial if every primitive character inducing it is linear. Isaacs has conjectured that if G is an M-group with odd order, then every irreducible character is super-monomial. We prove that the conjecture is true if G is an M-group of odd order where every irreducible character is a $\{p\}$-lift for some prime p. We say that a group where every irreducible character is super-monomial is a super M-group. We use our results to find an example of a super M-group that has a subgroup that is not a super M-group.

1. Introduction

Throughout this note, G will be a finite group. Following [2], we say that a character χ of G is monomial if it is induced from a linear character and that a group G is an M-group if every irreducible character of G is monomial. In [5], Isaacs has suggested that if G is an M-group of odd order and $\chi \in \mathrm{Irr}(G)$, then every primitive character inducing χ must be linear.

For ease of exposition, we introduce some notation from [8]: the character $\chi \in \mathrm{Irr}(G)$ is *super-monomial* if every primitive character inducing χ is linear. It is not hard to show that χ is super-monomial if and only if every character inducing χ is monomial. We say a group G is a *super M-group* if every character in $\mathrm{Irr}(G)$ is super-monomial. Hence, Isaacs is suggesting that every M-group of odd order is a super M-group. As far as we can tell, there has been little progress on this problem, and the only class of groups which are known to be super M-groups are groups where every subgroup is an M-group.

In this note, we obtain partial results along the lines suggested by Professor Isaacs. In particular, we will find a class of groups whose members that have odd order and are M-groups will be super M-groups. To define this class of groups, we need to introduce Isaacs' π-theory. Isaac introduced his π-theory in [3]. Two good surveys of π-theory are [5] and [6]. An introduction to π-theory can be in Section 40 of [1] or Section 21 of [10]. Both of these sources contain many of the results from π-theory that we need.

We now review the basics of π-theory. Let π be a set of primes, and let G be a π-separable group. Let G^π be the set of elements of G whose orders are π-numbers.

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(Usually, one uses G^0 to denote the π-elements of G, but since we will be restricting both π- and π'-special characters, it is useful to distinguish the π-elements and the π'-elements of G.) Notice that G^π is a union of conjugacy classes of G. Given a character χ of G, we define χ^π to be the restriction of χ to G^π. Following Isaacs, we say that these are the π-partial characters of G.

Isaacs has defined $\mathrm{I}_\pi(G)$ to be the irreducible π-partial characters of G. These are the π-partial characters that cannot be written as the sum of other π-partial characters. Isaacs has proved that $\mathrm{I}_\pi(G)$ forms a basis for the vector space of all complex valued functions on G^π. When $\pi = \{p\}'$, the set $\mathrm{I}_\pi(G)$ is the set of irreducible p-Brauer characters of G; so the π-partial characters can be viewed as generalizing Brauer characters to sets of primes. Isaacs has shown that induction can be defined for π-partial characters, and hence, a π-partial character can be said to be primitive if it is not induced from any proper subgroup. It can be called monomial if it is induced from a linear π-partial character. It is super-monomial if every primitive π-partial character inducing it is linear. Again, it is not difficult to show that a π-partial character is super-monomial if and only if every π-partial character inducing it is monomial.

Given a π-partial character ϕ, a *π-lift* of ϕ is a character χ so that $\chi^\pi = \phi$. If $\phi \in \mathrm{I}_\pi(G)$, then it is easy to see that $\chi \in \mathrm{Irr}(G)$. On the other hand, $\chi \in \mathrm{Irr}(G)$ does not imply that χ^π is irreducible. Thus, not every irreducible character is a lift of an irreducible π-partial character. On the other hand, Isaacs has found certain subsets of $\mathrm{Irr}(G)$ that do consist entirely of lifts. For example, he defines the set $\mathrm{B}_\pi(G) \subseteq \mathrm{Irr}(G)$, and he proves that the map $\chi \mapsto \chi^\pi$ is a bijection from $\mathrm{B}_\pi(G)$ to $\mathrm{I}_\pi(G)$. He has shown that this correspondence transports information between $\mathrm{B}_\pi(G)$ and $\mathrm{I}_\pi(G)$. In this note, we will show that when $|G|$ is odd, such information can be translated to any lift.

In particular, we will prove that if G is a group with odd order and $\chi \in \mathrm{Irr}(G)$ is a π-lift, then χ is (super-)monomial if and only if χ^π is (super-)monomial. We will also give an example to show that this is not true if $|G|$ is even. With the result on lift we will prove the following result regarding M-groups of odd order.

Theorem 1. *Let G be a group of odd order. Assume that every character in* $\mathrm{Irr}(G)$ *is a $\{p\}$-lift for possibly different primes p. If G is an M-group, then G is a super M-group.*

We will use this theorem to produce an example of a super M-group that has a subgroup that is not an M-group.

2. Super M-groups and normal subgroups

Isaacs has conjectured that if G is an M-group with $|G|$ odd, then G is a super M-group. If true, this conjecture would imply that normal subgroups of odd order M-groups are M-groups. We will see that this is a consequence of a more general result. The results in this section have been known (at least to Isaacs), but as far as we know they have not been proved elsewhere.

Lemma 2.1. *Let $\mathcal{G}$ be a class of groups that is closed under normal subgroups, and let $\mathcal{M}$ be the class of M-groups in $\mathcal{G}$. Suppose that every group in $\mathcal{M}$ is a super M-group. Then $\mathcal{M}$ is closed under normal subgroups.*

Proof. Fix a group $G \in \mathcal{M}$. Let M be a normal subgroup of G. If $M = G$, then the result is trivial. Thus, we may assume that $M < G$, and we can find a subgroup

N so that $M \leq N < G$, where N is a maximal normal subgroup of G. Since G is an M-group, it is solvable, and so $|G\colon N| = p$ for some prime p. As $\mathcal{G}$ is closed under normal subgroups, $N \in \mathcal{G}$. Consider a character $\theta \in \mathrm{Irr}(N)$. We know from [2] that either θ extends to G or θ induces irreducibly to G. If θ extends to G, then there is a character $\chi \in \mathrm{Irr}(G)$ so that $\chi_N = \theta$. In [2], it shown that if χ is monomial, then θ is monomial. On the other hand, if θ induces irreducibly to G, then there is a character $\chi \in \mathrm{Irr}(G)$ so that $\theta^G = \chi$. Since G is a super M-group, χ is super-monomial, so θ is monomial. This implies that N is an M-group, so $N \in \mathcal{M}$. Since $|N| < |G|$, we can use induction on $|G\colon M|$ to see that $M \in \mathcal{M}$. □

Notice that this does not imply that normal subgroups of super M-groups are necessarily M-groups. This will only follow if all M-groups in the class are super M-groups.

We can now show that Isaacs' conjecture implies that normal subgroups of odd order M-groups are M-groups.

Corollary 2.2. *Suppose that every odd order M-group is a super M-group. Then every normal subgroup of an odd order M-group is an M-group.*

Proof. The class of odd order groups is closed under normal subgroups. Hence, Lemma 2.1 applies to give the result. □

At this time, there have been few examples of super M-groups. Perhaps the easiest way to produce a super M-group is the following. Since all groups that are supersolvable-by-A-groups are M-groups, this implies that all of these groups are super M-groups. (Recall that an A-group is a solvable group where all the Sylow subgroups are abelian.) In particular, nilpotent groups, supersolvable groups, and A-groups are all super M-groups.

Lemma 2.3. *Let G be a group where all the primitive characters are linear and every proper subgroup of G is an M-group. Then G is a super M-group.*

Proof. Consider a character $\chi \in \mathrm{Irr}(G)$. If χ is primitive, then χ is linear and hence super-monomial. Thus, we may assume that χ is not primitive. Let $H < G$ and $\theta \in \mathrm{Irr}(H)$ so that $\theta^G = \chi$. Since all proper subgroups of G are M-groups, H is an M-group and θ is monomial. Therefore, χ is super-monomial. □

3. Lifts in odd order groups

We now work to prove Theorem 1. We begin by studying the characters in $\mathrm{B}_{\pi'}(G)$ that happen to be π-lifts. We prove that any such character must be real.

Lemma 3.1. *Let π be a set of primes and let G be a π-solvable group. Consider the partial character $\phi \in I_\pi(G)$. Let $\chi \in B_\pi(G)$ so that $\chi^\pi = \phi$. If there is a character $\psi \in B_{\pi'}(G)$ so that $\psi^\pi = \phi$, then $\chi = \bar{\chi}$.*

Proof. Let $n = |G|_\pi$ and $m = |G|_{\pi'}$. Let $\mathbf{Q}_\pi$ be the field obtained by adjoining all complex nth roots of unity to $\mathbf{Q}$, and let $\mathbf{Q}_{\pi'}$ be the field obtained by adjoining all complex mth roots of unity to $\mathbf{Q}$. Using Theorem 20.12 of [7], $\mathbf{Q}_\pi \cap \mathbf{Q}_{\pi'} = \mathbf{Q}$. By Corollary 12.1 of [3], we know that the values of χ lie in $\mathbf{Q}_\pi$ and the values of ψ lie in $\mathbf{Q}_{\pi'}$. Since $\phi = \chi^\pi = \psi^\pi$, we conclude that the values of ϕ lie in $\mathbf{Q}_\pi \cap \mathbf{Q}_{\pi'} = \mathbf{Q}$. We now have $\bar{\chi}^\pi = \bar{\phi} = \phi$. We know that the map from $\mathrm{B}_\pi(G)$ to $\mathrm{I}_\pi(G)$ is a bijection, and it is not difficult to see that $\bar{\chi} \in \mathrm{B}_\pi(G)$. Therefore, $\bar{\chi} = \chi$, as desired. □

We use Lemma 3.1 to show that if $|G|$ is odd and $\psi \in \mathrm{B}_{\pi'}(G)$ is a π-lift, then ψ is linear and a π-lift of the principal character.

Corollary 3.2. *Let G be a group of odd order, and let π be a set of primes. If $\psi \in B_{\pi'}(G)$ and $\psi^\pi \in I_\pi(G)$, then $\psi(1) = 1$ and $\psi^\pi = (1_G)^\pi$.*

Proof. Let $\chi \in B_\pi(G)$ so that $\chi^\pi = \psi^\pi$. By Lemma 3.1, we know that $\chi = \bar{\chi}$. It is well-known that if $|G|$ is odd and $\chi = \bar{\chi}$, then $\chi = 1_G$ (see Exercise 3.16 of [2]). This proves that $\psi^\pi = (1_G)^\pi$ and thus $\psi(1) = 1$. □

We now show if χ is a π-lift and χ^π is primitive, then χ is primitive.

Lemma 3.3. *Let π be a set of primes and G be a π-separable group. Suppose $\chi \in \mathrm{Irr}(G)$ and $\chi^\pi = \phi \in I_\pi(G)$. If ϕ is primitive, then χ is primitive.*

Proof. Suppose that $H \subseteq G$ and $\theta \in \mathrm{Irr}(H)$ so that $\theta^G = \chi$. We see that $(\theta^\pi)^G = (\theta^G)^\pi = \chi^\pi = \phi$. Since ϕ is irreducible, θ^π is irreducible, and thus, the fact that ϕ is primitive implies $H = G$. Therefore, χ is primitive. □

Using Isaacs' theorem that primitive characters of π-separable groups are π-factored (see Theorem 40.8 of [1] or Corollary 21.8 of [10]), we have that if $\chi \in \mathrm{Irr}(G)$ is primitive, then χ is π-factored. If in addition $\chi \in \mathrm{B}_\pi(G)$, then by Lemma 5.4 of [3], this implies that χ is π-special. Using [4], it is not difficult to show that a π-special character is primitive if and only if it lifts a primitive π-partial character. Therefore, we see that $\chi \in B_\pi(G)$ is primitive if and only if χ^π is primitive. For groups of odd order, this is true for any lift.

Lemma 3.4. *Let G be a group of odd order, and π be a set of primes. Let $\chi \in \mathrm{Irr}(G)$ so that $\chi^\pi = \phi \in I_\pi(G)$. Then χ is primitive if and only if ϕ is primitive.*

Proof. We saw in Lemma 3.3 that if ϕ is primitive, then χ is primitive. Conversely, suppose that χ is primitive. We know that $\chi = \theta\psi$, where θ is π-special and ψ is π'-special and both θ and ψ is primitive. Now, $\psi \in B_{\pi'}(G)$. Also, $\phi = \theta^\pi\psi^\pi$. Since ϕ is irreducible, it follows that ψ^π must be irreducible, so we may use Corollary 3.2 to see that $\psi(1) = 1$ and $\psi^\pi = (1_G)^\pi$. It follows that $\phi = \theta^\pi$. As we mentioned before this lemma, θ^π is primitive if and only if θ is primitive. Because θ is primitive, we conclude that $\phi = \theta^\pi$ is primitive. □

We can now show that if $\chi \in \mathrm{Irr}(G)$ is a π-lift and χ is monomial, then χ^π is monomial.

Lemma 3.5. *Let π be a set of primes, and let G be a π-separable group. Suppose $\chi \in \mathrm{Irr}(G)$ with $\chi^\pi = \phi \in I_\pi(G)$. If χ is monomial, then ϕ is monomial.*

Proof. Since χ is monomial, we can find $H \subseteq G$ and $\lambda \in \mathrm{Irr}(H)$ so that $\lambda(1) = 1$ and $\lambda^G = \chi$. We have that $(\lambda^\pi)^G = (\lambda^G)^\pi = \chi^\pi = \phi$. It follows that ϕ is induced from λ^π, which is linear, so ϕ is monomial. □

We now consider the case where $|G|$ is odd. We prove that if $\chi \in \mathrm{Irr}(G)$ is a π-lift, then χ is super-monomial if and only if χ^π is super-monomial.

Lemma 3.6. *Let G be a group of odd order, and let π be a set of primes. Suppose $\chi \in \mathrm{Irr}(G)$ with $\chi^\pi = \phi \in I_\pi(G)$. If ϕ is super-monomial, then χ is super-monomial.*

Proof. Let $H \subseteq G$ and $\mu \in \mathrm{Irr}(H)$ such that $\mu^G = \chi$ and μ is primitive. We see that $(\mu^\pi)^G = \phi$, so $\mu^\pi \in I_\pi(H)$. By Lemma 3.4, μ^π is primitive. Since ϕ is super-monomial, this implies that μ^π is linear, and so μ is linear. We conclude that χ is super-monomial. □

We continue to work in the case where $|G|$ is odd. We show that if $\chi \in \mathrm{Irr}(G)$ is a $\{p\}$-lift where p is prime, then χ is monomial if and only if χ is super-monomial. This uses the fact that Isaacs proved in [9] for $\{p\}$-partial characters that monomial is equivalent to super-monomial. The only case where the hypothesis that $|G|$ is odd is used is to show that (3) implies (4).

Theorem 3.7. *Let p be an odd prime, and let G be a group of odd order. Suppose $\chi \in \mathrm{Irr}(G)$ with $\chi^{\{p\}} = \phi \in I_{\{p\}}(G)$. Then the following are equivalent:*

(1) *χ is monomial.*
(2) *ϕ is monomial.*
(3) *ϕ is super-monomial.*
(4) *χ is super-monomial.*

Proof. Note that (4) implies (1) is obvious, and (1) implies (2) is Lemma 3.5. Isaacs proved (2) implies (3) as Theorem F in [9], and (3) implies (4) is Lemma 3.6. □

We now prove a preliminary result for groups where all the $\{p\}$-partial characters are monomial and all the characters in $\mathrm{Irr}(G)$ are $\{p\}$-lifts for perhaps different primes p.

Theorem 3.8. *Let G be a group with odd order. Suppose for every prime p dividing the order of $|G|$ that the partial characters in $\mathrm{I}_p(G)$ are monomial. Assume that every character in $\mathrm{Irr}(G)$ is a $\{p\}$-lift for some prime p that divides $|G|$, not necessarily the same prime for all characters in $\mathrm{Irr}(G)$. Then G is a super M-group.*

Proof. Let $\chi \in \mathrm{Irr}(G)$. By hypothesis, there is a prime p dividing $|G|$ so that $\chi^{\{p\}}$ is irreducible. Also, by hypothesis, $\chi^{\{p\}}$ is monomial. We apply Theorem 3.7 to see that this implies that χ is super-monomial. This shows that every character in $\mathrm{Irr}(G)$ is super-monomial, and hence, G is a super M-group. □

Next, we obtain Theorem 1 as a corollary.

Corollary 3.9. *Let G be an M-group with odd order. Suppose every character in $\mathrm{Irr}(G)$ is a $\{p\}$-lift for some prime dividing $|G|$. Then G is a super M-group.*

Proof. In light of Theorem 3.8, it suffices to show that every character in $\mathrm{I}_{\{p\}}(G)$ is monomial for all primes p that divide $|G|$. However, we know that every character in $\mathrm{B}_{\{p\}}(G)$ is monomial. By Theorem 3.7, this implies that all partial characters in $\mathrm{I}_{\{p\}}(G)$ are monomial. □

Finally, the following lemma is motivated by a result of Navarro. (See Theorem A in [11].)

Lemma 3.10. *Let π be a set of primes, let G be a π-separable group, and let $H \leq G$. Suppose that μ is a π-special character of H, and let $\nu = \mu^\pi$. Assume that $\phi = \nu^G \in \mathrm{I}_\pi(G)$. If λ is a π'-special linear character of H, then $(\mu\lambda)^G$ lifts ϕ.*

Proof. We know that $(\mu^G)^\pi = (\mu^\pi)^G = \nu^G = \phi$. Hence, it follows that μ^G must be irreducible. Since λ is linear and π'-special, it follows that $\lambda^\pi = (1_G)^\pi$, and so

$(\mu\lambda)^\pi = \mu^\pi\lambda^\pi = \nu(1_G)^\pi = \nu$. We see that $((\mu\lambda)^G)^\pi = (\mu^G)^\pi = \phi$, and so $(\mu\lambda)^G$ lifts ϕ. □

We know that every character is induced from a primitive character. Suppose $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(H)$ is primitive so that $\theta^G = \chi$. If χ is a π-lift for some set of primes π, then θ is a π-lift. When $|G|$ is odd, Lemma 3.4 shows that θ^π must be primitive. We know that θ is π-factored. Hence, $\theta = \theta_\pi\theta_{\pi'}$, where θ_π is π-special and $\theta_{\pi'}$ is π'-special. By Corollary 3.2, $\theta_{\pi'}$ is linear. Hence, χ has the form of the lift found in Lemma 3.10. In particular, if $|G|$ is odd and $\phi \in \mathrm{I}_\pi(G)$, then all lifts of ϕ arise in the fashion of Lemma 3.10. We will see that when $|G|$ is not odd, they do not necessarily arise in this fashion.

4. Examples

We now find groups where all the irreducible characters are $\{p\}$-lifts for various primes p. We begin with a technical lemma that will be useful.

Lemma 4.1. *Let G be a p-solvable group for some prime p. Suppose that G has a normal subgroup N so that G/N is abelian. Let $\theta \in \mathrm{B}_{\{p\}}(N)$ and let T be the stabilizer of θ in G. Assume that every character in $\mathrm{Irr}(T \mid \theta)$ is an extension of θ. Then every character in $\mathrm{Irr}(G \mid \theta)$ is a $\{p\}$-lift.*

Proof. Consider a character $\chi \in \mathrm{Irr}(G \mid \theta)$. We use Corollary 6.4 of [3] to see that $\mathrm{Irr}(G \mid \theta)$ contains a character $\psi \in \mathrm{B}_{\{p\}}(G)$. Let $\gamma, \delta \in \mathrm{Irr}(T \mid \theta)$ so that $\gamma^G = \psi$ and $\delta^G = \chi$. By Gallagher's theorem, there exists $\lambda \in \mathrm{Irr}(T/N)$ so that $\delta = \gamma\lambda$. Since G/N is abelian, λ extends to $\epsilon \in \mathrm{Irr}(G/N)$. By Problem 5.3 of [2], we have $\chi = \delta^G = (\gamma\lambda)^G = \gamma^G\epsilon = \psi\epsilon$. We conclude that $\chi^{\{p\}} = \psi^{\{p\}}\epsilon^{\{p\}}$. Since $\psi^{\{p\}}$ is irreducible and ϵ is linear, it follows that $\chi^{\{p\}}$ is an irreducible $\{p\}$-partial character. □

In this first example, all the irreducible characters are $\{p\}$-lifts for a single prime p. Notice that this includes all Frobenius groups whose Frobenius kernels are p-groups and whose Frobenius complements are abelian.

Lemma 4.2. *Let G be a group. Suppose that G has a normal subgroup N so that N is a p-group for some prime p and G/N is an abelian p'-group. Then every character in $\mathrm{Irr}(G)$ is a $\{p\}$-lift.*

Proof. Consider a character $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_N. By Corollary 8.16 of [2], we know that θ has an extension in $\mathrm{Irr}(T)$, and by Gallagher's theorem (Corollary 6.17 of [2]), every character in $\mathrm{Irr}(T \mid \theta)$ is an extension of θ. By Lemma 4.1, we conclude that χ is a $\{p\}$-lift as desired. □

We now have an example of a super M-group that has a subgroup that is not an M-group.

Example 4.3. Let p and q be odd primes, and suppose that q divides $p+1$. Take E to be an extra-special group of order p^3 of exponent p. Let E_1 and E_2 be copies of E, and take D to be the central product of E_1 and E_2 so that D is extra-special of order p^5 and exponent p. Now, E has an automorphism σ of order q that centralizes $\mathbf{Z}(E)$, and there is an automorphism s on D so that s acts like σ on E_1 and σ^{-1} on E_2. Let G be the semi-direct product of $\langle s\rangle$ acting on D.

For $i = 1, 2$, let K_i be the subgroup generated by E_i and s. We claim that K_i is not an M-group. To see that K_i is not an M-group, observe that E_i has irreducible characters of degree p that extend to K_i and that K_i does not have any subgroups of index p. Next, we claim that G is an M-group. It is not difficult to see that the degrees of the irreducible characters of G are 1, q, and p^2 and that the characters of degree 1 and q are monomial. To see that the characters of degree p^2 are monomial, we note that s normalizes an abelian subgroup A of index p^2 in E, and it is not difficult to see that the character is induced from a linear character of $\langle s \rangle A$.

By Lemma 4.2, every character in $\mathrm{Irr}(G)$ is a $\{p\}$-lift. By Corollary 3.9, G is a super M-group. Notice that we now have an example of a super M-group that has a subgroup that is not an M-group.

We present an example of group with Fitting height 3 so that all the irreducible characters are $\{p\}$-lifts for (perhaps different) primes p.

Lemma 4.4. *Let G be a group. Suppose that G has normal subgroups $N \leq M$ so that N is a p-group, M is a Frobenius group whose Frobenius kernel in N, M/N is a q-group for some prime q, and G/M is a cyclic q'-group. Then every character in* $\mathrm{Irr}(G)$ *is either a $\{p\}$-lift or a $\{q\}$-lift.*

Proof. Notice that Lemma 4.2 can be applied to G/N to see that every irreducible character in $\mathrm{Irr}(G/N)$ is a $\{q\}$-lift. Assume that $\chi \in \mathrm{Irr}(G)$ and that N is not contained in the kernel of χ. Let ν be an irreducible constituent of χ_N. Since M is a Frobenius group, we have $\nu^M \in \mathrm{Irr}(M)$, and by Corollary 6.4 of [3], we conclude that $\nu^M \in \mathrm{B}_{\{p\}}(M)$. Because T/M is cyclic, we know from Corollary 11.22 of [2] that all of the characters in $\mathrm{Irr}(T \mid \nu)$ are extensions of ν^M. By Lemma 4.1, it follows that every character in $\mathrm{Irr}(G \mid \nu)$, in particular χ, is a $\{p\}$-lift. □

We next present an example that shows that the hypothesis that $|G|$ is odd is necessary in Corollary 3.2, Lemma 3.4, and Lemma 3.6.

Example 4.5. Take G to be $\mathrm{GL}_2(3)$, and write S and Q for the normal subgroups of G corresponding to $\mathrm{SL}_2(3)$ and the quaternions. Let χ be the unique character in $\mathrm{Irr}(G/Q)$ that has degree 2. It is not difficult to see that χ is induced by a linear character $\lambda \in \mathrm{Irr}(S/Q)$ and that λ is 3-special. In [3], Isaacs showed that $\mathrm{Irr}(G \mid \lambda)$ has a unique character in $\mathrm{B}_{\{3\}}(G)$, and since χ is the unique irreducible constituent of λ^G, it follows that $\chi \in \mathrm{B}_{\{3\}}(G)$. We write $\phi = \chi^{\{3\}} \in \mathrm{I}_{\{3\}}(G)$.

Let θ be the unique character in $\mathrm{Irr}(Q)$ having degree 2. We know that θ is invariant in G, so θ extends to S. Obviously, θ is $\{2\}$-special (i.e., θ is $\{3\}'$-special). Using [3], θ has a unique $\{2\}$-special extension $\hat{\theta} \in \mathrm{Irr}(S)$. Furthermore, by its uniqueness, $\hat{\theta}$ will be G-invariant, and so it has an extension $\psi \in \mathrm{Irr}(G)$. (In fact, it has two extensions, and either will work for our purposes.) Applying [3], we see that ψ is $\{2\}$-special, and so $\psi \in B_{\{3\}'}(G)$. Notice that ψ is now an extension of θ, so $\psi(1) = 2$. Now, G has two nontrivial conjugacy classes of 3-elements, and so it is not difficult to see that $\psi^{\{3\}} = \phi$. This shows that Corollary 3.2 is false without the oddness assumption. It is not difficult to see that ψ is in fact primitive. Since ϕ is not primitive, it follows that Lemma 3.4 is false when $|G|$ is not odd. Finally, ϕ is monomial, and so super-monomial. This shows that the oddness is needed in Lemma 3.6 and in showing that (3) implies (4) in Theorem 3.7.

5. Questions

(1) Is the converse of Lemma 2.1 true? In particular, if $\mathcal{M}$ is closed under normal subgroups, then is it true that all the groups in $\mathcal{M}$ are super M-groups?

(2) If G is a super M-group and N is a normal Hall subgroup of G, then is N a super M-group?

(3) Assume the situation of Lemma 3.10. If λ and δ are π'-special characters of H, under what hypotheses will $(\mu\lambda)^G = (\mu\delta)^G$ imply that $\lambda = \delta$? Navarro has shown this to be the case when H is the nucleus of the $B_\pi(G)$-character of G that lifts ϕ.

(4) What can be said regarding groups where every irreducible character is a $\{p\}$-lift for (different) primes p? Lemma 4.2 shows that there is no bound on the derived length of such groups. Lemma 4.4 gives examples with Fitting height 3. Do there exist examples with larger Fitting heights? Do there exist examples of such groups with Fitting height 3 that are M-groups and have subgroups that are not M-groups?

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