GLOBAL GORENSTEIN DIMENSIONS

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ABSTRACT. In this paper, we prove that the global Gorenstein projective dimension of a ring $R$ is equal to the global Gorenstein injective dimension of $R$ and that the global Gorenstein flat dimension of $R$ is smaller than the common value of the terms of this equality.

1. INTRODUCTION

Throughout this paper, $R$ denotes a non-trivial associative ring with identity and all modules are, if not specified otherwise, left $R$-modules. All the results, except Proposition 2.6, are formulated for left modules, and the corresponding results for right modules hold as well. For an $R$-module $M$, we use $\text{pd}_R(M)$, $\text{id}_R(M)$, and $\text{fd}_R(M)$ to denote, respectively, the classical projective, injective, and flat dimensions of $M$. We use $\text{l.gldim}(R)$ and $\text{r.gldim}(R)$ to denote, respectively, the classical left and right global dimensions of $R$, and $\text{w.gldim}(R)$ to denote the weak global dimension of $R$. Recall that the left finitistic projective dimension of $R$ is the quantity $\text{l.FPD}(R) = \sup \{\text{pd}_R(M) \mid M \text{ is an } R\text{-module with } \text{pd}_R(M) < \infty\}$.

Furthermore, we use $\text{Gpd}_R(M)$, $\text{Gid}_R(M)$, and $\text{Gfd}_R(M)$ to denote, respectively, the Gorenstein projective, injective, and flat dimensions of $M$ (see [3, 4, 8]).

The main result of this paper is an analog of a classical equality that is used to define the global dimension of $R$; see [12, Theorems 9 and 10]. For Noetherian rings the following theorem is proved in [4, Theorem 12.3.1].

**Theorem 1.1.** The following equality holds:

$$\sup \{\text{Gpd}_R(M) \mid M \text{ is an } R\text{-module}\} = \sup \{\text{Gid}_R(M) \mid M \text{ is an } R\text{-module}\}.$$  

We call the common value of the quantities in the theorem the **left Gorenstein global dimension of $R$** and denote it by $\text{l.Ggldim}(R)$. Similarly, we set $\text{l.wGgldim}(R) = \sup \{\text{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\}$ and call this quantity the **left weak Gorenstein global dimension of $R$**.

**Corollary 1.2.** The following inequalities hold:

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(1) \( \text{lwGgldim}(R) \leq \sup \{ \text{lgldim}(R), \text{rgGgldim}(R) \} \),
(2) \( \text{FPD}(R) \leq \text{lgldim}(R) \leq \text{lgldim}(R) \),
(3) \( \text{lwGgldim}(R) \leq \text{wgldim}(R) \).

Equalities hold in (2) and (3) if \( \text{wgldim}(R) < \infty \).

The theorem and its corollary are proved in Section 2.

2. Proofs of the main results

The proofs use the following results:

**Lemma 2.1.** If \( \sup \{ \text{Gpd}_R(M) \mid M \text{ is an } R\text{-module} \} < \infty \), then, for a positive integer \( n \), the following are equivalent:

1. \( \sup \{ \text{Gpd}_R(M) \mid M \text{ is an } R\text{-module} \} \leq n \),
2. \( \text{id}_R(P) \leq n \) for every \( R\text{-module} P \) with finite projective dimension.

**Proof.** Use [6, Theorem 2.20] and [12, Theorem 9.8]. \( \square \)

The proof of the main theorem depends on the notions of strong Gorenstein projectivity and injectivity, which were introduced in [1] as follows:

**Definition 2.2 ([1, Definition 2.1]).** An \( R\text{-module} M \) is called strongly Gorenstein projective if there exists an exact sequence of projective \( R\text{-modules} 
\[ P = \cdots \to P \to P \to P \to \cdots \]

such that \( M \cong \ker f \) and such that \( \text{Hom}_R(\cdot, Q) \) leaves the sequence \( P \) exact whenever \( Q \) is a projective \( R\text{-module} \).

Strongly Gorenstein injective modules are defined dually.

**Remark 2.3.** It is easy to see that an \( R\text{-module} M \) is strongly Gorenstein projective if and only if there exists a short exact sequence of \( R\text{-modules} 0 \to M \to P \to M \to 0 \), where \( P \) is projective, and \( \text{Ext}^i_R(M, Q) = 0 \) for some integer \( i > 0 \) and for every \( R\text{-module} Q \) with finite projective dimension (or for every projective \( R\text{-module} Q \)).

Strongly Gorenstein injective modules are characterized in similar terms.

The principal role of these modules is to characterize the Gorenstein projective and injective modules, as follows:

**Lemma 2.4 ([1 Theorems 2.7]).** An \( R\text{-module} M \) is Gorenstein projective (resp., injective) if and only if it is a direct summand of a strongly Gorenstein projective (resp., injective) \( R\text{-module} \).

**Proof of Theorem 1.1.** For every integer \( n \) we need to show:

\[
\text{Gpd}_R(M) \leq n \text{ for every } R\text{-module } M \iff \text{Gid}_R(M) \leq n \text{ for every } R\text{-module } M.
\]

We prove only the direct implication; the converse one has a dual proof.

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1 In [1], the base ring is assumed to be commutative. However, for the result needed here, one can show easily that this assumption is not necessary.
Assume first that $M$ is strongly Gorenstein projective. By Remark 2.3 there is a short exact sequence $0 \to M \to P \to M \to 0$ with $P$ is projective. The Horseshoe Lemma (see [10, Remark, page 187]) gives a commutative diagram

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & M & P & M \\
\downarrow & \downarrow & \downarrow & \\
0 & I_0 & I_0 \oplus I_0 & I_0 \\
\downarrow & \downarrow & \downarrow & \\
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
0 & I_n & E_n & I_n \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
$$

where $I_i$ is injective for $i = 0, \ldots, n - 1$. Since $P$ is projective, $\text{id}_R(P) \leq n$ (by Lemma 2.1); hence $E_n$ is injective. On the other hand, from [7, Theorem 2.2], $\text{pd}_R(E) \leq n$ for every injective $R$-module $E$. Then, $\text{Ext}^1_R(E, I_n) = 0$ for all $i \geq n+1$. Then, from Remark 2.3, $I_n$ is strongly Gorenstein injective, and so $\text{Gid}_R(M) \leq n$.

Finally, consider an $R$-module $M$ with $\text{Gpd}_R(M) = m \leq n$. We can assume that $\text{Gpd}_R(M) \neq 0$. Then, there exists a short exact sequence $0 \to K \to N \to M \to 0$ such that $N$ is Gorenstein projective and $\text{Gpd}_R(K) \leq m - 1$ [6, Proposition 2.18]. By induction, $\text{Gid}_R(K) \leq n$ and $\text{Gid}_R(N) \leq n$. Therefore, using [6, Theorems 2.22 and 2.25] and the long exact sequence of Ext, we get that $\text{Gid}_R(M) \leq n$. \hfill \Box

**Proof of Corollary 1.2.** (1) We may assume that $\text{sup}\{l.\text{Ggldim}(R), r.\text{Ggldim}(R)\} < \infty$. Then, the character module, $I^* = \text{Hom}_R(I, \mathbb{Q}/\mathbb{Z})$, of every injective right $R$-module $I$ has finite projective dimension (by [7, Theorem 2.2] and [12, Theorem 3.52]). Then, similarly to the proof of [6, Proposition 3.4], we get that every Gorenstein projective $R$-module is Gorenstein flat. Therefore, $l.\text{wGgldim}(R) \leq \text{sup}\{l.\text{Ggldim}(R), r.\text{Ggldim}(R)\}$.

(2) and (3) The inequality $l.\text{FPD}(R) \leq l.\text{Ggldim}(R)$ follows from [6, Theorem 2.28].

The inequalities $l.\text{Ggldim}(R) \leq l.\text{gldim}(R)$ and $l.\text{wGgldim}(R) \leq \text{wgldim}(R)$ hold true since every projective (resp., flat) module is Gorenstein projective (resp., Gorenstein flat).

If $\text{wgldim}(R) < \infty$, then, from [10, Corollary 3], $l.\text{FPD}(R) = l.\text{Ggldim}(R) = l.\text{gldim}(R)$ and, from [11, Corollary 3.8], $l.\text{wGgldim}(R) = \text{wgldim}(R)$. \hfill \Box

**Remark 2.5.** It is well-known that there are examples of rings for which the left and right global dimensions differ (see [14, pages 74-75] and [9]). Then, by Corollary 1.2 the same examples show that there are also examples of rings for which the left and right Gorenstein global dimensions differ. However, as the classical case [12, Corollary 9.23], we have $l.\text{Ggldim}(R) = r.\text{Ggldim}(R)$ if $R$ is Noetherian [4, Theorem 12.3.1].
For the case where \( l.\text{Ggldim}(R) = 0 \) or \( r.\text{Ggldim}(R) = 0 \), we have the following result, which is [2, Theorem 2.2] in a non-commutative setting. Recall that a ring is called quasi-Frobenius if it is Noetherian and both left and right self-injective (see [11]).

**Proposition 2.6.** The following are equivalent:

1. \( R \) is quasi-Frobenius,
2. \( l.\text{Ggldim}(R) = 0 \),
3. \( r.\text{Ggldim}(R) = 0 \).

**Proof.** The implications 1 \( \Rightarrow \) 2 and 1 \( \Rightarrow \) 3 are well-known (see, for example, [4, Exercise 5, page 257]).

The implication 2 \( \Rightarrow \) 1 follows from Lemma 2.1 and the Faith-Walker Theorem [11, Theorem 7.56]. The implication 3 \( \Rightarrow \) 1 is proved similarly. \( \square \)

We finish with a generalization of a result of Iwanaga; see [4, Proposition 9.1.10].

**Corollary 2.7.** Assume that \( l.\text{Ggldim}(R) \leq n \) holds for some non-negative integer \( n \). If for an \( R \)-module \( M \) one of the numbers \( \text{pd}_R(M), \text{id}_R(M), \) or \( \text{fd}_R(M) \) is finite, then all of them are less than or equal to \( n \).

**Proof.** If \( \text{pd}_R(M) \) is finite, then [4, Proposition 2.27] and the assumption give \( \text{pd}_R(M) = \text{Gpd}_R(M) \leq n \). The argument for \( \text{id}_R(M) < \infty \) is similar. Finally, Corollary 1.2(2) and the assumption give \( \text{l.FPD}(R) \leq n \), and then \( \text{fd}_R(M) < \infty \) implies \( \text{pd}_R(M) < \infty \) by [10, Proposition 6]. \( \square \)

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**References**


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