SOME REMARKS ON THE POINCARÉ-BIRKHOFF THEOREM

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Abstract. We define the notion of a *positive path* of a homeomorphism of a topological space. It seems to be a natural object to understand Birkhoff's arguments in his proof of the celebrated Poincaré-Birkhoff theorem. We write the proof of this theorem, by using positive paths, and the proof of its generalization due to P. Carter. We will also explain the links with the free disk chains introduced in the subject by J. Franks. We will finish the paper by studying the local versions where the upper curve is not invariant and will explain why this curve or its image must be a graph to get such a generalization.

0. Introduction

In 1913, Birkhoff [Bi1] proved a celebrated result, conjectured and proved in special cases by Poincaré [P], that asserts that an area-preserving homeomorphism of the closed annulus that satisfies some "twist condition" admits at least two fixed points. Its proof used an index argument that was valid to find one fixed point but was incorrect to get the second one. A small modification of the argument was necessary, and Birkhoff corrected this minor error in a paper [Bi2] published in 1925 (see also the well-detailed expository paper of Brown and Newman [BN]), where he made some generalizations: the area-preserving hypothesis was replaced by a topological intersection property, and the hypothesis about the invariance of the annulus was weakened. A generalization of the Poincaré-Birkhoff theorem, with a nicer intersection property, was obtained by Carter [C]. Looking at noninvariant annuli, which are useful in the study of nonautonomous planar Hamiltonian systems to show the existence of periodic orbits, further generalizations have been obtained by Jacobowitz [J1], [J2], Ding [D] and other authors (see the paper of Dalbono and Rebelo [DR] on the subject).

More recently, Franks introduced new ideas by giving a proof of the Poincaré-Birkhoff theorem that uses Brouwer's lemma on translation arcs (Kerékjártó already observed the link between Brouwer's theory and the Poincaré-Birkhoff theorem [K]). His proof uses only the fact that the map has no wandering point, and the twist property can be extended to other interesting situations. His ideas have been developed by many authors.

The goal of this paper is to introduce the notion of a *positive path* of a homeomorphism, which seems to be a natural object for understanding Birkhoff's arguments. In the first section we will give a proof of the (classical) Poincaré-Birkhoff theorem using existence of positive paths that cross the annulus. In the second section we will make a link with Franks' ideas. We will recall a natural construction, in the spirit of Franks' ideas, and will observe that it permits us to obtain positive paths that cross the annulus and so to get the Poincaré-Birkhoff theorem without the help of Brouwer's theory. This method stays valid to prove Carter's theorem in the case of an invariant annulus. We will see in the last section that the arguments may be generalized in the situation where the annulus is not invariant, and so we will give a new proof of Carter's theorem in its generality.

1. Statement and proof of the Poincaré-Birkhoff theorem

In what follows, a *path* on a topological space X is a continuous map $\gamma : I \to X$ defined on a segment $I = [a,b] \subset \mathbf{R}$. The *origin* and the *extremity* of γ are respectively $\gamma(a)$ and $\gamma(b)$. If X_1 and X_2 are two subsets of X, we will say that γ *joins* X_1 to X_2 if its origin belongs to X_1 and its extremity belongs to X_2. The restriction of γ to a compact interval $J \subset I$ is a *subpath* of γ. If γ is one-to-one, γ is an *arc*; if $\gamma(a) = \gamma(b)$, it is a *loop*; if $\gamma(a) = \gamma(b)$ and γ is one-to-one on $[a,b)$, it is a *simple loop*. The concatenation of two paths (when it is defined) is denoted by $\gamma_1\gamma_2$. As is usually done, we often will not make any distinction between a path and its image. In particular, if Y is a subset of X, we will write $\gamma \subset Y$ if the image of γ is included in Y.

We write $\mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$ and we fix in this section and the next one a homeomorphism F of $\mathbf{A} = \mathbf{T}^1 \times [0,1]$ homotopic to the identity and a lift f of F to the universal cover $\widetilde{\mathbf{A}} = \mathbf{R} \times [0,1]$. We suppose that f satisfies the *boundary twist condition*:

$$\text{for} \quad \text{every} \ x \in \mathbf{R}, \ \ p_1 \circ f(x,0) < x < p_1 \circ f(x,1),$$

where $p_1 : \widetilde{\mathbf{A}} \to \mathbf{R}$ is the first projection. We write $\mathrm{Fix}_*(F)$ for the set of fixed points of F that are lifted to fixed points of f. Let us state the Poincaré-Birkhoff theorem.

Theorem 1. *If F preserves the measure induced by $dx \wedge dy$, then $\sharp\mathrm{Fix}_*(F) \geq 2$.*

Let us recall the ideas of Birkhoff. The vector field $\widetilde{X} : z \mapsto f(z) - z$ is invariant under the covering automorphism $T : (x,y) \mapsto (x+1,y)$ and lifts a vector field X on $\mathbf{A}$ whose singular set is exactly $\mathrm{Fix}_*(F)$. If γ is a path in $\widetilde{\mathbf{A}} \setminus \mathrm{Fix}(f)$, one may define the *variation of angle*

$$i_\gamma f = \int_{\widetilde{X}\circ\gamma} d\theta,$$

where

$$d\theta = \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2 + y^2}$$

is the usual polar form on $\mathbf{R}^2 \setminus \{0\}$. The form $d\theta$ being closed can be integrated on any (even nonsmooth) path in $\mathbf{R}^2 \setminus \{0\}$. The theorem will be proved if one finds a loop Γ in $\widetilde{\mathbf{A}} \setminus \mathrm{Fix}(f)$ such that $i_\Gamma \neq 0$. Indeed if $\mathrm{Fix}_*(F)$ is finite (equivalently if $\mathrm{Fix}(f)$ is discrete), then

$$i_\Gamma f = \sum_{z\in\mathrm{Fix}(f)} i(\widetilde{X}, z) \int_{\xi_z\circ\Gamma} d\theta,$$

where $i(\widetilde{X}, z)$ denotes the Poincaré index of $\widetilde{X}$ at z and ξ_z is the vector field $z' \mapsto z' - z$. This implies that $\sharp \mathrm{Fix}_*(F) \geq 2$ because $i(\widetilde{X}, z) = i(X, \pi(z))$ and because the Poincaré-Hopf formula asserts that

$$\sum_{z \in \mathrm{Fix}_*(F)} i(X, z) = \chi(\mathbf{A}) = 0.$$

If we can find two paths γ and γ' such that $i_\gamma f = i_{\gamma'} f \neq 0$, the first one joining $\mathbf{R} \times \{0\}$ to $\mathbf{R} \times \{1\}$, the second one joining $\mathbf{R} \times \{1\}$ to $\mathbf{R} \times \{0\}$, then the loop $\Gamma = \gamma \delta \gamma' \delta'$ obtained by adding horizontal segments on each boundary line will satisfy $i_\Gamma f = 2 i_\gamma f \neq 0$. In [Bi1] Birkhoff composes F with a small vertical translation to build such paths. Suppose that the displacement is positive, the iterates by the perturbed map G of $\mathbf{T}^1 \times \{0\}$ are pairwise disjoint, and they are not all included in the annulus (because G also preserves the area). Birkhoff chooses an arc α that joins a point $z \in \mathbf{T}^1 \times \{0\}$ to $G(z)$ and by concatenation of the iterates of α constructs an arc that joins $\mathbf{T}^1 \times \{0\}$ to $\mathbf{T}^1 \times \{1\}$ and that is lifted into an arc γ satisfying $i_\gamma f = -\dfrac{1}{2}$. We will give here a simple construction, in the spirit of Birkhoff's ideas, that does not need any perturbation and that is still valid under a weaker hypothesis than the preservation of the area.

Definition 1. Let G be a homeomorphism of a topological space X. A *positive path* of G is a path $\gamma : I \to X$ such that for every t, t' in I,

$$t' \geq t \Rightarrow G(\gamma(t')) \neq \gamma(t).$$

Observe that a positive path γ does not meet the fixed point set, that any sub-path of γ is positive and that the images $G^k \circ \gamma$, $k \in \mathbf{Z}$, are also positive.

Proposition 1. *If γ is a positive path of f that joins a boundary line of $\widetilde{\mathbf{A}}$ to the other one, then $i_\gamma f = -\dfrac{1}{2}$.*

Proof. We write the proof in the case where γ joins $\mathbf{R} \times \{0\}$ to $\mathbf{R} \times \{1\}$, the other case being similar. The boundary of the simplex

$$\Delta = \{(t, t') \in I^2 \,|\, t' \geq t\}$$

may be written $\partial \Delta = \delta_d \delta_h \delta_v$ where δ_d is the diagonal, δ_h a horizontal segment and δ_v a vertical one. The path γ being positive, the map

$$\Phi : (t, t') \mapsto f(\gamma(t')) - \gamma(t)$$

does not vanish on Δ, and one has

$$\int_{\Phi \circ \delta_d} d\theta + \int_{\Phi \circ (\delta_h \delta_v)} d\theta = \int_{\Phi \circ \partial \Delta} d\theta = 0.$$

Observe now that the image by Φ of each segment δ_h and δ_v does not intersect the vertical half-line $\{0\} \times (-\infty, 0]$. This implies that

$$i_\gamma f = \int_{\Phi \circ \delta_d} d\theta = - \int_{\Phi \circ (\delta_h \delta_v)} d\theta = -\frac{1}{2}. \qquad \square$$

Recall that a *wandering point* of a homeomorphism f of a topological space X is a point that admits a *wandering neighborhood U*, which means a neighborhood U such that the $f^k(U)$, $k \geq 0$, are pairwise disjoint. Recall that a *Urysohn space* is a topological space such that two distinct points may be separated by closed neighborhoods.

Proposition 2. *Suppose that X is a connected and locally path-connected Urysohn space and that G is a fixed point free homeomorphism of X with no wandering point. If $Z \subset X$ satisfies $G(Z) \subset Z$, then for every $z \in X$ there exists a positive path of G that joins Z to z.*

Proof. One must prove the equality $Y = X$, where Y is the set of points that may be joined by a positive path of G whose origin belongs to Z. The space X being connected, it is sufficient to prove that $\overline{Y} \subset \text{Int}(Y)$. Fix $z_0 \in \overline{Y}$. By hypothesis, one can find a path-connected neighborhood V of z_0 such that $\overline{V} \cap G(\overline{V}) = \emptyset$. We will prove that $V \subset Y$.

The fact that $z_0 \in \overline{Y}$ implies that there exists a positive path $\gamma_0 : I \to X$ from Z to V. The closures of the subsets $J = \gamma_0^{-1}(V)$ and $J' = \gamma_0^{-1}(G(V))$ do not intersect because $\overline{V} \cap G(\overline{V}) = \emptyset$. This implies that $\inf J \neq \inf J'$.

Suppose first that $\inf J < \inf J'$ (this includes the case where $J' = \emptyset$). In that case, there is a subpath γ_1 of γ_0 from Z to V that does not meet $G(V)$. For every $z \in V$ one can find a path γ inside V that joins the extremity z_1 of γ_1 to z. The path $\gamma_2 = \gamma_1 \gamma$ is positive because γ_1 is positive and $G(\gamma)$ is disjoint both from γ and γ_1. This implies that $z \in Y$.

Suppose now that $\inf J' < \inf J$. In that case, there is a subpath γ_1 of γ_0 from Z to $G(V)$ that does not meet V. We denote by z_1 its extremity. The point $G(z_1)$ does not belong to γ_1 because this path is positive. The path being compact (X is Hausdorff), one can find a path-connected neighborhood $U \subset G(V)$ of z_1 such that $G(U)$ does not intersect γ_1. The set U being nonwandering, one can find a point $z_2 \in U$ whose positive orbit meets $G^{-1}(U) \subset V$. Choose a path γ inside U that joins z_1 to z_2. The path $\gamma_2 = \gamma_1 \gamma$ does not meet V and is positive because γ_1 is positive and $G(\gamma)$ is disjoint from γ_1 and γ. Let us consider the integer $k \geq 1$ such that $G^k(\gamma_2) \cap V \neq \emptyset$ and $G^{k'}(\gamma_2) \cap V = \emptyset$ if $0 \leq k' < k$. Since $G(Z) \subset Z$, the path $G^k(\gamma_2)$ is a positive path from Z to V that does not meet $G(V)$. We conclude as in the first case. $\square$

Remark 1. A classical result asserts that if $\gamma : I \to X$ is a path in a Hausdorff space X, then there exists an arc $\gamma' : J \to X$ joining the origin of γ to its extremity such that for every couple $(s, s') \in J^2$ satisfying $s < s'$, there exists a couple $(t, t') \in I^2$ such that $\gamma'(s) = \gamma(t)$, $\gamma'(s') = \gamma(t')$ and $t < t'$. Therefore in Proposition 2, one may replace path with arc in the conclusion. For the same reason a locally path-connected space is locally arc-connected. In the case where X is "obviously" arc-connected (e.g. if X is a manifold) one can get directly the improved version of the theorem. In the proof above one may suppose that γ_0, γ_1 and γ are arcs. To construct an arc γ_2 joining Z to z_2, one considers the "last point" z_3 where γ meets γ_1. Then one concatenates the subpath of γ_1 that joins its origin to z_3 and the subpath of γ that joins z_3 to its extremity.

Remark 2. If one applies the proposition with Z equal to the positive orbit of a point z, one gets a positive path from a point $G^{k'}(z)$, $k' \geq 0$, to $G^{-1}(z)$. Therefore, for every point z one may find $k < 0$ and a positive arc that joins z to $G^k(z)$ (of course $k \neq -1$).

Let us explain how to deduce Theorem 1 from Propositions 1 and 2.

Proof of Theorem 1. One may suppose that $\mathrm{Fix}_*(F)$ does not separate the two boundary circles (otherwise $\sharp\mathrm{Fix}_*(F) = +\infty$). If n is large enough, the homeomorphism F' of the annulus $\mathbf{A}' = \mathbf{R}/n\mathbf{Z} \times [0,1]$ lifted by f has no fixed points except for the ones that are lifted to fixed points of f. Therefore $\mathrm{Fix}(F') = \mathrm{Fix}_*(F')$ does not separate the boundary circles of $\mathbf{A}'$, and one may consider the connected component W of $\mathbf{A}' \setminus \mathrm{Fix}(F')$ that contains the boundary. Moreover F' has no wandering point because it preserves the area. Applying Proposition 2 to $X = W$, to $G = F'|_W$ and to $Z = \mathbf{R}/n\mathbf{Z} \times \{0\}$ or $Z = \mathbf{R}/n\mathbf{Z} \times \{1\}$, one constructs a positive path of F' from one of the boundary circles of $\mathbf{A}'$ to the other one. Such a path is lifted to a positive path of f from the corresponding boundary line to the other one. Theorem 1 follows from Proposition 1. $\qquad\square$

Remark 3. One can prove that F' has no wandering point if it is the case for F. Indeed, let T' be a generator of the (finite) group of automorphisms of the covering space $\mathbf{A}'$. The fact that F has no wandering point implies that for every nonempty open set $U \subset \mathbf{A}'$, there exists $q \geq 1$ and $p \in \mathbf{Z}$ such that $F'^q(U) \cap T'^p(U) \neq \emptyset$. Let us fix a nonempty open set $U_0 \subset \mathbf{A}'$ and define a sequence $(U_k)_{k \geq 0}$ of nonempty open sets where U_{k+1} may be written $U_{k+1} = F'^{q_k}(U_k) \cap T'^{p_k}(U_k)$. One deduces that for every $k' > k$, one has $U_{k'} \subset F'^{q_k+\cdots q_{k'-1}}(U_k) \cap T'^{p_k+\cdots p_{k'-1}}(U_k)$. One can find $k' > k$ such that $p_k + \cdots + p_{k'-1} = 0 \pmod{n}$. This implies that U_k is nonwandering and therefore that U_0 itself is nonwandering. So Theorem 1 is valid if instead of supposing that F preserves the area, one supposes that F has no wandering point. Anyway, instead of working in a finite cover of $\mathbf{A}$, one can prove directly that the conclusion of Proposition 2 occurs if X is the connected component of $\widetilde{\mathbf{A}} \setminus \mathrm{Fix}(f)$ that contains the boundary, if $G = f|_X$ and if Z satisfies $f(Z) \subset Z$ and $T(Z) = Z$.

2. Links with Franks' method

In [F1], Franks gave a proof of Theorem 1 valid in the case where F has no wandering point and that uses the following result of Brouwer [Br] : *if an orientation-preserving homeomorphism g of $\mathbf{R}^2$ has a periodic point of period $q \geq 2$, then one can construct a loop Γ such that $i_\Gamma g = 1$*. By a simple perturbation argument, he observed that such a loop still exists if there exists a *periodic free disk chain*, which means a family $(U_r)_{r \in \mathbf{Z}/n\mathbf{Z}}$ of pairwise disjoint free (i.e., disjoint from their image) topological open disks such that for every $r \in \mathbf{Z}/n\mathbf{Z}$, one of the positive iterates of U_r meets U_{r+1}. The facts that F has no wandering point and that f satisfies the boundary twist condition permit him to construct such a chain under the condition of finiteness of $\mathrm{Fix}_*(F)$. A periodic free disk chain may be obtained under more general twist conditions, which permits Franks to state very useful generalizations of the Poincaré-Birkhoff theorem. We will recall now a natural construction for getting periodic free disk chains and will observe that it gives positive paths that cross the annulus.

Alternate proof of Theorem 1. Here again we suppose that $\mathrm{Fix}_*(F)$ does not separate the two boundary circles and write W for the connected component of $\mathbf{A} \setminus \mathrm{Fix}_*(F)$ that contains the boundary. Let us consider a triangulation $\mathcal{T} = (\sigma_i)_{i \in I}$ of W. Subdividing some simplices if necessary, one may assume that each 2-cell σ_i is free by f when lifted to $\widetilde{\mathbf{A}}$.

Lemma 1. *For every i, i' in I, there exists a sequence $(\sigma_{i_k})_{0 \leq k \leq l}$ such that $\sigma_{i_0} = \sigma_i$, $\sigma_{i_l} = \sigma_{i'}$ and $f(\sigma_{i_k}) \cap \sigma_{i_{k+1}} \neq \emptyset$ if $0 \leq k < l$.*

Proof. Fix $i \in I$ and define a subset $J \subset I$ by writing $i' \in J$ if there exists a sequence $(\sigma_{i_k})_{0 \leq k \leq l}$ such that $\sigma_{i_0} = \sigma_i$, $\sigma_{i_l} = \sigma_{i'}$ and $f(\sigma_{i_k}) \cap \sigma_{i_{k+1}} \neq \emptyset$ if $0 \leq k < l$. Observe now that $X = \bigcup_{i' \in J} \sigma_{i'}$ is a closed subset of W such that $F(X) \subset \mathrm{Int}_W(X)$. This implies that $\mathrm{Int}_W(X) \setminus F(X)$ is a wandering open set. By hypothesis, one deduces that $F(X) = \mathrm{Int}_W(X)$. We deduce that X is a closed and open subset of W, which implies by connectedness of W that it is equal to W. $\qquad\square$

This transitivity property and the twist condition could permit us to construct a periodic chain $(\widetilde{\sigma}_{i_r})_{r \in \mathbf{Z}/n\mathbf{Z}}$ of f made of distinct cells of the lifted triangulation $(\widetilde{\sigma}_i)_{i \in \widetilde{I}}$. Guillou and Le Roux [L] observed that here again, one may find a loop Γ such that $i_\Gamma f = 1$. We will see below how to construct a positive path from $\mathbf{R} \times \{0\}$ to $\mathbf{R} \times \{1\}$. We have found a sequence $(\sigma_i)_{0 \leq i \leq n}$ of free closed disks of $\widetilde{\mathbf{A}}$ whose interiors are pairwise disjoint such that $\sigma_0 \cap (\mathbf{R} \times \{0\}) \neq \emptyset$, $\sigma_n \cap (\mathbf{R} \times \{1\}) \neq \emptyset$ and $f(\sigma_i) \cap \sigma_{i+1} \neq \emptyset$ for every $i \in \{0, \ldots, n-1\}$. Among all such sequences (the cells are not necessarily cells of the triangulation), choose a sequence $(\sigma_i)_{0 \leq i \leq n}$ for which the integer n is the smallest possible (note that $n \geq 1$ because of the boundary twist condition) and observe that $f^k(\sigma_i) \cap \sigma_j = \emptyset$ if $j > i + 1$ and $k \geq 1$. Indeed suppose that

$$E = \{(i, j, k) \mid j > i + 1, \; k \geq 1, \; f^k(\sigma_i) \cap \sigma_j \neq \emptyset\}$$

is not empty and define

$$i_0 = \min\{i \mid \text{there exist } j > i + 1, \; k \geq 1 \text{ such that } (i, j, k) \in E\},$$
$$j_0 = \max\{j > i_0 + 1 \mid \text{there exists } k \geq 1 \text{ such that } (i_0, j, k) \in E\},$$
$$k_0 = \min\{k \geq 1 \mid \; (i_0, j_0, k) \in E\}.$$

Observe now that the sequence

$$(f^{k_0-1}(\sigma_0), \ldots, f^{k_0-1}(\sigma_{i_0}), \sigma_{j_0}, \ldots, \sigma_n)$$

satisfies the same properties as the sequence $(\sigma_i)_{0 \leq i \leq n}$, contradicting the minimality of n. This minimality also implies that σ_0 is the only disk that meets $\mathbf{R} \times \{0\}$ and that σ_n is the only disk that meets $\mathbf{R} \times \{1\}$.

The set $\bigcup_{0 \leq i \leq n} f^{n-i}(\sigma_i)$ is connected and intersects the two boundary lines. Choose a point $z_0 \in f^n(\sigma_0) \cap (\mathbf{R} \times \{0\})$, a point $z_{n+1} \in \sigma_n \cap (\mathbf{R} \times \{1\})$ and for every $i \in \{1, \ldots, n\}$ a point $z_i \in f^{n-i+1}(\sigma_{i-1}) \cap f^{n-i}(\sigma_i)$. Then choose for every $i \in \{0, \ldots, n\}$ an arc γ_i joining z_i to z_{i+1} such that $(\gamma_i \setminus \{z_i, z_{i+1}\}) \subset \mathrm{Int}(f^{n-i}(\sigma_i))$. The path $\gamma_0 \gamma_1 \ldots \gamma_n$ is positive. Indeed suppose that there exists $z \in \gamma_j$ such that $f(z) \in \gamma_i$, where $i \leq j$. The fact that γ_j is free implies that $i < j$ and that $z \neq z_j$ if $i = j - 1$. The fact that $f^k(\sigma_i) \cap \sigma_j = \emptyset$ if $j > i + 1$ and $k \geq 1$ implies that $i \geq j - 1$ and that $z \neq z_{j+1}$ if $i = j - 1$ (indeed $f(\gamma_j) \cap \gamma_i \subset f^{n-j+1}(\sigma_j) \cap f^{n-i}(\sigma_i)$). But if $i = j - 1$ and $z \notin \{z_j, z_{j+1}\}$, then $f(z) \in \mathrm{Int}(f^{n-j+1}(\sigma_j)) \cap f^{n-j+1}(\sigma_{j-1}) = \emptyset$. Therefore there is no such point z. $\qquad\square$

A slight modification of the argument will give us the following result. The first part of the theorem was already observed in [Bi2] and [K]. The second part is Carter's theorem [C] for an invariant annulus (see also [F2], [G1], [G2] for proofs that use Brouwer's theory). Recall that a simple loop in $\mathbf{A}$ is *essential* if it is not null-homotopic.

Theorem 2. *If every essential simple loop meets its image by F, then $\sharp\mathrm{Fix}_*(F) \geq 1$; if every essential simple loop meets its image by F in at least two points, then $\sharp\mathrm{Fix}_*(F) \geq 2$.*

Proof. Suppose that $\sharp\mathrm{Fix}_*(F) \leq 1$ and keep the same notation. Either there is no positive arc from the lower boundary circle to the upper one or there is no positive arc from the upper boundary circle to the lower one. Assume that we are in the first case and instead of considering a triangulation of W start with a *brick decomposition* $\mathcal{B} = (\sigma_i)_{i \in I}$. In this dual notion, the bricks are closed disks obtained as the closures of the connected components of the complement in W of a locally finite graph Σ whose vertices are locally the extremities of exactly three edges. Here again, one decomposes each brick to get a brick decomposition with bricks that are free when lifted to $\widetilde{\mathbf{A}}$. Define a subset $J \subset I$ by writing $i \in J$ if there exists a sequence $(\sigma_{i_k})_{0 \leq k \leq l}$ such that $\sigma_{i_0} \cap \mathbf{T}^1 \times \{0\}$, $\sigma_{i_l} = \sigma_j$ and $f(\sigma_{i_k}) \cap \sigma_{i_{k+1}} \neq \emptyset$. Here again $X = \bigcup_{j \in J} \sigma_j$ is a closed subset of W such that $F(X) \subset \mathrm{Int}_W(X)$, and one knows that X is neighborhood of $\mathbf{T}^1 \times \{0\}$ that does not intersect the upper boundary (otherwise the previous proof will give us a positive path from $\mathbf{T}^1 \times \{0\}$ to $\mathbf{T}^1 \times \{1\}$). Moreover one knows that X is a subsurface with boundary of W (every union of bricks of a brick decomposition has this property).

In the case where $\mathrm{Fix}_*(F) = \emptyset$, then ∂X is compact and may be written as a disjoint union of finitely many simple loops. As ∂X separates the two boundary lines, one of them must be essential. This essential loop is disjoint from its image because $F(\partial X) \subset \mathrm{Int}(X)$.

In the case where $\mathrm{Fix}_*(F) = \{z_0\}$, then each connected component of ∂X that is not a simple loop becomes a simple loop when one adds z_0. Here again, even if there may be infinitely many such loops, one of them must be essential because $\partial X \cup \{z_0\}$ separates the boundary circles. This implies the existence of an essential simple loop that meets its image in fewer than two points. $\square$

3. Local versions

We conclude with the versions of the Poincaré-Birkhoff theorem on noninvariant annuli. We fix two essential simple loops C_1 and C_2 in $\mathbf{T}^1 \times (0, +\infty)$ that project injectively onto $\mathbf{T}^1 \times \{0\}$. We denote by $\mathbf{A}_i$ the closed annulus bounded by $\mathbf{T}^1 \times \{0\}$ and C_i and write $\mathrm{Int}(\mathbf{A}_i) = \mathbf{A}_i \backslash C_i$. We denote by $\widetilde{C}_i$, $\widetilde{\mathbf{A}}_i$ and $\mathrm{Int}(\widetilde{\mathbf{A}}_i)$ the respective preimages of C_i, $\mathbf{A}_i$ and $\mathrm{Int}(\mathbf{A}_i)$ in the universal cover $\mathbf{R} \times [0, +\infty)$. We fix in this section a homeomorphism $F : \mathbf{A}_1 \to \mathbf{A}_2$ that preserves the orientation and leaves invariant the circle $\mathbf{T}^1 \times \{0\}$ and a lift $f : \widetilde{\mathbf{A}}_1 \to \widetilde{\mathbf{A}}_2$ that satisfies the following twist condition:

$$z \in \mathbf{R} \times \{0\} \Rightarrow p_1 \circ f(z) < p_1(z), \ \ z \in \widetilde{C}_1 \Rightarrow p_1 \circ f(z) > p_1(z).$$

Here again we write $\mathrm{Fix}_*(F)$ for the set of fixed points of F lifted to fixed points of f.

The first result is the version of Theorem 1 on a noninvariant annulus when F is supposed to be area-preserving. Under this condition, it is a particular case of Ding's theorem [D].

Theorem 3. *Let us suppose that for every nonempty open set $U \subset \mathbf{A}_1$ (resp. $U \subset \mathbf{A}_2$), there exists $n > 0$ (resp. $n < 0$) such that $f^n(U) \not\subset \mathbf{A}_1$ (resp. $f^n(U) \not\subset \mathbf{A}_2$) or $f^n(U) \cap U \neq \emptyset$. Then $\sharp\mathrm{Fix}_*(F) \geq 2$.*

The second result is stronger; it is the most general version of Carter's theorem [C].

Theorem 4. *If every essential simple loop in* $\mathbf{A}_1$ *meets its image by* F, *then* $\sharp\mathrm{Fix}_*(F) \geq 1$; *if every essential simple loop in* $\mathbf{A}_1$ *meets its image by* F *in at least two points, then* $\sharp\mathrm{Fix}_*(F) \geq 2$.

As in section 1, if we can find two paths γ and γ' in $\widetilde{\mathbf{A}}_1 \setminus \mathrm{Fix}(f)$ such that $i_\gamma f$ and $i_{\gamma'} f$ both belong to $(-\frac{3}{4}, -\frac{1}{4})$—the first one joining $\mathbf{R} \times \{0\}$ to $\widetilde{C}_1$, the second one joining $\widetilde{C}_1$ to $\mathbf{R} \times \{0\}$—then the loop $\Gamma = \gamma\delta\gamma'\delta'$ obtained by adding "segments" of $\widetilde{C}_1$ and $\mathbf{R} \times \{0\}$ will satisfy $i_\Gamma f = -1$ and we will deduce as in section 1 that $\sharp\mathrm{Fix}_*(F) \geq 2$.

Let us begin with the following results.

Lemma 2. *If there is a positive path* γ *of* f *in* $\widetilde{\mathbf{A}}_1$ *whose origin is on* $\mathbf{R} \times \{0\}$ *such that* $f \circ \gamma$ *is not included in* $\mathrm{Int}(\widetilde{\mathbf{A}}_1)$, *then there is a positive path* γ' *in* $\widetilde{\mathbf{A}}_1$ *joining* $\mathbf{R} \times \{0\}$ *to* $\widetilde{C}_1$ *such that the image of its extremity does not belong to* $\mathrm{Int}(\widetilde{\mathbf{A}}_1)$. *We obtain a similar result by replacing* f *with* f^{-1} *and* $\widetilde{C}_1$ *with* $\widetilde{C}_2$.

Lemma 3. *If* γ *is a positive path of* f *in* $\widetilde{\mathbf{A}}_1$ *joining* $\mathbf{R} \times \{0\}$ *to* $\widetilde{C}_1$ *such that the image of its extremity does not belong to* $\mathrm{Int}(\widetilde{\mathbf{A}}_1)$, *then* $i_\gamma f \in (-\frac{3}{4}, -\frac{1}{4})$. *Similarly, if* γ *is a positive path of* f^{-1} *in* $\widetilde{\mathbf{A}}_2$ *joining* $\mathbf{R} \times \{0\}$ *to* $\widetilde{C}_2$ *such that the inverse image of its extremity does not belong to* $\mathrm{Int}(\widetilde{\mathbf{A}}_2)$, *then* $i_{f^{-1}\gamma} f \in (\frac{1}{4}, \frac{3}{4})$.

Proof of Lemma 2. Suppose that $\gamma \in \widetilde{\mathbf{A}}_1$ is a positive path of f whose origin is on $\mathbf{R} \times \{0\}$ and such that $f \circ \gamma \not\subset \mathrm{Int}(\widetilde{\mathbf{A}}_1)$. Taking a subpath if necessary, one may suppose that the extremity z of $f(\gamma)$ does not belong to $\mathrm{Int}(\widetilde{\mathbf{A}}_1)$. Since C_1 projects injectively on $\mathbf{T}^1 \times \{0\}$, the vertical arc γ_v joining z to $\widetilde{C}_2$ does not intersect $\widetilde{\mathbf{A}}_1$ if $z \notin \widetilde{C}_1$ and intersects $\widetilde{\mathbf{A}}_1$ only at z if $z \in \widetilde{C}_1$. This implies that $\gamma' = \gamma f^{-1}(\gamma_v)$ is a positive path in $\widetilde{\mathbf{A}}_1$ joining $\mathbf{R} \times \{0\}$ to $\widetilde{C}_1$ and that the image by f of its extremity does not belong to $\mathrm{Int}(\widetilde{\mathbf{A}}_1)$. $\square$

Proof of Lemma 3. Let $\gamma \in \widetilde{\mathbf{A}}_1$ be a positive path of f joining $\mathbf{R} \times \{0\}$ to $\widetilde{C}_1$ such that the image of its extremity does not belong to $\mathrm{Int}(\widetilde{\mathbf{A}}_1)$. We keep the same notation as in Proposition 1. The map Φ does not reach the vertical $\{0\} \times (-\infty, 0]$, either on δ_v or on δ_h. This implies that $i_\gamma f \in (-\frac{3}{4}, -\frac{1}{4})$. Similarly, if $\gamma \in \widetilde{\mathbf{A}}_2$ is a positive path of f^{-1} whose origin is on $\mathbf{R} \times \{0\}$ and such that $f^{-1} \circ \gamma$ is not in $\mathrm{Int}(\widetilde{\mathbf{A}}_2)$, then one has $i_{f^{-1}(\gamma)} f = i_\gamma f^{-1} \in (\frac{1}{4}, \frac{3}{4})$. $\square$

Proof of Theorem 3. More precisely, in the case where $\mathrm{Fix}_*(F)$ does not separate the two boundary components of $\mathbf{A}_1$, we will prove the existence of a positive path of f that joins $\mathbf{R} \times \{0\}$ to $\widetilde{C}_1$ such that the image by f of its extremity is not in $\mathrm{Int}(\widetilde{\mathbf{A}}_1)$, and of course we will get a similar result with f^{-1} instead of f and $\widetilde{\mathbf{A}}_2$ instead of $\widetilde{\mathbf{A}}_1$. Taking a finite cover of $\mathbf{A}_1$ if necessary, one may suppose that $\mathrm{Fix}_*(F) = \mathrm{Fix}(F)$ and write $W = \mathbf{A}_1 \setminus \mathrm{Fix}(F)$.

Now look at the proof of Proposition 2 in the case where $X = W$, $G = F|_W$ and $Z = \mathbf{T}^1 \times \{0\}$ and keep the same notation (it makes sense to define the neighborhood $V \subset W$). The first situation is similar, and we conclude that $V \subset Y$. In the second situation, if $F(z_1) \notin \mathrm{Int}(\mathbf{A}_1)$, we are done by Lemmas 2 and 3. If not, U can

be defined similarly, and by hypothesis the point z_2 can be chosen such that its positive orbit either meets $F^{-1}(V)$ or is not included in $\text{Int}(\mathbf{A}_1)$. If one of the positive iterates of γ_2 is not included in $\text{Int}(\mathbf{A}_1)$, then we conclude using Lemmas 2 and 3; otherwise the argument is similar to the one in Proposition 2, and we conclude that $V \subset Y$. Note now that if $Y = W$, we can find a positive arc γ of F in $\mathbf{A}_1$ from $\mathbf{T}^1 \times \{0\}$ to C_1 such that $F \circ \gamma$ is not in $\text{Int}(\mathbf{A}_1)$. Otherwise $\mathbf{A}_2 \subset \text{Int}(\mathbf{A}_1)$, which contradicts the hypothesis. $\qquad\square$

Proof of Theorem 4. We suppose that $\text{Fix}_*(F) < 2$. Replacing f with f^{-1} if necessary, let us assume that there is no positive path that joins $\mathbf{R} \times \{0\}$ to $\widetilde{C}_1$ such that the image by f of its extremity is not in $\text{Int}(\widetilde{\mathbf{A}}_1)$. This implies that there is no sequence $(\delta_i)_{0 \leq i \leq n}$ of free closed disks of $\widetilde{\mathbf{A}}_1 \setminus \text{Fix}(f)$ whose interiors are pairwise disjoint such that $\delta_0 \cap (\mathbf{R} \times \{0\}) \neq \emptyset$, $f(\delta_n) \not\subset \text{Int}(\widetilde{\mathbf{A}}_1)$ and $f(\delta_i) \cap \delta_{i+1} \neq \emptyset$ for every $i \in \{0, \ldots, n-1\}$. Indeed if n is minimal among such sequences, then $f^k(\delta_i) \subset \text{Int}(\widetilde{\mathbf{A}}_1)$ for every integer $k \geq 1$, if $0 \leq i < n$, and the positive iterates of δ_i are all defined. So the argument of section 2 is still valid: one knows that $f^k(\delta_i) \cap \delta_j = \emptyset$ if $j > i+1$ and $k \geq 1$. Choose $z_{n+1} \in \delta_n$ such that $f(z_{n+1}) \notin \text{Int}(\widetilde{\mathbf{A}}_1)$ and do the same construction as in section 2. We will get a positive path in $\widetilde{\mathbf{A}}_1$ whose origin is on $\mathbf{R} \times \{0\}$ such that its image by f is not included in $\text{Int}(\widetilde{\mathbf{A}}_1)$. This would contradict Lemma 2.

Let us consider now a brick decomposition of $W = \mathbf{A}_1 \setminus \text{Fix}_*(F)$ whose bricks are free when lifted to $\widetilde{\mathbf{A}}_1$ and define a set $J \subset I$ by the same formula as in the proof of section 2. By the remark just above, one knows that $f(\delta_j) \subset \text{Int}(\mathbf{A}_1)$ for every $j \in J$. We deduce that the subsurface $X = \bigcup_{j \in J} \delta_j$ satisfies $F(X) \subset \text{Int}_W(X)$, and we can conclude as in the proof of Theorem 2. $\qquad\square$

Remark 4. We have used the fact that the loops C_1 and C_2 project injectively onto $\mathbf{T}^1 \times \{0\}$ only in the proofs of Lemma 2 and Lemma 3. As we will see below, these lemmas are still valid if one of the loops C_1 or C_2 projects injectively onto $\mathbf{T}^1 \times \{0\}$. Therefore, both Theorems 3 and 4 are still valid under this weaker assumption. This was already known ([D], [G1]). The proof of Lemma 2 is valid without any assumption on C_1 and C_2: one may find an arc γ_v from z to $\widetilde{C}_2$ included in $\widetilde{\mathbf{A}}_2$ that does not meet $\widetilde{\mathbf{A}}_1$ if $z \notin \widetilde{C}_1$ and that meets $\widetilde{\mathbf{A}}_1$ only at z if $z \in \widetilde{C}_1$ (but it will not necessarily be vertical if C_1 does not project injectively on $\mathbf{T}^1 \times \{0\}$).

The first statement of Lemma 3 is valid if C_1 projects injectively on $\mathbf{T}^1 \times \{0\}$ with exactly the same proof. Due to the symmetry of the problem, we still have to prove that the first statement of Lemma 3 is also true if C_2 projects injectively on $\mathbf{T}^1 \times \{0\}$. Let $\gamma : [0,1] \to \widetilde{\mathbf{A}}_1$ be a positive path satisfying the hypothesis of the lemma. Modifying γ in the uniform topology (the property of being positive is obviously stable by perturbation), one may suppose that $\gamma(t) \notin \mathbf{R} \times \{0\}$ if $t \neq 0$ and $\gamma(t) \notin \widetilde{C}_1$ if $t \neq 1$. One cannot compute $i_\gamma f$ directly using the arguments of Proposition 1. Indeed the map Φ does not reach the vertical $\{0\} \times (-\infty, 0]$ on δ_v, but it may reach it on δ_h. One needs an indirect argument. Write $\widetilde{C}_2$ as the graph of a continuous function $\varphi : \mathbf{R} \to (0, +\infty)$ and define, for every $r \in \mathbf{R}$, the

homeomorphism

$$
\begin{aligned}
T_r : \widetilde{\mathbf{A}}_2 &\rightarrow \widetilde{\mathbf{A}}_2, \\
(x, s\varphi(x)) &\mapsto (x + r, s\varphi(x + r)),
\end{aligned}
$$

where $s \in [0, 1]$. Then consider the map

$$
\begin{aligned}
\Psi : \Delta \times [0, +\infty[&\rightarrow \mathbf{R}^2, \\
(t, t', r) &\mapsto T_{-(1-t')r} \circ f(\gamma(t')) - f^{-1} \circ T_{-r} \circ f(\gamma(t)).
\end{aligned}
$$

This map may vanish on $\Delta \times [0, +\infty[$. However, observe that $\Psi(t, 1, r) \neq 0$ if $t \neq 1$ (because $f(\gamma(1)) \notin \mathrm{Int}(\widetilde{\mathbf{A}}_1)$ and $f^{-1} \circ T_{-r} \circ f(\gamma(t)) \in \mathrm{Int}(\widetilde{\mathbf{A}}_1)$), that $\Psi(0, t', r) \in \mathbf{R} \times (0, +\infty)$ if $t' \neq 0$ (because $\gamma(t') \notin \mathbf{R} \times \{0\}$), that $\Psi(0, 0, r) \in (-\infty, 0) \times \{0\}$ (because of the twist condition on the lower boundary), that $\Psi(1, 1, r) \in (0, +\infty) \times \mathbf{R}$ (because of the twist condition on the upper boundary) and finally that there exists $r_0 \geq 0$ such that $\Psi(t, 1, r) \in (0, +\infty) \times \mathbf{R}$ if $r \geq r_0$ (for obvious reasons).
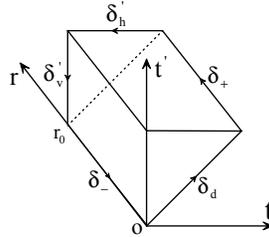

FIGURE 1. Integral illustration

Write (see Figure 1)

- δ_d for the segment joining $(0, 0, 0)$ to $(1, 1, 0)$;
- δ_+ for the segment joining $(1, 1, 0)$ to $(1, 1, r_0)$;
- δ'_h for the segment joining $(1, 1, r_0)$ to $(0, 1, r_0)$;
- δ'_v for the segment joining $(0, 1, r_0)$ to $(0, 0, r_0)$;
- δ_- for the segment joining $(0, 0, r_0)$ to $(0, 0, 0)$.

Observe that the loop $\delta_d \delta_+ \delta'_h \delta'_v \delta_-$ bounds a disk where Ψ does not vanish. This implies that

$$
i_\gamma f = \int_{\Psi \circ \delta_d} d\theta = -\int_{\Psi \circ (\delta_+ \delta'_h \delta'_v \delta_-)} d\theta,
$$

but the last integral belongs to $(\frac{1}{4}, \frac{3}{4})$ because Ψ does not reach the vertical $\{0\} \times (-\infty, 0]$ on $\delta_+ \delta'_h \delta'_v \delta_-$.

Lemma 3 may fail if none of the loops project injectively onto $\mathbf{T}^1 \times \{0\}$, unlike what is said in [D] and [J1], and then Theorems 3 and 4 are no longer valid (see also [MU]). We will conclude with the construction of an area-preserving and fixed point free homeomorphism satisfying a twist condition.

Let us consider the polygon Π whose vertices are (see Figure 2)

$$
P = (1, 0), \quad A = (1, 1), \quad B = (\frac{1}{2}, 1), \quad T^{-1}(A) = (0, 1), \quad T^{-1}(P) = (0, 0),
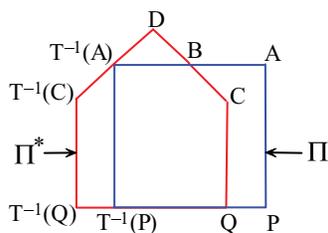$$

FIGURE 2. The construction of f_0

and the polygon Π^* whose vertices are

$$Q = (\frac{3}{4}, 0), \;\; C = (\frac{3}{4}, \frac{3}{4}), \;\; D = (\frac{1}{4}, \frac{5}{4}), \;\; T^{-1}(C) = (-\frac{1}{4}, \frac{3}{4}), \;\; T^{-1}(Q) = (-\frac{1}{4}, 0).$$

The bounded domains U and U^* enclosed by Π and Π^* having the same area, the piecewise affine map from Π to Π^*, sending P on Q, A on C, B on D, $T^{-1}(A)$ on $T^{-1}(C)$, $T^{-1}(P)$ on $T^{-1}(Q)$ and affine on each edge of Π, may be extended to an area-preserving homeomorphism from $\overline{U}$ to $\overline{U}^*$ (this is a consequence of Schoenflies theorem and of a classical result of Oxtoby-Ulam about topologically equivalent measures; see for example the appendix of [BCL]). This homeomorphism may be naturally extended to an area-preserving homeomorphism

$$f_0 : \widetilde{\mathbf{A}}_1 = \bigcup_{k \in \mathbf{Z}} T^k(\overline{U}) = \mathbf{R} \times [0,1] \to \widetilde{\mathbf{A}}_2 = \bigcup_{k \in \mathbf{Z}} T^k(\overline{U}^*)$$

that commutes with T. Let us fix $\varepsilon \in (0,1)$ and $M > 0$ such that

$$\begin{cases} p_1 \circ f_0(z) < p_1(z) & \text{for every } z \in \mathbf{R} \times [1-\varepsilon, 1]; \\ p_1 \circ f_0(z) < p_1(z) + M & \text{for every } z \in \mathbf{R} \times [0,1]. \end{cases}$$

The map

$$f_1 : (x,y) \mapsto f_0\left(x + \frac{M}{\varepsilon}(y-1), y\right)$$

coincides with f_0 on $\mathbf{R} \times \{1\}$, is a translation on $\mathbf{R} \times \{0\}$, preserves the area, commutes with T and satisfies

$$p_1 \circ f_1(z) < p_1(z) \quad \text{for every } z \in \mathbf{R} \times [0,1].$$

As a consequence f_1 has no fixed point. Observe that it sends $E = (\frac{3}{4}, 1)$ on B and $F = (\frac{1}{4}, 1)$ on $T^{-1}(A)$.

Now, let us consider (see Figure 3) :

-- the polygon Π_1 whose vertices are A, E, B, C;
-- the polygon Π_2 whose vertices are B, D, $T^{-1}(A)$, F;
-- the polygon Π_3 whose vertices are Q, C, B, F, $T^{-1}(A)$, $T^{-1}(C)$, $T^{-1}(Q)$;

and write U_i for the bounded domain enclosed by Π_i.

Similarly let us consider

-- the polygon Π_1' whose vertices are A', E', B', C';
-- the polygon Π_2' whose vertices are B', D', $T^{-1}(A')$, F';
-- the polygon Π_3' whose vertices are Q', C', B', F', $T^{-1}(A')$, $T^{-1}(C')$, $T^{-1}(Q')$,
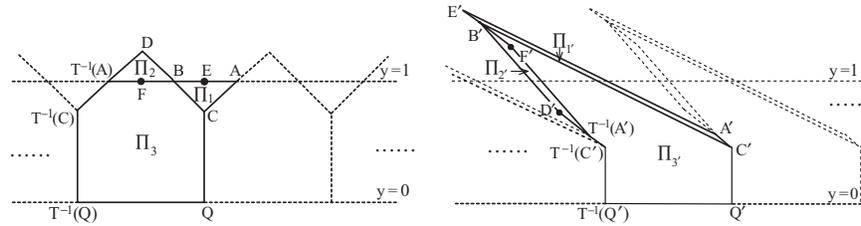
FIGURE 3. The construction of h

where $A' = (\frac{9}{8}, \frac{35}{64})$, $B' = (-\frac{3}{4}, \frac{93}{64})$, $C' = (\frac{5}{4}, \frac{29}{64})$, $D' = (-\frac{1}{8}, \frac{47}{64})$, $E' = (-\frac{7}{8}, \frac{99}{64})$, $F' = (-\frac{1}{2}, \frac{81}{64})$, $Q' = (\frac{5}{4}, 0)$. The bounded domains U'_1, U'_2, U'_3 enclosed respectively by Π'_1, Π'_2, Π'_3 are pairwise disjoint, and a simple computation gives us

$$\mu(U'_1) = \mu(U'_2) = \mu(U_1) = \mu(U_2) = \frac{1}{16}, \ \ \mu(U'_3) = \mu(U_3) = \frac{7}{8}.$$

Note that the second coordinate of the middle point of the diagonals of Π'_1 and Π'_2 is 1.

The piecewise affine map $\bigcup_{1 \leq i \leq 3} \Pi_i \to \bigcup_{1 \leq i \leq 3} \Pi'_i$ sending each vertex X on the vertex X' may be extended first to an area-preserving homeomorphism $\bigcup_{1 \leq i \leq 3} \overline{U_i} \to \bigcup_{1 \leq i \leq 3} \overline{U'_i}$ and then to an area-preserving homeomorphism

$$h : \bigcup_{k \in \mathbf{Z}} T^k \big(\bigcup_{1 \leq i \leq 3} \overline{U_i} \big) \to \bigcup_{k \in \mathbf{Z}} T^k \big(\bigcup_{1 \leq i \leq 3} \overline{U'_i} \big)$$

that commutes with T. The homeomorphism

$$f_2 : h \circ f_1 \circ h^{-1} : h(\widetilde{\mathbf{A}}_1) \to h(\widetilde{\mathbf{A}}_2)$$

preserves the area, commutes with T and is fixed point free. Nevertheless it satisfies the twist condition. Indeed one has $p_1 \circ f_2(z) = p_1 \circ f_1(z) < p_1(z)$ for every $z \in \mathbf{R} \times \{0\}$ and $p_1(f_2(z)) > p_1(z)$ for every z on the upper boundary $\widetilde{C}_1$ of $h(\widetilde{\mathbf{A}}_1)$. Indeed, the last property is true for each vertex A', E', B', F', $T^{-1}(A')$ (respectively sent onto C', B', D', $T^{-1}(A')$, $T^{-1}(C')$), and the map is affine between two vertices.

To get a smooth example, one considers a smooth essential simple loop $C'_1 \subset \pi(h(\widetilde{\mathbf{A}}_1))$ very close to $C_1 = \pi(\widetilde{C}_1)$. Then, on the annulus bounded by $\mathbf{T}^1 \times \{0\}$ and C'_1, one approximates the diffeomorphism F_2 lifted by f_2 by a smooth area-preserving diffeomorphism.

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