ON THE JAMES AND VON NEUMANN-JORDAN CONSTANTS IN BANACH SPACES

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Abstract. Recently Alonso, Martín and Papini conjectured that the value of the von Neumann-Jordan constant is less than or equal to that of the James constant. This paper presents an affirmative answer to such a conjecture. Moreover, we obtain a sharp estimate for the von Neumann-Jordan constant.

1. Introduction

Both the James constant $J(X)$ and the von Neumann-Jordan constant $C_{NJ}(X)$ play an important role in the description of various geometric structures. It is therefore worthwhile to clarify the relation between them. Kato, Maligranda and Takahashi [13] are the first who discussed their relation by the following inequality:

$$C_{NJ}(X) \leq \frac{[J(X)]^2}{1 + [J(X) - 1]^2}.$$ 

Since then, many authors have conducted worthwhile research on improving the above estimate. In 2003 Maligranda [15] formulated the following conjecture:

$$C_{NJ}(X) \leq 1 + [J(X)]^2/4.$$ 

Later on, some weak inequalities were obtained by several authors (see e.g. Maligranda et al. [16], Saejung [17], Takahashi [18]).

Recently Alonso, Martín and Papini [2] presented an inequality:

$$C_{NJ}(X) \leq 2 \left(1 + J(X) - \sqrt{2J(X)}\right),$$

which is a strong improvement of Maligranda’s conjecture. Wang and Pang [19] strengthened this inequality as

$$C_{NJ}(X) \leq J(X) + \sqrt{J(X) - 1} \left(\sqrt{1 + (1 - \sqrt{J(X) - 1})^2} - 1\right),$$

but it is weaker than the inequality $C_{NJ}(X) \leq J(X)$, conjectured by Alonso, Martín and Papini [2].

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This paper is devoted to an investigation of the relation between the James and von Neumann-Jordan constants. We first state an inequality concerning the constants $A_2(X)$ and $J(X)$, which enables us to clarify the relation between $J(X)$ and $J(X^*)$. Another inequality related to the constants $E(X)$ and $J(X)$ is also obtained. This allows us to get a better estimate for the von Neumann-Jordan constant.

2. Definitions and notation

Let $X$ be a real Banach space with $\dim X \geq 2$ and denote by $S_X$ and $B_X$ the unit sphere and the unit ball, respectively. The James constant

$$J(X) = \sup \left\{ \|x + y\| \wedge \|x - y\| : x, y \in S_X \right\}$$

and the von Neumann-Jordan constant

$$C_{NJ}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X \right\}$$

have been extensively studied. For more details, we refer to [4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 19]. We also need some other constants:

$$A_2(X) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} : x, y \in S_X \right\},$$

$$E(X) = \sup \left\{ \|x + y\|^2 + \|x - y\|^2 : x, y \in S_X \right\},$$

$$g(X) = \inf \left\{ \max(\|x + y\|, \|x - y\|) : x, y \in S_X \right\}.$$  

The first constant was defined by Baronti, Casini and Papini [3], and the other two were defined by Gao [9, 10]. It is worthwhile to mention that $J(X)g(X) = 2$ (see [10, 13]).

The modulus of convexity $\delta_X(\epsilon) : [0, 2] \to [0, 1]$ is defined as

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| \geq \epsilon \right\}.$$  

Obviously, $\delta_X(\epsilon)$ is nondecreasing on $[0, 2]$. Moreover, the function $\delta_X(\epsilon)/\epsilon$ is also nondecreasing on $(0, 2]$ (see [8]). Recall that a Banach space $X$ is called uniformly nonsquare if for any $x, y \in S_X$ there exists a $\delta > 0$ such that either $\|x - y\|/2 \leq 1 - \delta$, or $\|x + y\|/2 \leq 1 - \delta$. It is well known that $X$ is uniformly nonsquare if and only if $J(X) < 2$. The equality

$$J(X) = 2[1 - \delta_X(J(X))]$$

holds whenever $X$ is uniformly nonsquare (see [4]).

For simplicity we shall respectively denote $A_2(X), E(X), g(X)$ and $J(X)$ by $A, E, g$ and $J$ if it is required.

3. Some preliminary estimates

Let us first state an estimate of $A_2(X)$ in terms of $J(X)$. The first inequality between them was stated by Alonso and Llorens-Fuster [1] as

$$A_2(X) \leq 1 + \frac{J(X)}{2},$$

which has been improved by Wang and Pang [19] as

$$A_2(X) \leq 1 + \sqrt{J(X)} - 1.$$
The following is a further improvement of the above.

**Theorem 1.** For any Banach space $X$,

$$A_2(X) \leq \frac{3J(X) - 2}{J(X)}.$$  

**Proof.** We may assume that $X$ is uniformly nonsquare, since in the case $J = 2$ the inequality (2) is trivial. To show (2), we consider two cases for any $x, y \in S_X$.

**Case 1.** $J \leq \|x - y\| \leq 2$. Let $\|x - y\| = \epsilon$. According to the monotonicity of the function $\delta_X(\epsilon)/\epsilon$, one gets

$$\|x + y\| + \|x - y\| \leq \epsilon + 2(1 - \delta_X(\epsilon)) \leq \epsilon + 2 \left(1 - \frac{\delta_X(J)}{J}\epsilon\right)$$
$$= \epsilon + (2 - (g - 1)\epsilon) \leq \max_{J \leq \epsilon \leq 2} (2 - g)\epsilon + 2$$
$$= 6 - 2g,$$

where the first equality follows from (1).

**Case 2.** $\|x - y\| \leq J$. If $\|x + y\| \leq J$, then $\|x + y\| + \|x - y\| \leq 2J$. Since

$$J + \frac{2}{J} - 3 = \frac{(J - 2)(J - 1)}{J} \leq 0,$$

we have $J \leq 3 - 2/J = 3 - g$ and

$$\|x + y\| + \|x - y\| \leq 2J \leq 6 - 2g.$$  

Conversely, if $\|x + y\| \geq J$, let $\|x + y\| = \epsilon$. Then we get as in Case 1 that

$$\|x + y\| + \|x - y\| \leq 6 - 2g.$$  

Therefore, from both cases, we get (2). □

**Corollary 2.** For any Banach space $X$,

$$A_2(X) - J(X) \leq (\sqrt{2} - 1)^2.$$  

**Proof.** According to (2), we have

$$A_2(X) - J(X) \leq -J^2(X) + 3J(X) - 2 \leq \max_{\sqrt{2} \leq t \leq 2} \frac{-t^2 + 3t - 2}{t} = (\sqrt{2} - 1)^2,$$

which gives the result. □

Kato, Maligranda and Takahashi [13] discussed the relation between $J(X)$ and $J(X^*)$ and proved the following inequality:

$$2(J(X) - 1) \leq J(X^*) \leq J(X)/2 + 1.$$  

This gives an answer to the question posed by Gao and Lau [11]. Wang and Pang [19] improved this inequality as

$$1 + (J(X) - 1)^2 \leq J(X^*) \leq 1 + \sqrt{J(X) - 1}.$$  

Applying Theorem [1] we can give a further improvement of the above estimates.
Theorem 3. For any Banach space $X$, 
\begin{equation}
\frac{2}{3 - J(X)} \leq J(X^*) \leq \frac{3J(X) - 2}{J(X)}.
\end{equation}

Proof. It follows from Theorem [1] that 
\[\frac{2}{3 - J(X)} \leq \frac{2}{3 - A_2(X)} = \frac{2}{3 - A_2(X^*)} \leq J(X^*) \leq A_2(X^*) \leq A_2(X) \leq 3 - 2/J(X),\]
where one should note that $A_2(X) = A_2(X^*)$ [3, Proposition 2.2]. \hfill \Box

Corollary 4. For any Banach space $X$, 
\[|J(X) - J(X^*)| \leq \left(\sqrt{2} - 1\right)^2.\]

Proof. From (3) it follows that 
\begin{align*}
J(X) - J(X^*) &\leq J(X) - \max \left\{\sqrt{2}, \frac{2}{3 - J(X)}\right\} \\
&= \min \left\{J(X) - \sqrt{2}, \frac{-J^2(X) + 3J(X) - 2}{3 - J(X)}\right\} \\
&\leq \max_{\sqrt{2} \leq t \leq 2} \min \{f(t), g(t)\} \\
&= (\sqrt{2} - 1)^2,
\end{align*}
where $f(t) = t - \sqrt{2}$ and $g(t) = (-t^2 + 3t - 2)/(3 - t)$. On the other hand, also from (3), it follows that 
\begin{align*}
J(X^*) - J(X) &\leq \frac{-J^2(X) + 3J(X) - 2}{J(X)} \\
&\leq \max_{\sqrt{2} \leq t \leq 2} \frac{-t^2 + 3t - 2}{t} = (\sqrt{2} - 1)^2,
\end{align*}
which completes the proof. \hfill \Box

Next let us turn to the constant $E(X)$. Alonso, Martín and Papini [2] proved that 
\begin{equation}
2J^2(X) \leq E(X) \leq 4J(X),
\end{equation}
which can be strengthened as follows.

Theorem 5. For any Banach space $X$, 
\begin{equation}
E(X) \leq \frac{4(J^2(X) + 4(J(X) - 1)^2)}{J^2(X)}.
\end{equation}

Proof. Since $J(X)g(X) = 2$, it suffices to show that 
\[E(X) \leq 4(1 + (2 - g(X))^2).\]
Assume again that $X$ is uniformly nonsquare and consider two cases for any $x, y \in S_X$. 

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Case 1. \( J \leq \| x - y \| \leq 2 \). Let \( \| x - y \| = \epsilon \). The monotonicity of \( \delta_X(\epsilon)/\epsilon \) yields
\[
\| x + y \|^2 + \| x - y \|^2 \leq \epsilon^2 + 4(1 - \delta_X(\epsilon))^2 \\
\leq \epsilon^2 + 4 \left( 1 - \frac{\delta_X(J)}{J} \epsilon \right)^2 \\
= \epsilon^2 + (2 - (g - 1)\epsilon)^2.
\]
Since the function
\[
\varphi(\epsilon) := \epsilon^2 + (2 - (g - 1)\epsilon)^2
\]
is increasing on \([J, 2]\), we have
\[
\| x + y \|^2 + \| x - y \|^2 \leq \max_{\epsilon \leq 2} \varphi(\epsilon) = 4(1 + (2 - g)^2).
\]
Case 2. \( \| x - y \| \leq J \). If \( \| x + y \| \leq J \), then \( \| x + y \|^2 + \| x - y \|^2 \leq 2J^2 \). Note that
\[
J^4 - 8J^2 + 8J = J(J - 2)(J + 1 + \sqrt{5})(J + 1 - \sqrt{5}) \leq 0,
\]
which implies that
\[
J^2 \leq \frac{8J(J - 1)}{J^2} \leq \frac{2(J^2 + 4(J - 1)^2)}{J^2} = 2(1 + (2 - g)^2).
\]
Consequently,
\[
\| x + y \|^2 + \| x - y \|^2 \leq 2J^2 \leq 4(1 + (2 - g)^2).
\]
On the contrary, if \( \| x + y \| \geq J \), let \( \| x + y \| = \epsilon \). We get as in Case 1 that
\[
\| x + y \|^2 + \| x - y \|^2 \leq 4(1 + (2 - g)^2).
\]
Therefore, from both cases, we get (5). \( \square \)

Remark 1. It is easy to see that
\[
J^3 - (J^2 + 4(J - 1)^2) = (J - 1)(J - 2)^2 \geq 0,
\]
from which we have
\[
\frac{J^2 + 4(J - 1)^2}{J^2} \leq J.
\]
So Theorem 5 improves the inequality (3).

4. Estimate for the von Neumann-Jordan constant

Following the ideas in [17, 20], we can rewrite
\[
C_{NJ}(X) = \sup \{ C_{NJ}(t, X) : 0 \leq t \leq 1 \},
\]
where
\[
C_{NJ}(t, X) := \sup \left\{ \frac{\| x + ty \|^2 + \| x - ty \|^2}{2(1 + t^2)} : x, y \in S_X \right\}.
\]
Now let us state the main results.

Theorem 6. For any Banach space \( X \),
\[
C_{NJ}(X) \leq 1 + \frac{2(J(X) - 1)}{\sqrt{J^2(X) + (2 - J(X))^2 + 2 - J(X)}}.
\]

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Proof. Let \( x, y \in S_X \) and observe first that for every \( 0 \leq t \leq 1 \),
\[
\|x \pm ty\| \leq t\|x \pm y\| + (1 - t).
\]
This together with (2) and (5) yields
\[
C_{NJ}(t, X) = \sup \left\{ \frac{\|x + ty\|^2 + \|x - ty\|^2}{2(1 + t^2)} \right\} : x, y \in S_X \leq \frac{Et^2 + 4At(1 - t) + 2(1 - t)^2}{2(1 + t^2)} \leq 1 + \frac{2(2 - g)(1 - g)t^2 + t}{1 + t^2} := F(t).
\]
Let \( t_0 = \sqrt{(g - 1)^2 + 1 - (g - 1)} \). Then \( t_0 \in [0, 1] \) satisfies the equation \((1 - g)t + 1 = (1 + t^2)/2\). It is not difficult to deduce that
\[
C_{NJ}(X) \leq \max\{F(t) : 0 \leq t \leq 1\} = F(t_0) = 1 + \frac{2 - g}{\sqrt{1 + (g - 1)^2 + g - 1}}.
\]
Thus (6) follows from \( J(X)g(X) = 2 \). \( \square \)

In the paper [2] Alonso, Martín and Papini posed two questions: (1) Does the inequality \( C_{NJ}(X) \leq J(X) \) hold for any space? (2) Does the identity \( C_{NJ}(X) = J(X) \) hold only when both constants are equal to 2? By using Theorem 6 we can affirmatively answer these two questions.

Corollary 7. For any Banach space \( X \), \( C_{NJ}(X) \leq J(X) \).

Moreover the equality holds if and only if \( X \) is not uniformly nonsquare.

Proof. Since \( \sqrt{2} \leq J \leq 2 \), one gets
\[
1 + \frac{2(J - 1)}{\sqrt{J^2 + (2 - J)^2} + 2 - J} - J = \frac{(J - 1)(J - \sqrt{J^2 + (2 - J)^2})}{\sqrt{J^2 + (2 - J)^2} + 2 - J}.
\]
It is not hard to see that the right side above is less than or equal to zero and the equality holds if and only if \( J = 2 \), and thus the result follows. \( \square \)

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References


