AN OPERATOR EQUATION, KDV EQUATION AND INVARIANT SUBSPACES

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(Communicated by Nigel J. Kalton)

Abstract. Let $A$ be a bounded linear operator on a complex Banach space $X$. A problem, motivated by the operator method used to solve integrable systems such as the Korteweg-deVries (KdV), modified KdV, sine-Gordon, and Kadomtsev-Petviashvili (KP) equations, is whether there exists a bounded linear operator $B$ such that (i) $AB + BA$ is of rank one, and (ii) $(I + f(A))B$ is invertible for every function $f$ analytic in a neighborhood of the spectrum of $A$. We investigate solutions to this problem and discover an intriguing connection to the invariant subspace problem. Under the assumption that the convex hull of the spectrum of $A$ does not contain 0, we show that there exists a solution $B$ to (i) and (ii) if and only if $A$ has a non-trivial invariant subspace.

1. Introduction

Let $X$ be an infinite dimensional Banach space, and let $A$ be a bounded linear operator on $X$. Let $\sigma(A)$ denote the spectrum of $A$. It is well known that $\sigma(A)$ is a non-empty compact subset of the complex plane. Furthermore, $\sigma(A)$ is a disjoint union of the point spectrum $\sigma_p(A)$ (consisting of the eigenvalues of $A$), the continuous spectrum $\sigma_c(A)$, and the residual spectrum $\sigma_r(A)$. Recall that $\sigma_c(A) := \{ \lambda \in \mathbb{C} : (\lambda I - A) \text{ is injective, } \text{Range}(\lambda I - A) \text{ is dense in } X, \text{ but } \text{Range}(\lambda I - A) \neq X\}$ and $\sigma_r(A) := \{ \lambda \in \mathbb{C} : (\lambda I - A) \text{ is injective, but } \text{Range}(\lambda I - A) \text{ is not dense in } X\}$. For any operator $A$ on $X$, let $A^*$ denote the adjoint of $A$. That is, $A^*$ is the linear operator defined on the dual space $X'$ by $(A^* \phi)(x) = \phi(Ax)$ for each $x \in X$ and $\phi \in X'$.

In [1], Aden and Carl used a method known as the operator method to find solutions to the scalar Korteweg-deVries (KdV) equation $v_t = v_{xxx} + 3v^2$. A similar method was used in [1, 5, 6, 9] to solve some other non-linear partial differential equations such as the modified KdV, sine-Gordon, and KP equations. For the most general solution formula for the KP equation we refer to [13]. One of the main ingredients of the operator method to solve integrable systems involves solving the following problem: given a bounded linear operator $A$ on the Banach space $X$, is it possible to find an operator $B$ on $X$ such that (a) $AB + BA$ is of rank one, and (b) $I + e^{p(A)}B$ is invertible for any polynomial $p(A)$?
It was shown in [8] that if the point spectrum of $A$ or $A^*$ is non-empty for a bounded linear operator $A$ on a Banach space $X$ with $\dim(X) \geq 3$, then there exists a bounded linear operator $B$ on $X$ such that

(i) $AB + BA$ is of rank one, and
(ii) $I + f(A)B$ is invertible for every function $f$ analytic in a neighborhood of $\sigma(A)$.

Recall that the residual spectrum of $A$ is always contained in the point spectrum of the adjoint $A^*$ of $A$. Thus, if $\sigma_p(A)$ or $\sigma_r(A)$ is non-empty, then there exists a bounded linear operator $B$ on $X$ satisfying conditions (i) and (ii) given above. In particular, the above result is true when $X$ is a finite dimensional space as any linear operator on a finite dimensional space has a non-empty point spectrum. Therefore, it would be of interest to investigate the above problem when the space $X$ is infinite dimensional over the complex field $\mathbb{C}$ and the spectrum of the bounded linear operator $A$ on $X$ is precisely the continuous spectrum $\sigma_c(A)$ of $A$; i.e., $\sigma_c(A) = \sigma(A)$.

In this article we investigate solutions to (i) and (ii) given above under different assumptions. One of the major assumptions we impose is that $0$ not be in the convex hull of the spectrum of $A$. This assumption is natural in view of what is known about the Sylvester equation $A_1B + BA_2 = C$. We state the main facts presently after we introduce some standard notation.

For any complex normed spaces $X$ and $Y$, let $B(Y,X)$ denote the space of all bounded linear operators from $Y$ to $X$. The space $B(X,X)$ will be denoted simply by $B(X)$.

Let $X$ and $Y$ be Banach spaces, and let $A_1$ (respectively $A_2$) be bounded operators on $X$ (respectively $Y$). Let $\tau$ be the operator on $B(Y,X)$ defined by 

$$\tau(S) = A_1S + SA_2.$$ 

It is well known that

$$\sigma(\tau) = \sigma(A_1) + \sigma(A_2).$$

The proof of the inclusion $\sigma(\tau) \subseteq \sigma(A_1) + \sigma(A_2)$ is due to Lumer and Rosenblum [11] (see also [3] and the references therein). The reverse inclusion, as noted in [11], is due to Kleineke (unpublished). A complete proof of (1) may also be found in [2].

A corollary of the above is that the equation $A_1S + SA_2 = T$ has a solution $S$ for every $T$ if $0 \notin \sigma(A_1) + \sigma(A_2)$. When $A_1 = A_2 = A$, the spectral condition above is satisfied when the convex hull of the spectrum of $A$ does not include 0. In view of this, we shall seek solutions to (i) and (ii) under the assumption that

$$0 \notin \text{conv}(\sigma(A)),$$

where $\text{conv}(\Gamma)$ denotes the convex hull of the subset $\Gamma$ of the complex plane $\mathbb{C}$, i.e., the smallest convex subset of $\mathbb{C}$ that includes $\Gamma$.

In Section 2 we show, assuming the spectral condition (2), that a solution to (i) and (ii) exists if and only if $A$ has a non-trivial closed invariant subspace. In particular a solution exists if $A$ is a normal operator on a Hilbert space, and in this case, condition (ii) is true if the function $f$ is merely assumed to be continuous on the spectrum of $A$.

In section 3 we give some examples.

2. Main results

We start with an auxiliary proposition.
Proposition 2.1. Let $A \in \mathcal{B}(X)$, where $X$ is a complex Banach space. The spectrum $\sigma(A)$ is contained in $\{ z \in \mathbb{C} : Re z < 0 \}$ if and only if there exist positive real numbers $C$ and $\varepsilon$ such that $\|e^{tA}\| < Ce^{-\varepsilon t}$ for every $t > 0$.

Proof. First, assume that $\sigma(A) \subset \{ z \in \mathbb{C} : Re z < 0 \}$. Since $\sigma(A)$ is a compact set, there exists an $\varepsilon > 0$ such that $Re \lambda < -2\varepsilon$ for each $\lambda \in \sigma(A)$. Since $|e^\lambda| < e^{-2\varepsilon}$, by the spectral mapping theorem $r(e^\lambda) \leq e^{-2\varepsilon} < e^{-\varepsilon}$, where $r(e^\lambda)$ is the spectral radius of $e^\lambda$. However, it is well known that $r(e^\lambda) = \limsup_{t \to 0} \|e^{t\lambda}\|^{1/t}$. Therefore, there exists $t_0 > 0$ such that $\|e^{t\lambda}\| < e^{-\varepsilon t}$ for all $t > t_0$. Since the function $t \mapsto e^{t\lambda}\|e^{t\lambda}\|$ is continuous on the compact interval $[0, t_0]$, it follows that there exists $C > 1$ such that $\|e^{t\lambda}\|e^{-\varepsilon t} < C$ for all $t$ in $[0, t_0]$. Hence, $\|e^{t\lambda}\| < Ce^{-\varepsilon t}$ for all $t > 0$.

For the converse, suppose that the norm inequality in the statement is satisfied but that there exists a $\lambda_0 \in \sigma(A)$ such that $Re \lambda_0 \geq 0$. Then for any $t > 0$, $1 \leq |e^{\lambda_0 t}| \leq \|e^{t\lambda}\| \leq Ce^{-\varepsilon t}$. Obviously, this is false. \qed

In the following, by a non-trivial subspace of $X$, we shall mean a subspace other than $\{0\}$ or $X$. Recall that a subspace $M$ of $X$ is said to be invariant under $A$ if $A(M) \subseteq M$. For an operator $A$ and a function $f$ which is analytic in a neighborhood of the spectrum of $A$, the operator $f(A)$ is defined by the usual Riesz Functional Calculus ([7], VII.4).

Theorem 2.2. Let $A$ be a non-zero bounded linear operator on an infinite dimensional complex Banach space $X$ such that $0 \notin \text{conv}(\sigma(A))$, and assume that $A$ has a non-trivial closed invariant subspace. Then there exists a bounded linear operator $B$ on $X$ such that

(i) $AB + BA$ is of rank one, and
(ii) $I + f(A)B$ is invertible for every function $f$ analytic in a neighborhood of $\sigma(A)$.

Furthermore, the operator $B$ may be chosen so that $(f(A)B)^2 = 0$ for every $f$ in the class of functions described above and consequently $(I + f(A)B)^{-1} = I - f(A)B$.

Remark 1. Every convex subset of the plane is an intersection of half-planes. Therefore, the condition that $0 \notin \text{conv}(\sigma(A))$ is equivalent to the assertion that $\sigma(A)$ is included in a half-plane that does not include 0. We may then replace $A$ by $e^{i\theta}A$ for an appropriate real number $\theta$ to get $\sigma(e^{i\theta}A) \subset \{ z \in \mathbb{C} : Re z < 0 \}$. Solving the operator equation for $e^{i\theta}A$ yields a solution for $A$ itself. Consequently, we may assume, without loss of generality, that $\sigma(A) \subset \{ z \in \mathbb{C} : Re z < 0 \}$.

Remark 2. If either the point spectrum $\sigma_p(A)$ or the residual spectrum $\sigma_r(A)$ is non-empty, then the result follows from [8]. In particular, if the space $X$ is finite dimensional, the point spectrum of any bounded operator on $X$ is non-empty and hence the result follows from [8]. Therefore, in the above cases, the assumptions in Theorem 2.2 may be stated as “$A$ has an invariant subspace of dimension 1 or codimension 1.” Hence, in what follows, we may assume that $\sigma_p(A) = \sigma_r(A) = \emptyset$. We mention in passing that under these assumptions the invariant subspace $M$ and $X/M$ must be infinite dimensional, since otherwise it is easy to see that $A$ or $A^*$ has an eigenvalue.

Remark 3. In the following, two proofs of Theorem 2.2 will be presented. One is a non-constructive proof, which uses results on the spectra of operator equations
to assert the existence of the operator $B$. The other is a constructive proof, which gives a concrete integral representation for the operator $B$.

**First Proof (Non-constructive proof).** Let $M$ be a closed subspace of $X$ which is invariant under $A$.

For clarity of exposition, we first write the proof in the case that $M$ has a complement. Suppose $M$ has a complementary (closed) subspace $N$. Let $P$ be the projection of $X$ onto $M$ and $E := (I - P)$, where $I$ is the identity operator on $X$. The operator $A$ has a matrix representation of the form $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where $A_{11} \in \mathcal{B}(M)$, $A_{12} \in \mathcal{B}(N, M)$, $A_{22} \in \mathcal{B}(N)$. Obviously, $A_{11}x = Ax$ for all $x \in M$, $A_{12}x = PAx$ for all $x \in N$, and $A_{22}x = EAx$ for all $x \in N$. As noted in Remark 1, we shall assume that $\sigma(A) \subset \{z \in \mathbb{C} : \Re z < 0\}$. By Proposition 2.1 it is straightforward to conclude that the spectra of $A_{11}$ and $A_{12}$ are also included in $\{z \in \mathbb{C} : \Re z < 0\}$. The spectrum $\Sigma$ of the operator $S \mapsto A_{11}S + SA_{22}$ on $\mathcal{B}(N, M)$ is $\sigma(A_{11}) + \sigma(A_{22})$ (see eq. (1) above). It then follows that $0 \notin \Sigma$. Consequently, for any rank-one operator $R$ in $\mathcal{B}(N, M)$, there exists $S \in \mathcal{B}(N, M)$ such that $A_{11}S + SA_{22} = R$.

Now let

$$B = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}.$$ 

Obviously, $AB + BA$ is of rank one on $X$. Since $M$ is invariant under $A$, it is also invariant under $f(A)$. Therefore, $f(A)$ must also have a matrix representation of the form $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$. Clearly,

$$(f(A)B)^2 = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $(I + f(A)B)^{-1} = I - f(A)B$.

We now return to the general case. Let $A_1 := A|_M$, and let $A_2$ be the operator on the quotient space $X/M$ induced by $A$; i.e., $A_2(x + M) = Ax + M$ for each $x \in X$. As in the previous case, using Proposition 2.1 we may assume that the spectra of $A$, $A_1$, and let $A_2$ are included in the half-plane $\{z \in \mathbb{C} : \Re z < 0\}$. Again, we get that $0 \notin \sigma(A_1) + \sigma(A_2)$. Hence, there exists a bounded linear operator $S$ from $X/M$ to $M$ such that $A_1S + SA_2$ is of rank one. Let $B := JS\pi$, where $J : x \mapsto x$ is the natural injection from $M$ into $X$ and $\pi : x \mapsto x + M$ is the natural projection operator from $X$ onto $X/M$. Now, for any $x \in X$ we have

$$(AB + BA)x = (A_1S + SA_2)(x + M).$$

Hence, $AB + BA$ is of rank one. Furthermore, it is easy to see that $(f(A)B)^2 = 0$. \hfill \Box

**Second Proof (Constructive proof).** Let $M$ be a non-trivial invariant subspace of $A$. Note that $M$ is also invariant under $f(A)$ for any analytic function $f$ on a neighborhood of $\sigma(A)$. By the Hahn-Banach theorem, we may choose a non-zero bounded linear functional $\phi$ on $X$ that annihilates $M$. Let $v$ be a non-zero element...
in $M$. Define the operators $S_v : L^2(0, \infty) \to X$ and $R_\phi : X \to L^2(0, \infty)$ as follows:

$$ S_v(u) = \int_0^\infty u(s)(e^{sA}v) \, ds, \quad \forall u \in L^2(0, \infty), $$

$$ (R_\phi x)(s) = \phi(e^{sA}x), \quad \forall x \in X \text{ and } s \in (0, \infty). $$

By Proposition 2.1 the mapping $s \to \|e^{sA}\|$ is in $L^2(0, \infty)$. Hence, by Hölder’s inequality, $S_v$ is a well-defined bounded linear operator from $L^2(0, \infty)$ to $X$. Also, it is easy to show that $R_\phi$ is a well-defined bounded linear operator from $X$ to $L^2(0, \infty)$. Define

$$ B := S_v R_\phi. $$

We will show that operator $B$ satisfies conditions (i) and (ii) of the theorem. Note that $B$ has the following integral representation

$$ Bx = \int_0^\infty \phi(e^{sA}x) e^{sA}v \, ds \quad \forall x \in X. $$

For any $x \in X$,

$$ (AB + BA)x = \int_0^\infty \left( \phi(e^{sA}x) e^{sA}Av + \phi(e^{sA}Ax) e^{sA}v \right) \, ds $$

$$ = \int_0^\infty \left( \frac{d}{ds} \phi(e^{sA}x) e^{sA}v \right) \, ds $$

$$ = \phi(e^{sA}x) e^{sA}v \big|_0^\infty $$

$$ = -\phi(x) v. $$

Thus $AB + BA$ is a rank one operator. This proves (i). Since the range of $B$ is contained in $M$ and $\phi$ annihilates $M$, it follows that

$$ (R_\phi f(A)B)(x)(s) = 0 $$

for all $x \in X$ and $s > 0$ and hence $(f(A)B)^2 = 0$. Therefore, $(I + f(A)B)^{-1} = I - f(A)B$. This proves (ii).

**Theorem 2.3** (Converse of Theorem 2.2). Let $A$ be a non-zero bounded operator on an infinite dimensional Banach space $X$ such that $0 \notin \text{conv} \sigma(A)$. If there is a bounded linear operator $B$ on $X$ satisfying conditions (i) and (ii) of Theorem 2.2, then $A$ must have a non-trivial invariant subspace.

**Proof.** As mentioned earlier in the paper, we may assume that $\sigma(A)$ is contained in $\{ z \in \mathbb{C} : \text{Re} \, z < 0 \}$. Otherwise, we replace $A$ by $e^{i\theta}A$ for an appropriate real number $\theta$. Let $B$ be a bounded operator which satisfies conditions (i) and (ii) of Theorem 2.2 and let $L = AB + BA$. By Theorem VII.23 of [3] we have $B = \int_0^\infty e^{tA}Le^{tA} \, dt$. Since $L$ has rank one, $B$ is a norm limit of Riemann sums, each of which has finite rank (as a finite sum of rank one operators). Therefore, $B$ is a compact operator. Suppose that $A$ does not have an invariant subspace. Let $\mathcal{U}$ be the subalgebra of $B(X)$ consisting of all operators $h(A)$ where $h$ is a function analytic in a neighborhood of $\sigma(A)$. Obviously, $\mathcal{U}$ contains the identity operator. If $A$ does not have a proper invariant subspace, then the algebra $\mathcal{U}$ will not have any proper invariant subspace. Hence, by Lomonosov’s Lemma (see Lemma 8.22, [12]), for any compact operator $K$ on $X$ there exists an operator $T = h(A)$ in $\mathcal{U}$ such that the null space of $I - h(A)K$ is non-zero. This contradicts condition (ii) if we let $K = -B$. 

\[ \square \]
Remark 4. For a general operator, the Functional Calculus $f \mapsto f(A)$ is defined only for functions $f$ that are analytic in some neighborhood of $\sigma(A)$. On the other hand, for normal operators on Hilbert space, which are the subject of the next corollary, the spectral theorem provides a richer Functional Calculus defined for all Borel functions on $\sigma(A)$, in particular $f(A)$ is defined for $f \in C(\sigma(A))$, the space of continuous functions on the spectrum of $A$ ([7], IX.8). The above proof is easily seen to be valid for such functions; indeed all that is required of $f(A)$ is that it leaves $M$ invariant.

Corollary 2.4. Let $A$ be a normal operator on a complex Hilbert space $H$, and assume that $0 \notin \mathrm{conv}(\sigma(A))$. Then there exists a bounded linear operator $B$ on $H$ satisfying conditions (i) and (ii) of Theorem 2.2, where the function $f$ in (ii) is merely assumed to be continuous on $\sigma(A)$.

Proof. By the Spectral Theorem, every normal operator on a complex Hilbert space has a non-trivial invariant subspace. Hence, the result follows from the proof of Theorem 2.2.

3. Examples

Example 3.1. For $p \geq 1$, let $L^p[a, b]$ be the Banach space of all complex-valued measurable functions $f$ such that $|f|^p$ is integrable on the closed interval $[a, b]$. The multiplication operator $A : L^p[a, b] \to L^p[a, b]$ is defined by $Ax(t) = tx(t)$. It is known that $A$ is a bounded linear operator on $L^p[a, b]$ and that $\sigma(T) = \sigma_c(T) = [a, b]$. When $p = 2$, it is obvious that $A$ is a self-adjoint (and hence normal) operator.

If $a > 0$, then $A$ satisfies the spectral condition (2). Let $q = p/(p - 1)$, the conjugate transpose of $p$ (taken to be $\infty$ when $p = 1$ and to be 1 when $p = \infty$). Let $a < c < b$ and take nonzero functions $u \in L^p[a, b]$ and $v \in L^q[a, b]$ such that $u$ vanishes on $[c, b]$ and $v$ vanishes on $[a, c]$. If $B$ is the integral operator with kernel $k(s, t) = \frac{u(s)v(t)}{s + t}$, i.e.,

$$(Bx)(s) = \int_a^b k(s, t)x(t)dt,$$

then it is straightforward to verify that $B$ satisfies (i) and (ii).

Example 3.2. Let $H$ be a complex Hilbert space, and let $\{H_k\}_{k=\infty}^\infty$ be a sequence of mutually orthogonal subspaces of the same (finite or infinite) dimension $d$ such that $H = \sum_{k=\infty}^\infty \oplus H_k$. Let $U_k : H_k \to H_{k+1}$ be a sequence of unitary transformations.

For every $x = \sum_{k=-\infty}^\infty x_k$, $x_k \in H_k$, let $Sx = \sum_{k=-\infty}^\infty U_kx_k$. The operator $S$ is called the bilateral shift of multiplicity $d$ on $H$. It can be shown that $S$ is a normal operator on $H$ with empty point spectrum. Moreover,

$$\sigma(S) = \sigma_c(S) = \{z \in \mathbb{C} : |z| = 1\}.$$

Also, it is straightforward to see that $S^*(x) = \sum_{k=-\infty}^\infty U_{k+1}x_{k+1}$ and $\sigma(S^*) = \sigma_c(S^*) = \{z \in \mathbb{C} : |z| = 1\}$ (refer to p. 469 in [10] for details).

Let $\lambda$ be any complex number satisfying $|\lambda| > 1$. For any such $\lambda$, let $A_\lambda := S + \lambda I$. Obviously, $A_\lambda$ is a normal operator with empty point spectrum that satisfies the conditions of Theorem 2.2.

Example 3.3. Let $T$ be a bounded normal operator on a separable Hilbert space $\mathcal{H}$. Then by the spectral theory, there exist a finite measure space $(\tilde{X}, \mu)$, a bounded complex function $\varphi$ on $\tilde{X}$, and a unitary operator $U : \mathcal{H} \to L^2(\tilde{X}, \mu)$ such that
(UTU^{-1})(x) = (M_x f)(x) := \varphi(x)f(x) for each \( f \in L^2(\tilde{X}, \mu) \) and \( x \in \tilde{X} \), where \( \varphi \) is a bounded complex measurable function on \( \tilde{X} \). Here \( L^2(\tilde{X}, \mu) \) is the Hilbert space of complex square summable functions on \( \tilde{X} \). Therefore, any normal operator \( T \) on a separable Hilbert space \( \mathcal{H} \) is similar to a multiplication operator \( M_\varphi \) on \( L^2(\tilde{X}, \mu) \) of a finite measure space. Moreover, spectrum \( \sigma(T) \) of \( T \) is equal to the essential range of \( \varphi \). Furthermore, if the measure of \( \varphi^{-1}(\lambda) \) is zero for any complex number \( \lambda \) and the essential range of \( \varphi \) is properly contained in \( \{ z \in \mathbb{C} : \text{Re} \ z < 0 \} \), then the multiplication operator \( M_\varphi \) will satisfy the hypotheses of Theorem 2.2. Let \( P \) be a measurable subset of \( \tilde{X} \) such that \( \mu(P) \) and \( \mu(\tilde{X} - P) \) are non-zero. If such a measurable set \( P \) does not exist, then \( L^2(\tilde{X}, \mu) \) will be one dimensional. Let \( I_P \) be the set of all bounded measurable functions \( f \) such that \( f = 0 \) a.e. on \( P \). Obviously, \( I_P \) is an invariant subspace of \( M_\varphi \). Let \( g \) and \( v \) be bounded measurable functions such that the support of \( g \) is contained in \( P \) and the support of \( v \) is contained in \( \tilde{X} - P \). Obviously, \( v \in I_P \) and the bounded linear functional \( f \mapsto \langle f, g \rangle \) annihilates \( I_P \). By the integral formula in the constructive proof of the Theorem 2.2 the bounded linear operator \( B \) corresponding to \( M_\varphi \) (using the functional \( \phi : f \to \langle f, g \rangle \) and \( v \)) is given by
\[
(Bf)(x) = \int_0^\infty \left( \int_{\tilde{X}} (e^{sM_\varphi} f)(t)g(t) \, d\mu(t) \right) (e^{sM_\varphi} v)(x) \, ds
\]
for such \( f \in L^2(\tilde{X}, \mu) \) and \( x \in \tilde{X} \).

Remark 5. Finally, it is worth investigating whether Theorem 2.2 is true for any normal operator on a Hilbert space with empty point spectrum regardless of the location of the spectrum in the complex plane.

**Acknowledgments**

The authors would like to thank Peter Rosenthal for fruitful discussion about invariant subspaces and Lomonosov’s Lemma. Also, the authors would like to thank the anonymous referee for valuable suggestions.

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