SOLUTION TO FARHADI–GAHARAMANI’S MULTIPLIER PROBLEM

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Let $G$ be a locally compact group, and consider its group algebra $L_1(G)$ with convolution product. Let $A = L_1(G)^{**}$, endowed with the first Arens product, which we shall denote by $\cdot$. We write $\cdot$ for the first Arens product on $A^{**}$, as well as for the canonical left action of $A^{**}$ on $A^*$, and of $A^*$ on $A$. Recall that these are defined as follows, for $m, n \in A^{**}$, $f \in A^*$ and $a, b \in A$:
\[
\langle m \cdot n, f \rangle = \langle m, n \cdot f \rangle,
\]
\[
\langle n \cdot f, a \rangle = \langle n, f \cdot a \rangle,
\]
\[
\langle f \cdot a, b \rangle = \langle f, a \ast b \rangle.
\]
Analogously, one defines the second Arens product starting from the product $b \ast a$ (but we shall not need this here).

In [3], noting that for any $N \in L_1(G)^{****}$ the map
\[
R_N(f) := N \cdot f \quad (f \in L_1(G)^{***})
\]
is a right $L_1(G)$-module homomorphism on $L_1(G)^{***}$, the authors ask the following question [3, Problem 4.1]:

Is every $w^\ast$-continuous surjective right $L_1(G)$-module map
\[
(*) \quad R : L_1(G)^{***} \to L_1(G)^{***} \text{ equal to } R_N \text{ for some } N \in L_1(G)^{****}?
\]

In the present paper we shall show:

Theorem 1. The answer to the above multiplier problem is negative for all infinite, countable, discrete, abelian groups (e.g., for $G = \mathbb{Z}$).

To prepare the proof, let us first consider $A = \ell_1(G)^{**}$ for an arbitrary discrete group $G$. Denote by $B_{C^r}(A^*)$ the algebra of all bounded linear right $C^r$-module maps on $A^*$, where $C$ is a (closed) subalgebra of $A$, and by $B_{C^r}(A^*)$ the subalgebra of $w^*$-continuous module maps.
Since $\mathcal{A}$ is unital, by [8] Proposition 4.1, the map $R_N$ induces an isometric algebra isomorphism

$$\mathcal{A}^{**} = (A^* \cdot \mathcal{A})^* \cong B_{\mathcal{A},r}(\mathcal{A}^*)$$

So we need to find a surjective map

$$(1) \quad \Phi \in B_{\ell_1(G),r}(\ell_1(G)^{**}) \setminus B_{\ell_1(G)^{**},r}(\ell_1(G)^{**})$$

(which, as we shall see, exists whenever $G$ is infinite, countable, and abelian). Note that, as is easily checked, for any $N \in \ell_1(G)^{****}$ the map $R_N$ is in fact a right $\ell_1(G)^{**}$-module homomorphism, i.e., belongs to $B_{\ell_1(G)^{**},r}(\ell_1(G)^{**})$.

For later use we shall give below some background on semigroup compactifications; for a full meal, we refer the reader to [5] and [1]. Recall that the spectrum of the commutative $C^*$-algebra $\ell_\infty(G)$ is the Stone–Čech compactification $\beta G$, which is a compact right topological semigroup with the first Arens product (inherited from $\ell_1(G)^{**}$). Denoting the closure in $\beta G$ of a subset $S$ of $G$ by $\overline{S}$, one defines the growth or remainder of $S$ as

$$S^* := \overline{S} \cap (\beta G \setminus G);$$

then $S^*$ is compact, and $G^*$ is a compact right topological semigroup. An element $s \in \beta G$ is called left cancellable if left multiplication by $s$ is injective on $\beta G$.

For $m \in \mathcal{A}$, write $\lambda_m$ for left multiplication by $m$ in $\mathcal{A}$. We note the following.

**Lemma 2.** If $m$ is a left cancellable element of $\beta G$, then $\lambda_m : \ell_1(G)^{**} \to \ell_1(G)^{**}$ is an isometry.

**Proof.** This can be shown by transferring verbatim the argument given in [1] Proposition 4.4 for right multiplication to our situation (the compact right topological semigroup $V$ in [1] Proposition 4.4 is $\beta G$ in our case). $\square$

We are now ready to derive Theorem [1]

**Proof.** Let $G$ be an infinite, countable, discrete, abelian group, and recall that we write $\mathcal{A} = \ell_1(G)^{**}$ (endowed with the first Arens product). Then, by [5] Section 8.4, there exists $m \in G^* \subseteq \mathcal{A}$ such that $m$ is left cancellable in $\beta G$. In view of Lemma [2], $\Phi := \lambda_m \in B^{**}(\mathcal{A}^*)$ is surjective. To establish (1), we need to show that

1. $\Phi$ is a right $\ell_1(G)$-module map,
2. $\Phi$ is not a right $\ell_1(G)^{**}$-module map.

To see (i), let $H \in \mathcal{A}^*$, $a \in \ell_1(G) \subseteq \mathcal{A}$ and $b \in \mathcal{A}$. Then we have

$$\langle \Phi(H \cdot a), b \rangle = \langle H, a \ast m \ast b \rangle.$$

But $a \in \ell_1(G)$, and since $G$ is abelian, $\ell_1(G)$ is contained in the algebraic centre of $\ell_1(G)^{**}$. Hence $a$ commutes with $m$, and we get

$$\langle \Phi(H \cdot a), b \rangle = \langle H, m \ast a \ast b \rangle = \langle \Phi(H) \cdot a, b \rangle,$$

as desired.

As for (ii), assume towards a contradiction that $\Phi$ is a right $\mathcal{A}$-module map. Then we obtain for all $H \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$ that

$$\langle H, a \ast m \ast b \rangle = \langle \Phi(H \cdot a), b \rangle = \langle \Phi(H) \cdot a, b \rangle = \langle H, m \ast a \ast b \rangle.$$

Thus, we have $a \ast m \ast b = m \ast a \ast b$ for all $a, b \in \mathcal{A}$, whence (choosing $b = \delta_e$, the unit of $\mathcal{A}$) we see that $m$ lies in the algebraic centre $Z_\alpha(\mathcal{A})$ of $\mathcal{A} = \ell_1(G)^{**}$. But since $\ell_1(G)$ is commutative, $Z_\alpha(\mathcal{A})$ equals the first topological centre $Z^1(\mathcal{A})$ of $\mathcal{A}$, i.e., the
set of all elements $n \in A$ such that $\lambda_n$ is $w^*$-continuous on $A$. By [7, Theorem 1], $Z_1^*(\ell_1(G)^{**}) = \ell_1(G)$. Hence we obtain that $m \in \ell_1(G)$, which contradicts the fact that $m \in G^*$.

\[ \square \]

Remark 3. The motivation for question (\ast), i.e., [3, Problem 4.1], stems from its link to a long-standing open problem raised by Duncan and Hosseiniun [2, p. 323] asking if, for a locally compact group $G$, the natural involution on $L_1(G)$ extends to an involution on its bidual. Indeed, Farhadi–Ghahramani show in [3, Theorem 4.2] that the latter has a negative answer for all infinite, locally compact groups $G$ satisfying (\ast). Note that the existence of an extension of the involution forces $G$ to be discrete (cf. [4, Theorem 2] or [3, Proposition 3.1]). Now, on the one hand, our Theorem 1 shows that there are discrete groups which violate (\ast). On the other hand, [3, Theorem 3.2(a)] answers Duncan–Hosseiniun’s question in the negative for all (discrete) groups which admit an infinite, amenable subgroup. It follows from work by Ol’shanski˘ı [9] that there are infinite, countable, discrete groups without infinite, amenable subgroups (cf. [6, Remark 11]). So, the Duncan–Hosseiniun problem remains open for this class of groups, and the approach proposed through [3, Theorem 4.2] may very well prove fruitful since these groups, being of course radically different from the ones appearing in Theorem 1, may satisfy (\ast).

References


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