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A NOTE ON MAXIMAL AVERAGES IN THE PLANE

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ABSTRACT. Let $\mathcal{B}_\delta$ be the class of all $h \times \delta h$ rectangles in the plane with $h > 0$ and $0 < \delta < \frac{1}{2}$. The orientation of the rectangles is arbitrary. Form the maximal operator

$$GMf(x) = \sup_{0<\delta<\frac{1}{2}} \sup_{x \in R \in \mathcal{B}_\delta} \frac{1}{|\log \delta| \cdot |R|} \int_R |f(y)|\, dy.$$

Note the logarithmic term in the average. It is shown that GM is a bounded maximal operator in $L^2(\mathbb{R}^2)$. The case of a fixed δ is due to Córdoba.

1. Introduction

For δ in $(0,1)$, let $\mathcal{B}_\delta$ denote the basis of all $h \times \delta h$ rectangles in $\mathbb{R}^2$. For a locally integrable function f, define the maximal operator M_δ by

$$M_\delta f(x) = \sup_{x \in R \in \mathcal{B}_\delta} \frac{1}{|R|} \int_R |f(y)|\, dy. \tag{1.1}$$

Córdoba proved in [3] that M_δ satisfies the following sharp $L^2(\mathbb{R}^2)$ estimate:

$$||\, M_\delta \,|| \le C|\log \delta|. \tag{1.2}$$

Define the following *grand* maximal operator:

$$GMf(x) = \sup_{0<\delta<1/2} \frac{1}{|\log \delta|} M_\delta f(x). \tag{1.3}$$

We prove

Theorem 1. *GM is bounded on $L^2(\mathbb{R}^2)$.*

Observe that the factor $\frac{1}{|\log \delta|}$ in (1.3) is sharp. For if it is replaced by A_δ satisfying $\limsup_{\delta \to 0^+} A_\delta \,|\log \delta| = \infty$, then GM is unbounded.

We use the following convention: If A and B are positive operators, $A \lesssim B$ means that there is an absolute constant C such that $Af(x) \le CBf(x)$ for all nonnegative f and a.e. x. Also norms of vectors $||f||, ||Tf||$ are Lebesgue spaces L^2 norms while norms $||S||$ of operators are the uniform operator norm.

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2. Remarks about the TT^* technique

The TT^* method is a well-known technique for proving the boundedness of a positive linear operator on L^2. It calls for one to prove the estimate

$$TT^* \lesssim T + T^*. \tag{2.1}$$

This, with the Hilbertian fact that $\|\, T\,T^*\,\| = \|T\|^2 = \|T^*\|^2$, shows that $\|T\| \lesssim 1$. This principle can be applied to maximal operators which are only sublinear as follows.

Let $\{\, T_m : m \in \mathcal{M}\,\}$ be a collection of positive linear operators, and form the maximal operator by setting

$$T_{\max} f(x) = \sup_{m\in\mathcal{M}} |T_m f(x)|. \tag{2.2}$$

The inequality

$$\|\, T_{\max}\, f\,\|_{L^2} \lesssim \|\, f\,\|_{L^2} \tag{2.3}$$

is equivalent to the uniform boundedness of the family of linear maps $f \to T_{\mathfrak{M}(x)} f(x)$, where $\mathfrak{M}$ is *any* measurable map from the underlying measure space to $\mathcal{M}$.

Suppose that one can show that

$$T_m\, T_n \lesssim T_m + T_n^*, \qquad m, n \in \mathcal{M}.$$

Take $\mathfrak{M}$ to be a measurable map from the underlying measure space into $\mathcal{M}$. It follows that

$$T_{\mathfrak{M}}\, T_n^* \lesssim T_{\mathfrak{M}} + T_n^*, \qquad n \in \mathcal{M}.$$

Positivity is preserved under adjoints; hence

$$T_n\, T_{\mathfrak{M}}^* \lesssim T_{\mathfrak{M}}^* + T_n, \qquad n \in \mathcal{M}.$$

Then we can replace n above by $\mathfrak{M}$ to deduce (2.3) for the linear operator $T_{\mathfrak{M}}$. Then one has a bound on $\|\, T_{\mathfrak{M}}\,\|$ that is independent of the choice of measurable map $\mathfrak{M}$. Hence, we deduce the boundedness of the maximal function $T_{\max}$. Arguments of this type appeared first in [5]. Other variants can be found in [1], [2], and [4].

There is an elaboration of this method here. Let T_m be as in (2.2).

Lemma 2. *Suppose that $\{T_m : m \in \mathcal{M}\}$ are positive uniformly bounded operators and there is an auxiliary collection of positive operators $\{U_m : m \in \mathcal{M}\}$ for which we can show*

$$T_m\, T_n^* \lesssim T_{\mu(m)} + T_{\mu(n)}^* + T_{\mu(m)} U_n^* + U_m T_{\mu(n)}^*, \qquad m, n \in \mathcal{M}. \tag{2.4}$$

Here, $\mu : \mathcal{M} \longrightarrow \mathcal{M}$ is an arbitrary fixed map. If, in addition, $U_{\max} f = \sup_{m\in\mathcal{M}} U_m f$ is a bounded maximal operator on L^2, we can then conclude the same for $T_{\max}$.

Proof. Fix an integer l, and define

$$\Lambda = \sup_{\substack{\mathcal{M}' \subset \mathcal{M} \\ \sharp\mathcal{M}' = l}} \ \sup_{\|\, f\,\|=1} \left\| \sup_{m\in\mathcal{M}'} T_m(|f|) \right\|.$$

By assumption, $\Lambda \lesssim l$, and we seek a bound on Λ that is independent of l. By a limiting argument, this will finish the proof of the lemma.

Fix $\mathcal{M}' \subset \mathcal{M}$ with $^\sharp\mathcal{M}' = l$ and $\varphi \geq 0$ with $||\varphi|| = 1$ so that

$$\Lambda < 2 \left|\left| \sup_{m \in \mathcal{M}'} T_m(\varphi) \right|\right|.$$

Further select a measurable map $\mathfrak{M}$ from the underlying measure space to $\mathcal{M}'$ such that

$$\Lambda < 4 \, || \, T_{\mathfrak{M}}(\varphi) \, || \, .$$

We provide an estimate for the norm of the linear operator $T_{\mathfrak{M}}$.

Note that uniformity in (2.4) implies that

$$T_{\mathfrak{M}} T_n^* \lesssim T_{\mu \circ \mathfrak{M}} + T_{\mu(n)}^* + T_{\mu \circ \mathfrak{M}} U_n^* + U_{\mathfrak{M}} T_{\mu(n)}^*,$$

where $U_{\mathfrak{M}}$ is the linear operator defined analogously to $T_{\mathfrak{M}}$. Positivity is preserved under adjoints—and we can take adjoints as all operators are linear—so we see that

$$T_n T_{\mathfrak{M}}^* \lesssim T_{\mu \circ \mathfrak{M}}^* + T_{\mu(n)} + U_n T_{\mu \circ \mathfrak{M}}^* + T_{\mu(n)} U_{\mathfrak{M}}^*.$$

This is a uniform inequality; hence

$$T_{\mathfrak{M}} T_{\mathfrak{M}}^* \lesssim T_{\mu \circ \mathfrak{M}}^* + T_{\mu \circ \mathfrak{M}} + U_{\mathfrak{M}} T_{\mu \circ \mathfrak{M}}^* + T_{\mu \circ \mathfrak{M}} U_{\mathfrak{M}}^*.$$

The operator norm of the term on the left is, by construction, comparable to Λ^2. Note that $\mu \circ \mathfrak{M}$ is a map from the underlying measure space to $\mu(\mathcal{M}')$, which is a subset of $\mathcal{M}$ of cardinality l. Thus, each term on the right, by selection of an extremal $\mathcal{M}'$, has operator norm dominated by a constant times $\Lambda(1 + ||U_{\max}||)$. So we have a bound on Λ that is independent of l, and the proof is finished. □

3. Principal line of argument

Let $\mathcal{M} = (0, \infty) \times [0, \frac{1}{10}] \times (\delta_0, \frac{1}{2})$. For $m = (h, \theta, \delta) \in \mathcal{M}$, let R_m be the $h \times \delta h$ rectangle centered at the origin, with longest side having an angle θ with the x-axis. Define

$$\begin{aligned} T_m f(x) &= \frac{1}{|\log \delta| \, |R_m|} \int_{R_m} f(x - y) \, dy, \\ GM f(x) &= \sup_{m \in \mathcal{M}} |T_m f(x)|. \end{aligned}$$

We prove

$$T_m \, T_n \lesssim T_{m'} + T_{n'} + T_{m'} \, W_n + W_m \, T_{n'}, \tag{3.1}$$

where for $m = (h, \theta, \delta) \in \mathcal{M}$ we set $m' = (2h, \theta, \delta)$. We now define the operators W_m:

$$\begin{aligned} W_m f(x) &= \frac{1}{2 \, |\log \delta|} \sum_{j=0}^{2|\log \delta|} H_{m,j} \, f(x), \\ H_{m,j} f(x) &= \frac{1}{\delta \, h \, 2^{j+1}} \int_{-\delta h 2^j}^{\delta h 2^j} f(x + t\vec{\mathfrak{e}}_2) \, dt, \end{aligned}$$

where $\vec{\mathfrak{e}}_2 = (0, 1)$ is the second coordinate direction in the plane. By construction, the rectangles we deal with essentially point in the first coordinate direction in the plane. If M^2 denotes the one-dimensional maximal operator computed in the second coordinate, then we have $W_m f \lesssim M^2 f$. In view of (2.4), this will prove the main theorem.

Turning to the proof of (3.1), we set $m = (h, \theta, \delta)$ and $n = (h', \theta', \delta')$. Let us assume that $h > h'$, which assumption we return to below.

Lemma 3. *For $f \geq 0$ we have the pointwise inequality*

$$T_m\, T_n f(x) \lesssim \frac{1}{|\log\delta| \cdot |\log\delta'| \cdot |R|} \int_R f(x-y)\, dy,$$

where R is the rectangle centered at the origin, of dimensions $2h \times 2w$, where

$$w = \max\{\delta h, \delta' h', h' \sin|\theta - \theta'|\},$$

and the longest side of R is $2h$, which forms an angle θ with the x-axis.

Proof. Since T_m has kernel

$$K_m(x,y) = \frac{1}{|\log\delta| \cdot |R_m|}\, \mathbf{1}_{R_m}(x-y),$$

the kernel of $T_m T_n$ is

$$\begin{aligned} K_{m,n}(x,y) &= \frac{1}{|\log\delta| \cdot |\log\delta'| \cdot |R_m| \cdot |R_n|} \int \mathbf{1}_{R_m}(x-s)\, \mathbf{1}_{R_n}(s-y)\, ds \\ &= \frac{|\,(R_m + x) \cap (R_n + y)\,|}{|\log\delta| \cdot |\log\delta'| \cdot |R_m| \cdot |R_n|}. \end{aligned}$$

First note that this implies that

$$\operatorname{supp} K_{m,n} \subset \big\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x - y \in R_m + R_n\big\} \subset \big\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x - y \in R\big\}$$

where R is the rectangle described in the statement of the lemma. Second, if $(x,y) \in \operatorname{supp} K_{m,n}$, the above implies

$$\begin{aligned} K_{m,n}(x,y) &\lesssim \frac{|\,R_m \cap R_n\,|}{|\log\delta| \cdot |\log\delta'| \cdot |R_m| \cdot |R_n|} \\ &\lesssim \frac{1}{|\log\delta| \cdot |\log\delta'| \cdot |\,R\,|}, \end{aligned}$$

and the lemma follows from these two observations. $\square$

The argument for (3.1) depends upon the value of w. In the case that $w = \delta h$, it follows that R is a $2h \times 2\delta h$ rectangle; hence

$$T_m\, T_n \lesssim |\log\delta'|^{-1}\, T_{m'} \lesssim T_{m'}.$$

In the case $w = \max\{\delta' h', h' \sin|\theta - \theta'|\}$, there is a choice of integer $j = 1, \ldots, 2[\log\delta']$ such that $2^{j-1}\delta' h' \leq w < 2^j \delta' h'$, whence,

$$\begin{aligned} T_m\, T_n &\lesssim |\log\delta'|^{-1}\, T_m\, H_{n,j} \\ &\lesssim |\log\delta'|^{-1}\, T_{m'} \sum_{j=1}^{2[|\log\delta'|]} H_{n,j} \\ &\lesssim T_{m'}\, W_n. \end{aligned}$$

We have passed to the larger sum in the middle step in order to get an estimate that depends solely on n. To recap, in the case that $h > h'$, we have shown

$$T_m\, T_n \lesssim T_{m'} + T_{m'} W_n, \tag{3.2}$$

which proves (3.1) in this case.

Finally, if $h' > h$, the operators T_m and T_n commute, so that the argument just given supplies us with the estimate

$$T_m T_n \lesssim T_{n'} + T_{n'} W_m.$$

Summing this inequality and (3.2) proves (3.1) as stated.

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