STRICTLY SINGULAR OPERATORS ON $L_p$ SPACES AND INTERPOLATION

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Abstract. We study the class $V_p$ of strictly singular non-compact operators on $L_p$ spaces. This allows us to obtain interpolation results for strictly singular operators on $L_p$ spaces. Given $1 \leq p < q \leq \infty$, it is shown that if an operator $T$ bounded on $L_p$ and $L_q$ is strictly singular on $L_r$ for some $p \leq r \leq q$, then it is compact on $L_s$ for every $p < s < q$.

1. Introduction

Given Banach spaces $E$ and $F$, a bounded operator $T : E \to F$ is strictly singular (or Kato) if the restriction of $T$ to any infinite-dimensional subspace of $E$ is not an isomorphism. This class was introduced by T. Kato in [K] as an extension of compact operators and in connection with the perturbation theory of Fredholm operators. Strictly singular operators form a closed operator ideal which in certain aspects behaves in a different way compared to that of compact operators. Thus, in general, strictly singular operators are not stable under duality (cf. [P], [Whi]), they are not suitable for interpolation properties (cf. [H], [H]) and fail to have invariant subspaces ([R]).

However, in the setting of operators on $L_p$ spaces ($1 \leq p < \infty$) the behaviour of strictly singular operators is somehow closer to that of compact operators. For example, concerning endomorphisms on $L_p$ spaces, it is known that an operator $T : L_p \to L_p$ is strictly singular if and only if $T^* : L_p^* \to L_p^*$ is strictly singular. One implication of this result was given by V. Milman in [M], and it was completely proved by L. Weis in [W1]. This same fact for $L_1$ and $C(K)$ spaces was already known, since in these cases the class of strictly singular operators coincides with that of weakly compact ones (see [P]). Moreover, recall that the square of a strictly singular operator $T : L_p \to L_p$ is always a compact operator ([M]).

The aim of this paper is to study interpolation properties of strictly singular operators on $L_p$ spaces ($1 \leq p \leq \infty$). In particular, we present an extension of Krasnoselskii’s result [Kr] on interpolation of compact operators on $L_p$ spaces. To
this end, we first study the properties of the class $V_p$ of strictly singular non-compact operators on an $L_p$ space.

As a starting point, we will show that for $p > 2$ strictly singular non-compact operators behave “locally” as inclusions $i_{2,p} : \ell_2 \hookrightarrow \ell_p$, and from this fact some structural properties of the operator class $V_p$ will follow. Thus, in Section 4 we give a version of Kato’s result that $S(L_2) = K(L_2)$ for operators which are simultaneously bounded on different $L_p$ spaces (see Corollary 3.3). This is deduced from an extrapolation type result for strict singularity (see Theorem 3.3). The connection of an operator $T \in V_p$ with boundedness in the scale of $L_q$ spaces will also be explored (see Theorem 3.7).

In Section 4 we present an extension of Krasnoselskii’s result on interpolation of compact operators on $L_p$ spaces to strictly singular operators. Namely, we will show that if an operator is strictly singular in $L_r$ and bounded in some $L_s$ for $1 \leq r, s \leq \infty$, then the operator is compact in $L_p$ for every $p$ strictly between $r$ and $s$ (Theorem 4.2).

2. Preliminaries

In this section we fix the terminology and include some results that will be needed later. A bounded operator $T : E \to F$ between Banach spaces is called strictly singular if the restriction of $T$ to any (closed) infinite-dimensional subspace of $E$ is not an isomorphism. Strictly singular operators form a closed operator ideal that contains the ideal of compact operators. It is well-known that an operator $T : E \to F$ is strictly singular if and only if for every infinite-dimensional subspace $X$ of $E$, there exists another infinite-dimensional subspace $Y$ of $X$ such that the restriction $T_{|Y}$ is compact (cf. [LT, Prop. 2.c.4]).

We denote by $S(E)$ and $K(E)$ the sets of strictly singular and compact operators on a Banach space $E$. It holds that $K(E) \subset S(E) \subset L(E)$. In the case when $E$ is a sequence space $\ell_p$ ($1 \leq p < \infty$) or $c_0$, it is well-known that the space of all bounded operators $L(E)$ contains only a unique non-trivial closed two-sided ideal $([C], \text{GME})$. From this it follows that $K(\ell_p) = S(\ell_p)$ and $K(c_0) = S(c_0)$. The simplest examples of strictly singular non-compact operators are the formal inclusion mappings $i_{p,q} : \ell_p \hookrightarrow \ell_q$, with $p < q$.

Given $1 \leq p \leq \infty$, let $L_p$ denote the function space $L_p[0,1]$ with the Lebesgue measure $\mu$. In [K] Kato showed that for Hilbert spaces strictly singular and compact operators coincide, so $S(L_2) = K(L_2)$ (this also follows from results about ideals in $L(\ell_2)$ given in [C]). However, for every $p \neq 2$ it holds that $S(L_p) \neq K(L_p)$ ([GME]). We will denote by $V_p$ the class $S(L_p) \setminus K(L_p)$. Let us recall some well-known examples of operators in the class $V_p$ for $1 \leq p \neq 2 \leq \infty$.

Let $1 \leq q < 2$. Consider a complemented subspace $F_q$ of $L_q$ isomorphic to $\ell_q$ (generated by disjointly supported functions), and denote by $P_q$ a projection from $L_q$ onto $F_q$. Let us take the inclusion $i_{q,2}$ and the operator $Q$ defined by $Qx = \sum_{k=1}^{\infty} x_k r_k(t)$, for $x \in \ell_2$, where $(r_k)$ are the Rademacher functions $(r_k(t) = \text{sign} \sin 2^k \pi t)$. By Khintchine’s inequality, the operator $Q$ is an isomorphic embedding of $\ell_2$ into $L_p$ for every $1 \leq p < \infty$. Clearly, the operator $A_q : L_q \to L_q$ given by

$$A_q = Q i_{q,2} P_q$$

belongs to $V_q$. 


Now, let $2 < p < \infty$. It is well-known that the orthogonal projection $R$ on the span $[r_k]$ acts from $L_p$ ($p > 1$) into $L_2$ which is isomorphic to $\ell_2$. Consider the inclusion $i_{2,p}$ and denote by $j_p$ an isometric embedding of $\ell_p$ into $L_p$. Then the operator $B_p : L_p \to L_p$
\begin{equation}
B_p = j_p i_{2,p} R
\end{equation}
belongs to $V_p$. Note that the operator $A_q \in \mathcal{L}(L_r)$ for every $r \in [q, \infty)$ and the operator $B_p \in \mathcal{L}(L_r)$ for every $r \in (1,p]$.

There also exist strictly singular and non-compact operators in $L_\infty$ and $C(0,1)$. For instance, consider the operator $T : L_\infty \to L_\infty$ given by $T = JR$, where $J : L_2 \to L_\infty$ is an isometric embedding and $R : L_\infty \hookrightarrow L_2$ is the formal inclusion.

Given $1 \leq p < \infty$, for each $\varepsilon > 0$ we will consider the Kadec-Pełczyński sets $\{K_P\}$:

$$M_p(\varepsilon) = \{f \in L_p : \mu(\{t : |f(t)| \geq \varepsilon\|f\|_p\}) \geq \varepsilon\}.$$  

**Theorem 2.1.** Let $X$ be a subspace of $L_p$ ($1 < p < \infty$). The following alternative holds:

1. If $X \subset M_p(\varepsilon)$ for some $\varepsilon > 0$, then the inclusion $i_X$ of $L_p$ into $L_1$ restricted to $X$ is an isomorphism (in this case we say that $X$ is a strongly embedded subspace).
2. If $X \not\subset M_p(\varepsilon)$ for any $\varepsilon > 0$, then $X$ contains an almost disjoint normalized sequence; that is, there exists a normalized sequence $(x_n) \subset X$ such that $x_n = u_n + v_n$, where $(u_n)$ is a disjoint sequence, $v_n \to 0$ in $L_p$, and $|u_n| \wedge |v_n| = 0$. In particular, $(x_n)$ can be taken to be equivalent to the unit vector basis of $\ell_p$.

The next result, due to L. Dor [D] (cf. [AO] Theorem 44]), will be used in the proof of Theorem 3.3.

**Theorem 2.2.** Let $1 \leq p \neq 2 < \infty$, $0 < \theta \leq 1$, and $(f_i)_{i=1}^\infty$ in $L_p$. Assume that either:

1. $1 \leq p < 2$, $\|f_i\| \leq 1$ for all $i$, and $\|\sum_{i=1}^n a_i f_i\|_p \geq \theta (\sum_{i=1}^n |a_i|^p)^{1/p}$ for scalars $(a_i)_{i=1}^n$ and every $n \in \mathbb{N}$, or
2. $2 < p < \infty$, $\|f_i\| \geq 1$ for all $i$, and $\|\sum_{i=1}^n a_i f_i\|_p \leq \theta^{-1} (\sum_{i=1}^n |a_i|^p)^{1/p}$ for scalars $(a_i)_{i=1}^n$ and every $n \in \mathbb{N}$.

Then there exist disjoint measurable sets $(A_i)_{i=1}^\infty$ in $[0,1]$ such that

$$\|f_i \chi_{A_i}\|_p \geq \theta^{2/(p-2)}.$$  

A classical interpolation result for compact operators on $L_p$ spaces proved by Krasnoselskii is the following [Kr] (see also [KZPS]).

**Theorem 2.3.** Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. If $T : L_{p_0} \to L_{q_0}$ is a compact operator and $T : L_{p_1} \to L_{q_1}$ is bounded, then $T : L_{p_0} \to L_{q_0}$ is compact, where $\frac{1}{p_0} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_0} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, for every $\theta \in (0,1)$.

An analogous result for interpolating strictly singular operators does not hold in general. Indeed, consider the formal inclusion $i : L_\infty \to L_1$ which is strictly singular by a result of Grothendieck (cf. [Rn] Theorem 5.2) and bounded as an operator $i : L_1 \to L_1$. However, for $1 < p < \infty$, $i : L_p \to L_1$ is not strictly singular (since it is an isomorphism on the span of the Rademacher functions). Apparently, positive results for one-sided interpolation of strictly singular operators are only
known in the degenerated case when the initial couple reduces to one single space (see [3, Prop. 2.1], [CMM], [H, Prop. 1.6]).

Recall that an operator $T$ between Banach spaces is compact if and only if its adjoint $T^*$ is compact (Schauder’s theorem). This fact is not true in general for strictly singular operators (cf. [P], [Whi]). However, for endomorphisms on $L_p$ spaces we have the following fact due to V. Milman [M] and L. Weis [W1]:

**Theorem 2.4.** Let $1 \leq p \leq \infty$. An operator $T : L_p \to L_p$ is strictly singular if and only if $T^* : L_p^* \to L_p^*$ is strictly singular.

We refer the reader to the monographs [AA], [G] and [LT] for unexplained terminologies.

### 3. Strictly singular non-compact operators

Let us start with some preliminary results about the operator sets $V_p = S(L_p) \setminus K(L_p)$, for $1 \leq p \neq 2 < \infty$ (recall that $V_2 = \emptyset$). Notice that unlike $K(L_p)$, the space $S(L_p)$ is not separable; thus neither is $V_p$.

**Lemma 3.1.** Let $2 < p < \infty$. If an operator $T \in V_p$, then there exists a Hilbertian subspace $H$ of $L_p$ which is complemented, such that the restriction $T|_H$ behaves, up to equivalence, like the inclusion $j_{p,2,p}$.

**Proof.** We proceed as in [LST, Lemma 2.10]. Since $T \notin K(L_p)$, there exists a sequence $(x_k)$ in $L_p$, such that $\|x_k\|_p = 1$, $x_k \overset{w}{\to} 0$ and $\|Tx_k\|_p \geq \varepsilon$ for some $\varepsilon > 0$.

By the Kadeč-Pelczynski theorem [KP] every weakly null seminormalized sequence in $L_p$ contains a subsequence equivalent to the unit vector basis of $\ell_2$ or $\ell_p$. Applying this theorem to the sequences $(x_k)$ and $(Tx_k)$, we may suppose that $(x_k)$ (resp. $(Tx_k)$) is equivalent to the unit vector basis of $\ell_q$ (resp. $\ell_r$) where $q, r \in \{2, p\}$.

The cases (i) $q = r = 2$, (ii) $q = r = p$, and (iii) $q = p$, $r = 2$ are impossible. Indeed, the restriction of $T$ on the subspace $[x_k]$ is an isomorphism in the cases (i) or (ii). This contradicts the assumption that $T \in S(L_p)$. While, if the case (iii) holds, then we clearly have

$$\left\| \sum_{k=1}^{n} x_k \right\|_p \approx n^{\frac{1}{p}} \quad \text{and} \quad \left\| \sum_{k=1}^{n} Tx_k \right\|_p \approx n^{\frac{1}{q}};$$

where the sign $\approx$ means two-side estimates with constants which do not depend on $n$. Then it follows that

$$\left\| \frac{T \left( \sum_{k=1}^{n} x_k \right)}{\| \sum_{k=1}^{n} x_k \|_p} \right\|_p \approx n^{\frac{1}{q} - \frac{1}{p}} \to \infty$$

as $n \to \infty$, which contradicts that $T$ is bounded in $L_p$.

Hence, $(x_k)$ is equivalent to the unit vector basis of $\ell_2$ and $(Tx_k)$ is equivalent to the unit vector basis of $\ell_p$. And since any Hilbertian subspace in $L_p$ is complemented if $2 < p < \infty$ ([KP]), we have that $[x_n]$ is complemented in $L_p$. \(\square\)

We need an improvement of Lemma 3.1. Recall that two measurable functions $f$ and $g$ are equi-measurable if for every $-\infty < s < \infty$ the distribution functions satisfy

$$\mu(\{t : f(t) > s\}) = \mu(\{t : g(t) > s\}).$$
Lemma 3.2. Let $2 < p < \infty$. If an operator $T$ belongs to $V_p$, then there exists a sequence $(y_k)$ in $L_p$ with $\|y_k\|_p \leq 1$ such that $(y_k)$ is equivalent to the unit vector basis of $\ell_2$, the sequence $(|y_k|)$ is equi-measurable, and $(Ty_k)$ is equivalent to the unit vector basis of $\ell_p$.

Proof. By Lemma 3.1 there exists a sequence $(x_n)$ in $L_p$, $\|x_n\|_p = 1$, $x_n \xrightarrow{w} 0$ such that $(x_n)$ is equivalent to the unit vector basis of $\ell_2$ and $(Tx_n)$ is equivalent to the unit vector basis of $\ell_p$. Denote by $K$ the basis constant of the sequence $(Tx_n)$. Using [SS] Theorem 3.2 we can choose a subsequence $(x_{n_k})$ such that $x_{n_k} = u_k + v_k + w_k$, where

1. $|u_k|$ are equi-measurable; i.e. there exists a function $u$ equi-measurable with $|u_k|$ for any $k \in \mathbb{N}$ and $\|u\|_p \leq 1$. Moreover, $u_k \xrightarrow{w} 0$;
2. $\text{supp } v_i \cap \text{supp } v_j = \emptyset$ for any $i \neq j$ in $\mathbb{N}$, with $\|v_k\|_p \leq 2$, and $v_k \xrightarrow{w} 0$;
3. $\lim_{k \to \infty} \|w_k\|_p = 0$.

It holds that $\lim_{k \to \infty} \|Ty_k\|_p = 0$. Indeed, otherwise we can select a subsequence $(v_{i_k})$ such that $\inf_k \|Tv_{i_k}\|_p > 0$. By the Kadeč-Pelczyński theorem [KP] some subsequence of $(Tv_{i_k})$ is equivalent to the unit vector basis of $\ell_2$ or $\ell_p$. Both cases are impossible because $(v_{i_k})$ is equivalent to the unit vector basis of $\ell_p$ (see Lemma 3.1).

Now, since $\lim_{k \to \infty} \|w_k\|_p = 0$ we have that $\lim_{k \to \infty} \|Tv_k\|_p = 0$, and so $\lim_{k \to \infty} (\|Tv_k\|_p + \|Tw_k\|_p) = 0$.

Thus, we can find an increasing sequence of integers $(j_k)$ such that $\|Tv_{j_k}\|_p + \|Tw_{j_k}\|_p < \frac{1}{2^{p+K}}$. Thus

$$\sum_{k=1}^{\infty} \|Tx_{n_{j_k}} - T_{j_k}\|_p \leq \sum_{k=1}^{\infty} (\|Tv_{j_k}\|_p + \|Tw_{j_k}\|_p) < \frac{1}{2^{p+K}}.$$ 

Hence, by the stability basis result [LT Thm. 1.a.9], it follows that $(Tv_{j_k})$ is also equivalent to the unit vector basis of $\ell_p$. And, since $u_k \xrightarrow{w} 0$ and $T \in S(L_p)$, we must have that $(u_{j_k})$ is equivalent to the unit vector basis of $\ell_2$. \qed

We can present now an extrapolation type result for strict singularity:

Theorem 3.3. Let $1 < q < r < \infty$. If an operator $T$ is bounded in $L_q$ and $L_r$, and strictly singular in $L_p$ for some $p \in (q, r)$, then $T$ is compact in $L_s$ for all $s \in (q, r)$.

Proof. Suppose the contrary. By Krasnoselskii’s Theorem 2.3 we deduce that $T$ is not compact in $L_s$ for any $s \in (q, r)$. In particular, $T$ is not compact in $L_p$, and so $T \in V_p$.

Without loss of generality we can assume that $p > 2$. Indeed, for $p = 2$ the result follows directly from the fact that $S(L_2) = K(L_2)$, while for $p < 2$ it follows from the dual counterpart for the adjoint operator $T^*$, since by Schauder’s theorem and [WI], compact and strictly singular operators on $L_p$ spaces are stable under taking adjoints.

Now, by Lemma 3.2 there exists a sequence $(y_k)$ in $L_p$ such that $(|y_k|)$ is equi-measurable and $(Ty_k)$ is equivalent to the unit vector basis of $\ell_p$. By Dor’s Theorem 2.2 there exist a constant $c > 0$ and a sequence of disjoint measurable sets $A_k \subset [0, 1]$ such that $\|(Ty_k)\chi_{A_k}\|_p \geq c$ for each $k \in \mathbb{N}$.
Since for every $x \in L_p$ we have
\[
\lim_{\varepsilon \to 0} \sup_{\mu(A) \leq \varepsilon} \|x_A\|_p = 0,
\]
and using the fact that $(|y_k|)$ is equi-measurable, we can find $\varepsilon > 0$ such that
\[
\|y_k \chi_A\|_p \leq \frac{c}{2\|T\|_p}
\]
for every $A \subset [0, 1]$ with $\mu(A) \leq \varepsilon$, and for every $k \in \mathbb{N}$. Moreover, the equi-measurability of $(|y_k|)$ also implies the existence of measurable subsets $B_k \subset [0, 1]$ with $\mu(B_k) \geq 1 - \varepsilon$, such that $y_k \chi_{B_k} \in L_\infty$ and $\|y_k \chi_{B_k}\|_\infty \leq y_1'(\varepsilon)$ for every $k \in \mathbb{N}$. Now, using Hölder’s inequality and the fact that $\|y_k \chi_{B_k}\|_r \leq \|y_k \chi_{B_k}\|_\infty \leq y_1'(\varepsilon)$ we have
\[
\|T(y_k \chi_{A_k})\|_p \leq \|(T(y_k \chi_{B_k})) \chi_{A_k}\|_p + \|(T(y_k \chi_{[0,1]\setminus B_k}))\|_p \\
\leq \|(T(y_k \chi_{B_k}))\|_r \|\chi_{A_k}\|_{\frac{r}{r-1}} + \|\|T\|_p\|y_k \chi_{[0,1]\setminus B_k}\|_p \\
\leq \|\|\|\|T\|_r y_1'(\varepsilon) \mu(A_k)\left(\frac{r}{r-1}\right) + \|\|T\|_p c \frac{\|T\|_p}{2\|T\|_p}.
\]
And, since $c \leq \|T y_k \chi_{A_k}\|_p$ and $\mu(A_k) \to 0$ as $k \to \infty$, we obtain $2c \leq c$, which is a contradiction. \hfill \Box

The following corollary can be regarded as a version of Kato’s result that $K(L_2) = \mathcal{S}(L_2)$ for operators that are simultaneously bounded on different $L_p$ spaces.

**Corollary 3.4.** Let $1 < q < r < \infty$, and let $T$ be an operator bounded in $L_q$ and $L_r$. The following statements are equivalent:

(i) $T \in K(L_p)$ for some $p \in (q, r)$;  
(ii) $T \in \mathcal{S}(L_p)$ for every $p \in (q, r)$;  
(iii) $T \in K(L_p)$ for every $p \in (q, r)$;  
(iv) $T \in \mathcal{S}(L_p)$ for some $p \in (q, r)$.

**Proof.** (i) $\Rightarrow$ (ii) follows from Krasnoselskii’s Theorem \[23\] (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial. (iv) $\Rightarrow$ (i) follows from Theorem 3.3 \hfill \Box

Notice that these facts are no longer true for operators on $L_p$ spaces of infinite measure:

**Example 3.1.** There exists a strictly singular non-compact operator $T$ on $L_p(0, \infty)$ for every $1 \leq p < 2$. Similarly, there exists a strictly singular non-compact operator $S$ on $L_p(0, \infty)$ for every $2 < p < \infty$.

**Proof.** For $1 \leq p < 2$, let $P : L_p(0, \infty) \to \ell_p$ be the operator given by $P(f) = (\int_{n-1}^{n} f\,d\mu)_{n=1}^\infty$, and let $Q : \ell_2 \to L_p(0, \infty)$ be the isomorphic embedding via the Rademacher functions in $[0, 1]$. Then, $T = Q i_{p,2} P$ is bounded on $L_p(0, \infty)$ for every $1 \leq p \leq 2$. Moreover, $T$ is strictly singular for $1 \leq p < 2$ since it factors through the inclusion $i_{p,2}$, but it is not compact on any $L_p(0, \infty)$ since the sequence $(\chi_{[n-1,n]})$ has norm one in every $L_p(0, \infty)$ and $T(\chi_{[n-1,n]}) = t_n$ does not have a convergent subsequence.

Similarly, for $2 < p < \infty$, we consider $R : L_p(0, \infty) \to \ell_2$ the projection onto the span of the Rademacher functions on $[0, 1]$, and $J : \ell_p \to L_p(0, \infty)$ given by $J(a_n) = \sum_{n=1}^{\infty} a_n \chi_{[n-1,n]}$. Clearly, the operator $S = J i_{2,p} R$ is strictly singular and not compact on $L_p(0, \infty)$ for every $2 < p < \infty$. \hfill \Box
As a consequence of Theorem 3.3, we can obtain a result of V. Caselles and M. González [CG] for regular operators (i.e. those which can be written as a difference of positive operators):

**Corollary 3.5.** Let $1 < p < \infty$, and let $T : L_p \to L_p$ be a regular operator. Then $T \in S(L_p)$ if and only if $T \in K(L_p)$.

**Proof.** Since $T$ is regular, by a result of Weis [W2, Theorem 2.1], there exists a positive isometry $J : L_p \to L_p$, such that the operator $JTJ^{-1} : L_q \to L_q$ is bounded for every $1 \leq q \leq \infty$. Hence, since $JTJ^{-1} : L_p \to L_p$ is strictly singular, by Theorem 3.3 we have that $JTJ^{-1}$ belongs to $K(L_p)$. Now, since $J$ is an isometry we have that $T$ belongs to $K(L_p)$.

Notice that this result is no longer true for $p = 1$. Indeed, let $T : L_1 \to L_1$ be given by $T = Q_{1,2}P$, where $P$ is a projection onto some subspace isomorphic to $\ell_1$ and $Q : \ell_2 \to L_1$ is the isomorphic embedding via the Rademacher functions. Clearly, $T$ belongs to the set $V_1$ and is a regular operator like every operator in $L_1$ (cf. [AA] Theorem 3.9).

It was proved by V. Milman in [M] that the composition of two strictly singular operators on $L_p$ is compact. We present below a converse to this result.

**Proposition 3.6.** Let $1 < p \neq 2 < \infty$. Given an operator $R \in L(L_p)$, it holds that $R \in S(L_p)$ if and only if $RT$ and $TR$ are compact for every $T \in S(L_p)$.

**Proof.** The “if” part was proved in [M]. Suppose $p > 2$ and $R \notin S(L_p)$. Then there exists a subspace $Q$ of $L_p$, such that the restriction $R|Q$ is an isomorphism, and by Theorem 2.1 we can suppose that $Q$ is isomorphic to $\ell_2$ or $\ell_p$ and complemented in $L_p$.

(1) If $Q \approx \ell_2$, then we can consider an operator $T \in L(L_p)$ defined as follows. Since $R(Q)$ is isomorphic to $\ell_2$ and complemented, there is a projection $P : L_p \to R(Q)$. Now, take an isomorphic embedding $J : \ell_p \to L_p$ and define $T = J_{i,2,p}P$. Clearly, there exists a sequence $(x_n)$ in $Q$, equivalent to the unit vector basis to $\ell_2$, such that $TR(x_n)$ does not have any convergent subsequence. Hence, $TR$ is not compact, which is a contradiction.

(2) If $Q \approx \ell_p$, then we consider a projection $P : L_p \to H$ onto some Hilbert subspace of $L_p$, and the isomorphic embedding $J$ of $\ell_p$ into $Q \subset L_p$. Hence, if we consider the operator $T = J_{i,2,p}P$, then $RT$ is not compact, which is again a contradiction.

This proves the statement for $p > 2$. By duality arguments (Theorem 2.1) the same fact is proved for $p < 2$.

Note that the assumption in Proposition 3.6 that $RT$ and $TR$ are compact cannot be relaxed to only one condition $RT$ (or respectively $TR$) being compact for every $T \in S(L_p)$.

Let $1 \leq p \neq q \leq \infty$, and let $T : L_p \to L_p$ be a bounded operator. If $q > p$, then $T$ is also defined acting from $L_q$. If $q < p$, then $T$ is defined on a dense subset of $L_q$. Thus, in both cases we can consider the quantity $\|T\|_q$ taking values in $[0, +\infty]$, and we can analyze the boundedness or unboundedness of $T$ from $L_q$ to $L_q$. Let us denote

$$O(T) = \{q \in [1, +\infty] : T \text{ is bounded in } L_q\}.$$  

It follows from M. Riesz's interpolation result that $O(T)$ is a convex subset of $[1, +\infty]$, which may or may not contain its endpoints.
Theorem 3.7. Let $1 < p < \infty$. If an operator $T \in V_p$, then $p$ is an endpoint of $O(T)$. Moreover, $p$ is the right (respectively left) endpoint of $O(T)$ when $p > 2$ (resp. $p < 2$).

Proof. It follows from Theorem 3.3 that $p$ is always an endpoint of $O(T)$.

First consider the case $p > 2$. By Lemma 3.2 there exists a sequence $(x_k)$ in $L_p$ which is equivalent to the unit vector basis of $\ell_2$ and with $(|x_k|)$ equi-measurable, such that $(Tx_k)$ is equivalent to the unit vector basis of $\ell_p$. Actually, since $(|x_k|)$ is equi-measurable and $\|T x_k\|_p \geq \alpha$ for some $\alpha > 0$, we can truncate $(x_k)$ considering $y_k = x_k \chi_{\{|x_k| \leq M\}}$. Since

$$\lim_{M \to \infty} \sup_k \|x_k \chi_{\{|x_k| > M\}}\|_p = 0,$$

then for large enough $M$, we have $\|Ty_k\|_p \geq \frac{\alpha}{2}$ for all $k \in \mathbb{N}$. Now, as in the proof of Lemma 3.1 by Theorem 2.1, we have that the sequence $(y_k)$ is equivalent to the unit vector basis of $\ell_2$ and $(Ty_k)$ is equivalent to the unit vector basis of $\ell_p$.

Now, suppose that $p$ is not the right endpoint of $O(T)$; that is, $T : L_q \to L_q$ is also bounded for some $q > p$. Since $(y_k)$ is also in $L_q$, and $\|Ty_k\|_q \geq \|Ty_k\|_p \geq \frac{\alpha}{2}$, by Theorem 2.1 we have that $(y_k)$ is equivalent in $L_q$ to the unit vector basis of $\ell_2$ and $(Ty_k)$ is equivalent to the unit vector basis of $\ell_q$. However, this yields

$$C_1 n^{\frac{1}{q}} \leq \left\| \sum_{k=1}^{n} Ty_k \right\|_p \leq \left\| \sum_{k=1}^{n} Ty_k \right\|_q \leq C_2 n^{\frac{1}{p}},$$

for certain constants $C_1, C_2 > 0$ and every $n \in \mathbb{N}$. This is a contradiction since $q > p$.

The case when $p < 2$ follows by duality. Indeed, if $T \in V_p$, then by Theorem 2.3 we have $T^* \in V_{p'}$, where $\frac{1}{p'} + \frac{1}{p} = 1$. Since $p' > 2$, by the first part of the proof we have that $p'$ is the right endpoint of $O(T^*)$, which means that $p$ is the left endpoint of $O(T)$. This finishes the proof. \(\square\)

The examples of operators in $V_p$ presented above always depend on the scalar $p$. The following result explains this phenomenon.

Proposition 3.8. Let $1 < q < p < \infty$. The set $V_q \cap V_p$ is not empty if and only if $q < 2 < p$.

Proof. Let $1 < q < 2 < p < \infty$. Let us consider the operators $A_q$ and $B_p$, defined above in (11) and (12). Also, consider the following operators acting on functions on $[0, 1]$: 

$$\begin{cases}
  U x(t) = x(2t), & 0 \leq t \leq \frac{1}{2}, \\
  W x(t) = x(2t - 1), & \frac{1}{2} < t \leq 1.
\end{cases}$$

Then the operators $UA_q U^{-1}$ and $WB_p W^{-1}$ act in the corresponding function spaces on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ respectively. Given a measurable function $x$ on $[0, 1]$, denote $x = y + z$, where $y$ and $z$ are the restriction of $x$ on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, and define the operator

$$T_{p,q}(x) = UA_q U^{-1}(y) + WB_p W^{-1}(z).$$

Since $A_q \in \mathcal{L}(L_r)$ for any $r \in [q, \infty)$ and $B_p \in \mathcal{L}(L_r)$ for any $r \in (1, p)$, $T_{p,q} \in \mathcal{L}(L_r)$ for any $r \in [q, p]$. Moreover, $A_q \in V_q$ and $B_p \in V_p$ clearly imply that $T_{p,q} \in V_q \cap V_p$. 

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Let us prove the converse. If \( T \in V_p \) and \( 1 < p < 2 \), then by Theorem 3.7, \( T \) does not belong to \( \mathcal{L}(L_q) \). Similarly, if \( T \in V_q \) and \( q > 2 \), then, by Theorem 3.7, \( T \notin \mathcal{L}(L_p) \).

4. Interpolation of Strictly Singular Operators

Let us denote by \( P_A \) the operator of multiplication by the characteristic function of a measurable set \( A \), i.e., \( P_A x(t) = x(t) \chi_A(t) \). Notice that \( \|P_A\|_{L_p} = 1 \) for every \( A \subset [0,1] \) with positive measure and every \( 1 \leq p \leq \infty \).

**Proposition 4.1.** Let \( 1 \leq p \leq \infty \) and let \( T : L_p \to L_p \) be an operator which is not an isomorphism when restricted to any subspace isomorphic to \( \ell_p \) (\( c_0 \) when \( p = \infty \)). Then for every sequence of disjoint measurable sets \( (A_n) \) the following holds:

1. If \( 2 \leq p \leq \infty \), then \( \lim_{n \to \infty} ||TP_{A_n}||_{L_p} = 0 \).
2. If \( 1 \leq p \leq 2 \), then \( \lim_{n \to \infty} ||P_{A_n}T||_{L_p} = 0 \).

**Proof.** Let us first prove the case (1). Suppose the contrary. Then there exist \( \alpha > 0 \), \( x_n \in L_p \), and pairwise disjoint sets \( A_n \subset [0,1] \) such that \( \|x_n\|_{L_p} \leq 1 \), \( \text{supp}(x_n) \subset A_n \), and \( \|Tx_n\|_{L_p} \geq \alpha \) for every \( n \in \mathbb{N} \).

Let \( p = \infty \). As \( (x_n) \) is semi-normalized and disjoint, \( (x_n) \) is equivalent to the unit vector basis of \( c_0 \). In particular, \( (Tx_n) \) is weakly null and semi-normalized; hence it has a basic subsequence \( (Tx_{n_k}) \). This yields that there exist constants \( c, C \) such that for every scalar sequence \( (a_k)_{k=1}^{\infty} \) it holds that

\[
\frac{c}{1} \sup_{1 \leq k \leq n} |a_k| \leq \left\| \sum_{k=1}^{n} a_k Tx_{n_k} \right\|_{L_\infty} \leq \left\| T \right\| \left\| \sum_{k=1}^{n} a_k x_{n_k} \right\|_{L_\infty} \leq C \sup_{1 \leq k \leq n} |a_k|,
\]

which is a contradiction to the fact that \( T \) is not an isomorphism on any subspace isomorphic to \( c_0 \).

Similarly, if \( p = 2 \), both \( (x_n) \) and \( (Tx_n) \) are weakly null semi-normalized sequences; hence extracting subsequences we can assume that both are equivalent to the unit vector basis of \( \ell_2 \). Again we obtain a contradiction.

Now, suppose \( 2 < p < \infty \). In this case, \( (x_n) \) is equivalent to the unit vector basis of \( \ell_p \). And, since \( \alpha \leq \|Tx_n\|_{L_p} \leq ||T||_{L_p} \) for every \( n \in \mathbb{N} \) and \( Tx_n \to 0 \) weakly, we have, by [KP] Corollary 5), that there exists an increasing sequence \( (n_k) \subset \mathbb{N} \) such that \( (Tx_{n_k}) \) is equivalent to the unit vector basis of \( \ell_2 \) or \( \ell_p \). Both cases will lead to a contradiction. Indeed, in the first case we would have

\[
n^{\frac{1}{p^*}} \approx \left\| \sum_{k=1}^{n} Tx_{n_k} \right\|_{L_p} \leq ||T||_{L_p} \left\| \sum_{k=1}^{n} x_{n_k} \right\|_{L_p} \approx ||T||_{L_p} n^{\frac{1}{p^*}},
\]

which is impossible for large \( n \in \mathbb{N} \). In the second case, the sequences \( (Tx_{n_k}) \) and \( (x_{n_k}) \) are both equivalent to the unit vector basis of \( \ell_p \). Hence, the operator \( T \) is an isomorphism on the span \( [x_{n_k}] \) in contradiction with the assumption on \( T \). This finishes the proof of case (1).

To prove (2), we will proceed by duality. First, notice that for \( 1 \leq p \leq 2 \), if an operator \( T : L_p \to L_p \) is not an isomorphism on any subspace isomorphic to \( \ell_p \), then \( T^* : L_p^* \to L_p^* \) is not an isomorphism on a subspace isomorphic to \( \ell_p^* \). Indeed, suppose that \( T^* \) is invertible in a subspace \( X \) of \( L_p^* \) isomorphic to \( \ell_p^* \). Then as \( p \leq 2 \) it follows that \( X \) and \( T^*(X) \) are complemented and isomorphic to \( \ell_p^* \) [KP]. This implies that \( T^{**} \) is also invertible in a subspace isomorphic to \( \ell_p \). In the case \( 1 < p \),
since $T = T^{**}$, the claim is proved. Now, for $p = 1$ recall that if $T : L_1 \to L_1$ is not an isomorphism on a subspace isomorphic to $\ell_1$, then $T$ is weakly compact and in particular $T^{**}(L_1) \subseteq L_1$. This proves the claim.

Therefore, by the case (1), we get that $\lim_{n \to \infty} \| T^* P_{A_n} \|_{L_p} = 0$ for every disjoint sequence $(A_n)$ in $[0, 1]$. And, since $(P_A)^* = P_A$, we obtain

$$\lim_{n \to \infty} \| P_{A_n} T \|_{L_p} = \lim_{n \to \infty} \| T^* P_{A_n} \|_{L_p} = 0.$$  

\[ \Box \]

**Theorem 4.2.** Let $1 \leq r, s \leq \infty$, $r \neq s$, and $T$ be an operator bounded on $L_s$. If $T \in S(L_r)$, then $T \in K(L_p)$ for every $p$ between $r$ and $s$.

**Proof.** Let us prove first the case $r < \infty$. By Theorem 3.3 it is enough to show that $T \in S(L_p)$ for some $p$ strictly between $r$ and $s$. So, let us suppose that $T \notin S(L_p)$ for any $p \neq 2$. Thus, for every $p$ between $r$ and $s$, $T$ is an isomorphism on a subspace $\mathbb{X}_p$ of $L_p$, which, by [VII], can be taken to be isomorphic either to $\ell_2$ or $\ell_p$, with both subspaces $\mathbb{X}_p$ and $T(\mathbb{X}_p)$ complemented in $L_p$. We distinguish two cases:

(A) Suppose that for some $p$ the subspace $\mathbb{X}_p$ is isomorphic to $\ell_2$. Let us denote $X = \mathbb{X}_p$. Then, by Theorem 2.1 both $X$ and $T(X)$ are strongly embedded subspaces of $L_p$. Thus, we can distinguish two subcases:

1. If $r < p$, then $X$ and $T(X)$ are also closed subspaces of $L_r$ and isomorphic to $\ell_2$ in the norm of $L_r$. This gives a contradiction to the fact that $T \notin S(L_r)$.

2. If $r > p$, then, since $X$ and $T(X)$ are complemented in $L_p$, it follows that $T^* : L_{p'} \to L_{p'}$ ($\frac{1}{p} + \frac{1}{p'} = 1$) is an isomorphism on a complemented subspace $Z$ of $L_p'$ isomorphic to $\ell_2$. Using again Theorem 2.1 we have that $Z$ and $T^*(Z)$ must be strongly embedded in $L_{p'}$. Now since $r' < p'$, as in case (a), this yields that $T^* : L_{r'} \to L_{r'}$ ($\frac{1}{r} + \frac{1}{r'} = 1$) is also an isomorphism on a subspace isomorphic to $\ell_2$. Now, by [PR] Thm. 3.1, every such subspace contains another complemented subspace, so we get that $T^{**} = T : L_r \to L_r$ is an isomorphism on a subspace isomorphic to $\ell_2$.

This is a contradiction to the fact that $T \notin S(L_r)$.

(B) Otherwise, suppose that for every $p$ between $r$ and $s$ the subspace $\mathbb{X}_p$ is isomorphic to $\ell_p$. Then the subspaces $\mathbb{X}_p$ and $T(\mathbb{X}_p)$ are not included in $M_p(\varepsilon)$ for any $\varepsilon > 0$. Now, assume first $r > 2$, hence we can fix some $p > 2$ between $r$ and $s$. By Theorem 2.1 we can find a sequence $(x_n) \subset \mathbb{X}_p$, such that $\| x_n \|_{L_p} = 1$, $x_n = u_n + v_n$, where $(u_n)$ is a disjoint sequence in $L_p$ and $\lim_{n \to \infty} \| v_n \|_{L_p} = 0$. Hence, we can suppose that the operator $T$ is an isomorphism on the subspace $[u_k]$. In particular there exists a constant $c > 0$ such that $\| T(u_n) \|_{L_p} \geq c \| u_n \|_{L_p}$ for every $n \in \mathbb{N}$. Now, let us denote $A_n = \text{supp}(u_n)$ and let $\theta \in (0, 1)$ such that $\frac{1}{p} = \frac{1-\theta}{p_r} + \frac{\theta}{p_s}$. By the Riesz interpolation theorem we have that

$$\| T P_{A_n} \|_{L_p} \leq \| T P_{A_n} \|_{L_s}^{1-\theta} \| T P_{A_n} \|_{L_r}^\theta \leq \| T P_{A_n} \|_{L_r}^{1-\theta} \| T \|_{L_s}^\theta.$$  

Since $\lim_{n \to \infty} \mu(A_n) = 0$, we have, by Proposition 1.11 $\lim_{n \to \infty} \| T P_{A_n} \|_{L_s} = 0$. Therefore, $\lim_{n \to \infty} \| T P_{A_n} \|_{L_p} = 0$. However, we have that

$$\| T P_{A_n} \|_{L_p} \geq \frac{\| T P_{A_n}(u_n) \|_{L_p}}{\| u_n \|_{L_p}} = \frac{\| T(u_n) \|_{L_p}}{\| u_n \|_{L_p}} \geq c > 0,$$

which is a contradiction.
The proof when \( r < 2 \) is analogous. Indeed, in this case we can fix some \( p < 2 \), and by Theorem \( \text{[2.1]} \) we can find an almost disjoint normalized sequence \((y_n)\) in \( T(X_p)\); that is, \( y_n = u_n + v_n \), where \((u_n)\) is a disjoint sequence in \( L_p \), \( \lim_{n \to \infty} \|v_n\|_{L_p} = 0 \) and \( |u_n| \wedge |v_n| = 0 \) for every \( n \in \mathbb{N} \). Moreover, \( y_n = T(x_n) \) for some semi-normalized sequence \((x_n)\) in \( X_p \). As in the previous case, if we denote \( A_n = \text{supp}(u_n) \), then we have

\[
\|P_{A_n}T\|_{L_p} \geq \frac{\|P_{A_n}T(x_n)\|_{L_p}}{\|x_n\|_{L_p}} = \frac{\|u_n\|_{L_p}}{\|x_n\|_{L_p}} \geq \alpha
\]

for some \( \alpha > 0 \) and \( n \) large enough, because \( \|v_n\| \to 0 \). However, by the Riesz interpolation theorem, we have

\[
\|P_{A_n}T\|_{L_p} \leq \|P_{A_n}T\|_{L_r}^{1-\theta} \|P_{A_n}T\|_{L_q}^{\theta} \leq \|P_{A_n}T\|_{L_r}^{1-\theta} \|T\|_{L_{s'}}^{\theta},
\]

for the corresponding \( \theta \in (0, 1) \). Then apply Proposition \( \text{[4.1]} \) to conclude.

This finishes the proof for \( r < \infty \). The case \( r = \infty \) follows by duality. Indeed, if \( T : L_\infty \to L_\infty \) is strictly singular and bounded on \( L_s \) for some \( 1 < s < \infty \), then \( T^* : L_\infty^* \to L_\infty^* \) is strictly singular and bounded on \( L_s' \) (with \( \frac{1}{s} + \frac{1}{s'} = 1 \)). Therefore, we have

\[
T^*(L_1) = T^*(L_1') \subseteq T^*(L_s') \subseteq L_1'.
\]

In particular, the operator \( T^* : L_1 \to L_1 \) is also strictly singular. Now, by the previous part of the proof we conclude that \( T^* \in K(L_q) \) for every \( q \) between 1 and \( s' \). Hence, by Schauder’s theorem, the operator \( T \in K(L_p) \) for every \( s < p < \infty \). \( \square \)

References


