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AN OPERATOR EQUATION, KDV EQUATION AND INVARIANT SUBSPACES

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Abstract. Let A be a bounded linear operator on a complex Banach space X. A problem, motivated by the operator method used to solve integrable systems such as the Korteweg-deVries (KdV), modified KdV, sine-Gordon, and Kadomtsev-Petviashvili (KP) equations, is whether there exists a bounded linear operator B such that (i) $AB+BA$ is of rank one, and (ii) $(I+f(A)B)$ is invertible for every function f analytic in a neighborhood of the spectrum of A. We investigate solutions to this problem and discover an intriguing connection to the invariant subspace problem. Under the assumption that the convex hull of the spectrum of A does not contain 0, we show that there exists a solution B to (i) and (ii) if and only if A has a non-trivial invariant subspace.

1. Introduction

Let X be an infinite dimensional Banach space, and let A be a bounded linear operator on X. Let $\sigma(A)$ denote the spectrum of A. It is well known that $\sigma(A)$ is a non-empty compact subset of the complex plane. Furthermore, $\sigma(A)$ is a disjoint union of the point spectrum $\sigma_p(A)$ (consisting of the eigenvalues of A), the continuous spectrum $\sigma_c(A)$, and the residual spectrum $\sigma_r(A)$. Recall that $\sigma_c(A) := \{\lambda \in \mathbb{C} : (\lambda I - A) \text{ is injective}, Range(\lambda I - A) \text{ is dense in } X, \text{ but } Range(\lambda I - A) \neq X\}$ and $\sigma_r(A) := \{\lambda \in \mathbb{C} : (\lambda I - A) \text{ is injective, but } Range(\lambda I - A) \text{ is not dense in } X\}$. For any operator A on X, let A^* denote the adjoint of A. That is, A^* is the linear operator defined on the dual space X' by $(A^*\phi)(x) = \phi(Ax)$ for each $x \in X$ and $\phi \in X'$.

In [1], Aden and Carl used a method known as the operator method to find solutions to the scalar Korteweg-deVries (KdV) equation $v_t = v_{xxx} + 3v_x^2$. A similar method was used in [4, 5, 6, 9] to solve some other non-linear partial differential equations such as the modified KdV, sine-Gordon, and KP equations. For the most general solution formula for the KP equation we refer to [13]. One of the main ingredients of the operator method to solve integrable systems involves solving the following problem: given a bounded linear operator A on the Banach space X, is it possible to find an operator B on X such that (a) $AB + BA$ is of rank one, and (b) $I + e^{p(A)}B$ is invertible for any polynomial $p(A)$?

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It was shown in [8] that if the point spectrum of A or A^* is non-empty for a bounded linear operator A on a Banach space X with $\dim(X) \geq 3$, then there exists a bounded linear operator B on X such that

(i) $AB + BA$ is of rank one, and
(ii) $I + f(A)B$ is invertible for every function f analytic in a neighborhood of $\sigma(A)$.

Recall that the residual spectrum of A is always contained in the point spectrum of the adjoint A^* of A. Thus, if $\sigma_p(A)$ or $\sigma_r(A)$ is non-empty, then there exists a bounded linear operator B on X satisfying conditions (i) and (ii) given above. In particular, the above result is true when X is a finite dimensional space as any linear operator on a finite dimensional space has a non-empty point spectrum. Therefore, it would be of interest to investigate the above problem when the space X is infinite dimensional over the complex field $\mathbb{C}$ and the spectrum of the bounded linear operator A on X is precisely the continuous spectrum $\sigma_c(A)$ of A; i.e., $\sigma_c(A) = \sigma(A)$.

In this article we investigate solutions to (i) and (ii) given above under different assumptions. One of the major assumptions we impose is that 0 not be in the convex hull of the spectrum of A. This assumption is natural in view of what is known about the *Sylvester equation* $A_1B + BA_2 = C$. We state the main facts presently after we introduce some standard notation. For any complex normed spaces X and Y, let $\mathcal{B}(Y, X)$ denote the space of all bounded linear operators from Y to X. The space $\mathcal{B}(X, X)$ will be denoted simply by $\mathcal{B}(X)$.

Let X and Y be Banach spaces, and let A_1 (respectively A_2) be bounded operators on X (respectively Y). Let τ be the operator on $\mathcal{B}(Y, X)$ defined by

$$\tau(S) = A_1S + SA_2.$$

It is well known that

$$\sigma(\tau) = \sigma(A_1) + \sigma(A_2). \tag{1}$$

The proof of the inclusion $\sigma(\tau) \subseteq \sigma(A_1)+\sigma(A_2)$ is due to Lumer and Rosenblum [11] (see also [3] and the references therein). The reverse inclusion, as noted in [11], is due to Kleineke (unpublished). A complete proof of (1) may also be found in [2].

A corollary of the above is that the equation $A_1S + SA_2 = T$ has a solution S for every T if $0 \notin \sigma(A_1)+\sigma(A_2)$. When $A_1 = A_2 = A$, the spectral condition above is satisfied when the convex hull of the spectrum of A does not include 0. In view of this, we shall seek solutions to (i) and (ii) under the assumption that

$$0 \notin \operatorname{conv}(\sigma(A)), \tag{2}$$

where $\operatorname{conv}(\Gamma)$ denotes the *convex hull* of the subset Γ of the complex plane $\mathbb{C}$, i.e., the smallest convex subset of $\mathbb{C}$ that includes Γ.

In Section 2 we show, assuming the spectral condition (2), that a solution to (i) and (ii) exists if and only if A has a non-trivial closed invariant subspace. In particular a solution exists if A is a normal operator on a Hilbert space, and in this case, condition (ii) is true if the function f is merely assumed to be continuous on the spectrum of A.

In section 3, we give some examples.

2. Main results

We start with an auxiliary proposition.

Proposition 2.1. *Let $A \in \mathcal{B}(X)$, where X is a complex Banach space. The spectrum $\sigma(A)$ is contained in $\{z \in \mathbb{C} : Re\, z < 0\}$ if and only if there exist positive real numbers C and ε such that $\|e^{tA}\| < Ce^{-\varepsilon t}$ for every $t > 0$.*

Proof. First, assume that $\sigma(A) \subset \{z \in \mathbb{C} : Re\, z < 0\}$. Since $\sigma(A)$ is a compact set, there exists an $\varepsilon > 0$ such that $Re\, \lambda < -2\varepsilon$ for each $\lambda \in \sigma(A)$. Since $|e^{\lambda}| < e^{-2\varepsilon}$, by the spectral mapping theorem $r(e^A) \le e^{-2\varepsilon} < e^{-\varepsilon}$, where $r(e^A)$ is the spectral radius of e^A. However, it is well known that $r(e^A) = \limsup_{t>0} \|e^{tA}\|^{1/t}$. Therefore, there exists $t_0 > 0$ such that $\|e^{tA}\| < e^{-\varepsilon t}$ for all $t > t_0$. Since the function $t \mapsto e^{\varepsilon t}\|e^{tA}\|$ is continuous on the compact interval $[0, t_0]$, it follows that there exists $C > 1$ such that $\|e^{tA}\|e^{\varepsilon t} < C$ for all t in $[0, t_0]$. Hence, $\|e^{tA}\| < Ce^{-\varepsilon t}$ for all $t > 0$.

For the converse, suppose that the norm inequality in the statement is satisfied but that there exists a $\lambda_0 \in \sigma(A)$ such that $Re\, \lambda_0 \ge 0$. Then for any $t > 0$, $1 \le |e^{\lambda_0 t}| \le \|e^{tA}\| \le Ce^{-\varepsilon t}$. Obviously, this is false. □

In the following, by a non-trivial subspace of X, we shall mean a subspace other than $\{0\}$ or X. Recall that a subspace M of X is said to be *invariant* under A if $A(M) \subseteq M$. For an operator A and a function f which is analytic in a neighborhood of the spectrum of A, the operator $f(A)$ is defined by the usual Riesz Functional Calculus ([7], VII.4).

Theorem 2.2. *Let A be a non-zero bounded linear operator on an infinite dimensional complex Banach space X such that $0 \notin \operatorname{conv}(\sigma(A))$, and assume that A has a non-trivial closed invariant subspace. Then there exists a bounded linear operator B on X such that*

(i) *$AB + BA$ is of rank one, and*
(ii) *$I + f(A)B$ is invertible for every function f analytic in a neighborhood of $\sigma(A)$.*

Furthermore, the operator B may be chosen so that $(f(A)B)^2 = 0$ for every f in the class of functions described above and consequently $(I+f(A)B)^{-1} = I - f(A)B$.

Remark 1. Every convex subset of the plane is an intersection of half-planes. Therefore, the condition that $0 \notin \operatorname{conv}(\sigma(A))$ is equivalent to the assertion that $\sigma(A)$ is included in a half-plane that does not include 0. We may then replace A by $e^{i\theta}A$ for an appropriate real number θ to get $\sigma(e^{i\theta}A) \subset \{z \in \mathbb{C} : Re\, z < 0\}$. Solving the operator equation for $e^{i\theta}A$ yields a solution for A itself. Consequently, we may assume, without loss of generality, that $\sigma(A) \subset \{z \in \mathbb{C} : Re\, z < 0\}$.

Remark 2. If either the point spectrum $\sigma_p(A)$ or the residual spectrum $\sigma_r(A)$ is non-empty, then the result follows from [8]. In particular, if the space X is finite dimensional, the point spectrum of any bounded operator on X is non-empty and hence the result follows from [8]. Therefore, in the above cases, the assumptions in Theorem 2.2 may be stated as "A has an invariant subspace of dimension 1 or codimension 1." Hence, in what follows, we may assume that $\sigma_p(A) = \sigma_r(A) = \emptyset$. We mention in passing that under these assumptions the invariant subspace M and X/M must be infinite dimensional, since otherwise it is easy to see that A or A^* has an eigenvalue.

Remark 3. In the following, two proofs of Theorem 2.2 will be presented. One is a *non-constructive* proof, which uses results on the spectra of operator equations to assert the existence of the operator B. The other is a *constructive* proof, which gives a concrete integral representation for the operator B.

First Proof (Non-constructive proof). Let M be a closed subspace of X which is invariant under A.

For clarity of exposition, we first write the proof in the case that M has a complement. Suppose M has a complementary (closed) subspace N. Let P be the projection of X onto M and $E := (I - P)$, where I is the identity operator on X. The operator A has a matrix representation of the form $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where $A_{11} \in \mathcal{B}(M)$, $A_{12} \in \mathcal{B}(N, M)$, $A_{22} \in \mathcal{B}(N)$. Obviously, $A_{11}x = Ax$ for all $x \in M$, $A_{12}x = PAx$ for all $x \in N$, and $A_{22}x = EAx$ for all $x \in N$. As noted in Remark 1, we shall assume that $\sigma(A) \subset \{z \in \mathbb{C} : Re\, z < 0\}$. By Proposition 2.1, it is straightforward to conclude that the spectra of A_{11} and A_{12} are also included in $\{z \in \mathbb{C} : Re\, z < 0\}$. The spectrum Σ of the operator $S \mapsto A_{11}S + SA_{22}$ on $\mathcal{B}(N, M)$ is $\sigma(A_{11}) + \sigma(A_{22})$ (see eq. (1) above). It then follows that $0 \notin \Sigma$. Consequently, for any rank-one operator R in $\mathcal{B}(N, M)$, there exists $S \in \mathcal{B}(N, M)$ such that $A_{11}S + SA_{22} = R$.

Now let

$$B = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}.$$

Obviously, $AB + BA$ is of rank one on X. Since M is invariant under A, it is also invariant under $f(A)$. Therefore, $f(A)$ must also have a matrix representation of the form $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$. Clearly,

$$(f(A)B)^2 = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $(I + f(A)B)^{-1} = I - f(A)B$.

We now return to the general case. Let $A_1 := A|_M$, and let A_2 be the operator on the quotient space X/M induced by A; i.e., $A_2(x + M) = Ax + M$ for each $x \in X$. As in the previous case, using Proposition 2.1, we may assume that the spectra of A, A_1, and let A_2 are included in the half-plane $\{z \in \mathbb{C} : Re\, z < 0\}$. Again, we get that $0 \notin \sigma(A_1) + \sigma(A_2)$. Hence, there exists a bounded linear operator S from X/M to M such that $A_1S + SA_2$ is of rank one. Let $B := JS\pi$, where $J : x \mapsto x$ is the natural injection from M into X and $\pi : x \mapsto x + M$ is the natural projection operator from X onto X/M. Now, for any $x \in X$ we have

$$(AB + BA)x = (A_1S + SA_2)(x + M).$$

Hence, $AB + BA$ is of rank one. Furthermore, it is easy to see that $(f(A)B)^2 = 0$. □

Second Proof (Constructive proof). Let M be a non-trivial invariant subspace of A. Note that M is also invariant under $f(A)$ for any analytic function f on a neighborhood of $\sigma(A)$. By the Hahn-Banach theorem, we may choose a non-zero bounded linear functional ϕ on X that annihilates M. Let v be a non-zero element in M. Define the operators $S_v : L^2(0,\infty) \to X$ and $R_\phi : X \to L^2(0,\infty)$ as follows:

$$S_v(u) = \int_0^\infty u(s)(e^{sA}v)\,ds, \quad \forall\, u \in L^2(0,\infty),$$
$$(R_\phi x)(s) = \phi(e^{sA}x), \quad \forall\, x \in X \text{ and } s \in (0,\infty).$$

By Proposition 2.1, the mapping $s \to \|e^{sA}\|$ is in $L^2(0,\infty)$. Hence, by Hölder's inequality, S_v is a well-defined bounded linear operator from $L^2(0,\infty)$ to X. Also, it is easy to show that R_ϕ is a well-defined bounded linear operator from X to $L^2(0,\infty)$. Define

$$B := S_v R_\phi.$$

We will show that operator B satisfies conditions (i) and (ii) of the theorem. Note that B has the following integral representation

$$Bx = \int_0^\infty \phi(e^{sA}x)\, e^{sA}v\, ds \quad \forall\, x \in X.$$

For any $x \in X$,

$$\begin{aligned}(AB + BA)x &= \int_0^\infty \Big(\phi(e^{sA}x)\, e^{sA}Av + \phi(e^{sA}Ax)\, e^{sA}v\Big)\, ds \\ &= \int_0^\infty \Big(\frac{d}{ds}\,\phi(e^{sA}x)\, e^{sA}v\Big)\, ds \\ &= \phi(e^{sA}x)\, e^{sA}v\Big|_0^\infty \\ &= -\phi(x)\, v.\end{aligned}$$

Thus $AB + BA$ is a rank one operator. This proves (i). Since the range of B is contained in M and ϕ annihilates M, it follows that

$$(R_\phi f(A)B)(x)(s) = 0$$

for all $x \in X$ and $s > 0$ and hence $(f(A)B)^2 = 0$. Therefore, $(I + f(A)B)^{-1} = I - f(A)B$. This proves (ii). □

Theorem 2.3 (Converse of Theorem 2.2)**.** *Let A be a non-zero bounded operator on an infinite dimensional Banach space X such that $0 \notin \mathrm{conv}(\sigma(A))$. If there is a bounded linear operator B on X satisfying conditions* (i) *and* (ii) *of Theorem* 2.2*, then A must have a non-trivial invariant subspace.*

Proof. As mentioned earlier in the paper, we may assume that $\sigma(A)$ is contained in $\{z \in \mathbb{C} : Re\, z < 0\}$. Otherwise, we replace A by $e^{i\theta}A$ for an appropriate real number θ. Let B be a bounded operator which satisfies conditions (i) and (ii) of Theorem 2.2, and let $L = AB + BA$. By Theorem VII.23 of [3] we have $B = \int_0^\infty e^{tA}Le^{tA}\,dt$. Since L has rank one, B is a norm limit of Riemann sums, each of which has finite rank (as a finite sum of rank one operators). Therefore, B is a compact operator. Suppose that A does not have an invariant subspace. Let $\mathcal{U}$ be the subalgebra of $\mathcal{B}(X)$ consisting of all operators $h(A)$ where h is a function analytic in a neighborhood of $\sigma(A)$. Obviously, $\mathcal{U}$ contains the identity operator. If A does not have a proper invariant subspace, then the algebra $\mathcal{U}$ will not have any proper invariant subspace. Hence, by Lomonosov's Lemma (see Lemma 8.22, [12]), for any compact operator K on X there exists an operator $T = h(A)$ in $\mathcal{U}$ such that the null space of $I - h(A)K$ is non-zero. This contradicts condition (ii) if we let $K = -B$. □

Remark 4. For a general operator, the Functional Calculus $f \mapsto f(A)$ is defined only for functions f that are analytic in some neighborhood of $\sigma(A)$. On the other hand, for normal operators on Hilbert space, which are the subject of the next corollary, the spectral theorem provides a richer Functional Calculus defined for all Borel functions on $\sigma(A)$, in particular $f(A)$ is defined for $f \in C(\sigma(A))$, the space of continuous functions on the spectrum of A ([7], IX.8). The above proof is easily seen to be valid for such functions; indeed all that is required of $f(A)$ is that it leaves M invariant.

Corollary 2.4. *Let A be a normal operator on a complex Hilbert space H, and assume that $0 \notin \operatorname{conv}(\sigma(A))$. Then there exists a bounded linear operator B on H satisfying conditions* (i) *and* (ii) *of Theorem* 2.2 *where the function f in* (ii) *is merely assumed to be continuous on $\sigma(A)$.*

Proof. By the Spectral Theorem, every normal operator on a complex Hilbert space has a non-trivial invariant subspace. Hence, the result follows from the proof of Theorem 2.2. □

3. Examples

Example 3.1. For $p \geq 1$, let $L^p[a,b]$ be the Banach space of all complex-valued measurable functions f such that $|f|^p$ is integrable on the closed interval $[a,b]$. The multiplication operator $A : L^p[a,b] \to L^p[a,b]$ is defined by $Ax(t) = tx(t)$. It is known that A is a bounded linear operator on $L^p[a,b]$ and that $\sigma(T) = \sigma_c(T) = [a,b]$. When $p = 2$, it is obvious that A is a self-adjoint (and hence normal) operator.

If $a > 0$, then A satisfies the spectral condition (2). Let $q = p/(p-1)$, the conjugate transpose of p (taken to be ∞ when $p = 1$ and to be 1 when $p = \infty$). Let $a < c < b$ and take nonzero functions $u \in L^p[a,b]$ and $v \in L^q[a,b]$ such that u vanishes on $[c,b]$ and v vanishes on $[a,c]$. If B is the integral operator with kernel $k(s,t) = \frac{u(s)v(t)}{s+t}$, i.e.,

$$(Bx)(s) = \int_a^b k(s,t)x(t)dt,$$

then it is straightforward to verify that B satisfies (i) and (ii).

Example 3.2. Let H be a complex Hilbert space, and let $\{H_k\}_{-\infty}^{\infty}$ be a sequence of mutually orthogonal subspaces of the same (finite or infinite) dimension d such that $H = \sum_{k=-\infty}^{\infty} \oplus H_k$. Let $U_k : H_k \to H_{k+1}$ be a sequence of unitary transformations. For every $x = \sum_{k=-\infty}^{\infty} x_k$, $x_k \in H_k$, let $Sx = \sum_{k=-\infty}^{\infty} U_k x_k$. The operator S is called the bilateral shift of multiplicity d on H. It can be shown that S is a normal operator on H with empty point spectrum. Moreover,

$$\sigma(S) = \sigma_c(S) = \{z \in \mathbb{C} : |z| = 1\}.$$

Also, it is straightforward to see that $S^*(x) = \sum_{k=-\infty}^{\infty} U_{k+1}x_{k+1}$ and $\sigma(S^*) = \sigma_c(S^*) = \{z \in \mathbb{C} : |z| = 1\}$ (refer to p. 469 in [10] for details).

Let λ be any complex number satisfying $|\lambda| > 1$. For any such λ, let $A_\lambda := S + \lambda I$. Obviously, A_λ is a normal operator with empty point spectrum that satisfies the conditions of Theorem 2.2.

Example 3.3. Let T be a bounded normal operator on a separable Hilbert space $\mathcal{H}$. Then by the spectral theory, there exist a finite measure space $(\tilde{X}, \mu)$, a bounded complex function φ on $\tilde{X}$, and a unitary operator $U : \mathcal{H} \to L^2(\tilde{X}, \mu)$ such that

$(UTU^{-1})(x) = (M_\varphi f)(x) := \varphi(x)f(x)$ for each $f \in L^2(\tilde{X}, \mu)$ and $x \in \tilde{X}$, where φ is a bounded complex measurable function on $\tilde{X}$. Here $L^2(\tilde{X}, \mu)$ is the Hilbert space of complex square summable functions on $\tilde{X}$. Therefore, any normal operator T on a separable Hilbert space $\mathcal{H}$ is similar to a multiplication operator M_φ on $L^2(\tilde{X}, \mu)$ of a finite measure space. Moreover, spectrum $\sigma(T)$ of T is equal to the essential range of φ. Furthermore, if the measure of $\varphi^{-1}(\lambda)$ is zero for any complex number λ and the essential range of φ is properly contained in $\{z \in \mathbb{C} : Re\, z < 0\}$, then the multiplication operator M_φ will satisfy the hypotheses of Theorem 2.2. Let P be a measurable subset of $\tilde{X}$ such that $\mu(P)$ and $\mu(\tilde{X} - P)$ are non-zero. If such a measurable set P does not exist, then $L^2(\tilde{X}, \mu)$ will be one dimensional. Let I_P be the set of all bounded measurable functions f such that $f = 0$ a.e. on P. Obviously, I_P is an invariant subspace of M_φ. Let g and v be bounded measurable functions such that the support of g is contained in P and the support of v is contained in $\tilde{X} - P$. Obviously, $v \in I_P$ and the bounded linear functional $f \overset{\phi}{\mapsto} \langle f, g\rangle$ annihilates I_P. By the integral formula in the constructive proof of the Theorem 2.2, the bounded linear operator B corresponding to M_φ (using the functional $\phi : f \to \langle f, g\rangle$ and v) is given by

$$\begin{aligned}(Bf)(x) &= \int_0^\infty \left(\int_{\tilde{X}} (e^{sM_\varphi} f)(t)\overline{g(t)}\, d\mu(t) \right) (e^{sM_\varphi} v)(x)\, ds \\ &= \int_0^\infty \left(\int_{\tilde{X}} (e^{s\varphi(t)f(t)})\overline{g(t)}\, d\mu(t) \right) e^{s\varphi(x)v(x)}\, ds \\ &= \int_{\tilde{X}} \left(\int_0^\infty (e^{s(\varphi(t)f(t)+\varphi(x)v(x))}\, ds \right) \overline{g(t)}\, d\mu(t)\end{aligned}$$

for such $f \in L^2(\tilde{X}, \mu)$ and $x \in \tilde{X}$.

Remark 5. Finally, it is worth investigating whether Theorem 2.2 is true for any normal operator on a Hilbert space with empty point spectrum regardless of the location of the spectrum in the complex plane.

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