A NOTE ON THE BUCHSBAUM-RIM MULTIPlicITY OF A PARAMETER MODULE

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(Communicated by Bernd Ulrich)

ABSTRACT. In this article we prove that the Buchsbaum-Rim multiplicity $e(F/N)$ of a parameter module $N$ in a free module $F = A^r$ is bounded above by the colength $\ell_A(F/N)$. Moreover, we prove that once the equality $\ell_A(F/N) = e(F/N)$ holds true for some parameter module $N$ in $F$, then the base ring $A$ is Cohen-Macaulay.

1. Introduction

Let $(A, m)$ be a Noetherian local ring with the maximal ideal $m$ and $d = \dim A > 0$. Let $F = A^r$ be a free module of rank $r > 0$, and let $M$ be a submodule of $F$ such that $F/M$ has finite length and $M \subseteq mF$.

In their article [5] from 1964 Buchsbaum and Rim introduced and studied a multiplicity associated to a submodule of finite colength in a free module. This multiplicity, which generalizes the notion of Hilbert–Samuel multiplicity for ideals, is nowadays called the Buchsbaum-Rim multiplicity. In more detail, it first turns out that the function

$$\lambda(n) := \ell_A(S_n(F)/R_n(M))$$

is eventually a polynomial of degree $d + r - 1$, where $S_A(F) = \bigoplus_{n \geq 0} S_n(F)$ is the symmetric algebra of $F$ and $R(M) = \bigoplus_{n \geq 0} R_n(M)$ is the image of the natural homomorphism from $S_A(M)$ to $S_A(F)$. The polynomial $P(n)$ corresponding to $\lambda(n)$ can then be written in the form

$$P(n) = \sum_{i=0}^{d+r-1} (-1)^i e_i \binom{n+d+r-2-i}{d+r-1-i}$$

with integer coefficients $e_i$. The Buchsbaum-Rim multiplicity of $M$ in $F$, denoted by $e(F/M)$, is now defined to be the coefficient $e_0$.

Buchsbaum and Rim also introduced in their article the notion of a parameter module (matrix), which generalizes the notion of a parameter ideal (system of parameters). The module $N$ in $F$ is said to be a parameter module in $F$ if the following three conditions are satisfied: (i) $F/N$ has finite length, (ii) $N \subseteq mF$, (iii) $e(F/N) = \ell_A(F/N)$.
and (iii) $\mu_A(N) = d + r - 1$, where $\mu_A(N)$ is the minimal number of generators of $N$.

Buchsbaum and Rim utilized in their study the relationship between the Buchsbaum-Rim multiplicity and the Euler-Poincaré characteristic of a certain complex and proved the following:

**Theorem 1.1** (Buchsbaum-Rim [5, Corollary 4.5]). Let $(A, m)$ be a Noetherian local ring of dimension $d > 0$. Then the following statements are equivalent:

1. $A$ is a Cohen-Macaulay local ring.
2. For any rank $r > 0$, the equality $\ell_A(F/N) = e(F/N)$ holds true for every parameter module $N$ in $F = A^r$.

Then it is natural to ask the following:

**Question 1.2.** (1) Does the inequality $\ell_A(F/N) \geq e(F/N)$ hold true for any parameter module $N$ in $F$?

(2) Does the equality $\ell_A(F/N) = e(F/N)$ for some parameter module $N$ in $F$ imply that the ring $A$ is Cohen-Macaulay?

The purpose of this article is to give a complete answer to Question 1.2. Our results can be summarized as follows:

**Theorem 1.3.** Let $(A, m)$ be a Noetherian local ring of dimension $d > 0$.

1. For any rank $r > 0$, the two inequalities
   \[ \ell_A(F/N) \geq e(F/N) \text{ and } \ell_A(A/I(N)) \geq e(F/N) \]
   always hold true for every parameter module $N$ in $F = A^r$, where $I(N)$ is the 0-th Fitting ideal of $F/N$.

2. The following statements are equivalent:
   (i) $A$ is a Cohen-Macaulay local ring.
   (ii) For some rank $r > 0$, there exists a parameter module $N$ in $F = A^r$ such that the equality $\ell_A(F/N) = e(F/N)$ holds true.
   (iii) For some rank $r > 0$, there exists a parameter module $N$ in $F = A^r$ such that the equality $\ell_A(A/I(N)) = e(F/N)$ holds true.

When this is the case, the equality $\ell_A(F/N) = \ell_A(A/I(N)) = e(F/N)$ holds true for all parameter modules $N$ in $F = A^r$ of any rank $r > 0$.

Note that the equality $\ell_A(F/N) = \ell_A(A/I(N))$ is known by [1, 2.10].

The proof of our Theorem 1.3 will be completed in section 4. Section 2 is of a preliminary character. In that section we will recall the definition and some basic facts about the generalized Koszul complex. In order to prove Theorem 1.3 we will investigate in section 3 the higher Euler-Poincaré characteristics of the generalized Koszul complex and show that they are non-negative. Finally, in section 4, we will obtain Theorem 1.3 as a corollary of a more general result (Theorem 4.1).

2. Preliminaries

In this section we will recall the definition and some basic facts about the generalized Koszul complex introduced in [3, 8] (for more details, see also [7, Appendix A2.6]).

Let $A$ be a commutative Noetherian ring, and let $n \geq r > 0$ be integers. Let $a = (a_{ij})$ be an $r \times n$ matrix over $A$, and let $I_r(a)$ denote the ideal generated by
the maximal minors of \(a\). Let \(F\) and \(G\) be free modules with bases \(\{f_1, \ldots, f_r\}\) and \(\{e_1, \ldots, e_n\}\), respectively. Let \(S\) be the symmetric algebra of \(F\), and let \(S_\ell\) be the \(\ell\)-th symmetric power of \(F\). Let \(\wedge\) be the exterior algebra of \(G\), and let \(\wedge^\ell\) be the \(\ell\)-th exterior power of \(G\). Associated with the \(i\)-th row \([a_{i1} \cdots a_{in}]\) of \(a\), there is a differentiation homomorphism \(\delta_i : \wedge \to \wedge\) given by

\[
\delta_i(f_{j1} \wedge \cdots \wedge f_{jp}) = \sum_{k=1}^{p} (-1)^{k-1} a_{ijk} f_{j1} \wedge \cdots \wedge f_{jk} \wedge \cdots \wedge f_{jp}.
\]

Let \(f_i : S \to S\) and \(f_i^{-1} : S \to S\) denote the multiplication and division maps by \(f_i\), respectively, i.e.,

\[
f_i^{-1}(f_1^{\mu_1} \cdots f_i^{\mu_i} \cdots f_r^{\mu_r}) = \begin{cases} f_1^{\mu_1} \cdots f_i^{\mu_i} \cdots f_r^{\mu_r} & (\mu_i > 0) \\ 0 & (\mu_i = 0). \end{cases}
\]

Then the generalized Koszul complex \(K_\bullet(a; t)\) associated to a matrix \(a\) and an integer \(t\) is the complex

\[
K_\bullet(a; t) : \cdots \to K_{p+1}(a; t) \xrightarrow{d_{p+1}} K_p(a; t) \xrightarrow{d_p} K_{p-1}(a; t) \to \cdots
\]

defined by

\[
K_p(a; t) = \begin{cases} \wedge^{t+p-1} \otimes_A S_{p-t-1} & (p \geq t + 1) \\ \wedge^p \otimes_A S_{t-p} & (p \leq t) \end{cases}
\]

and

\[
d_{p+1} = \begin{cases} \sum_{j=1}^{r} \delta_j \otimes f_j^{-1} & (p > t) \\ \delta_0 \cdots \delta_1 \otimes 1 & (p = t) \\ \sum_{j=1}^{r} \delta_j \otimes f_j & (p < t). \end{cases}
\]

The generalized Koszul complex \(K_\bullet(a; t)\) is a free complex of \(A\)-modules. We note that it is of length \(n - r + 1\) when \(-1 \leq t \leq n - r + 1\). Also recall that \(K_\bullet(a; t)\) coincides with the ordinary Koszul complex for any \(t\) in the case \(r = 1\), whereas \(K_\bullet(a; 0)\) is the Eagon-Northcott complex and \(K_\bullet(a; 1)\) is the Buchsbaum-Rim complex. Moreover, the generalized Koszul complex has the following important properties (see [8] and [7 Appendix A2.6]):

**Proposition 2.1.** Let \(a\) be an \(r \times n\) matrix over \(A\) with \(n \geq r > 0\). Then

1. ([8] Theorem 1) For any \(t, p \in \mathbb{Z}\), \(\text{I}_t(a)H_p(K_\bullet(a; t)) = (0)\).
2. ([7] Theorem A2.10) If the grade of \(\text{I}_t(a)\) is at least \(n - r + 1\), then \(K_\bullet(a; t)\) is acyclic for all \(-1 \leq t \leq n - r + 1\). Furthermore, if \(a\) is a generic matrix, then \(K_\bullet(a; t)\) is acyclic for all \(t \geq -1\).

### 3. Higher Euler-Poincaré Characteristics

In this section we will investigate higher Euler-Poincaré characteristics of a generalized Koszul complex.

Throughout this section, let \((A, \mathfrak{m})\) be a Noetherian local ring of dimension \(d > 0\). Let \(F = A^r\) be a free module of rank \(r > 0\) with a basis \(\{f_1, \ldots, f_r\}\). Let \(M\) be a submodule of \(F\) generated by \(c_1, c_2, \ldots, c_n\), where \(n = \mu(A)(M)\) is the minimal number of generators of \(M\). Writing \(c_j = c_{ij} f_1 + \cdots + c_{rj} f_r\) for some \(c_{ij} \in A\), we have an \(r \times n\) matrix \(\{c_{ij}\}\) associated to \(M\). We call this matrix the matrix of \(M\) and denote it by \(\hat{M}\). Let \(I(M) = \text{Fitt}_0(F/M)\) be the 0-th Fitting ideal of \(F/M\). We assume that \(F/M\) has finite length and \(M \subseteq \mathfrak{m} F\). Then \(I(M)\) is an \(\mathfrak{m}\)-primary ideal, because \(\sqrt{I(M)} = \sqrt{\text{Ann}_A(F/M)}\). Hence each homology
module $H_p(K_\bullet(\tilde{M};t))$ has finite length by Proposition 2.1. So the Euler-Poincaré characteristics of $K_\bullet(\tilde{M};t)$ can be defined as follows:

**Definition 3.1.** For any integer $q \geq 0$, we set

$$\chi_q(K_\bullet(\tilde{M};t)) := \sum_{p \geq q} (-1)^{p-q} \ell_A(H_p(K_\bullet(\tilde{M};t)))$$

and call it the $q$-th partial Euler-Poincaré characteristic of $K_\bullet(\tilde{M};t)$. When $q = 0$, we simply write $\chi(K_\bullet(\tilde{M};t))$ for $\chi_0(K_\bullet(\tilde{M};t))$ and call it the Euler-Poincaré characteristic of $K_\bullet(\tilde{M};t)$.

Buchsbaum and Rim studied in [5] the Euler-Poincaré characteristic of the Buchsbaum-Rim complex in analogy with the Euler-Poincaré characteristic of the ordinary Koszul complex in the case of usual multiplicities. In 1985 Kirby investigated in [9] Euler-Poincaré characteristics of the complex $K_\bullet(\tilde{M};t)$ for all $t$ and proved the following:

**Theorem 3.2** (Buchsbaum-Rim, Kirby). For any integer $t \in \mathbb{Z}$, we have

$$\chi(K_\bullet(\tilde{M};t)) = \left\{ \begin{array}{ll} e(F/M) & (n = d + r - 1), \\ 0 & (n > d + r - 1), \end{array} \right.$$  

where $n = \mu_A(M)$ is the minimal number of generators of $M$. In particular, $\chi(K_\bullet(\tilde{M};t)) \geq 0$ for all $t \in \mathbb{Z}$.

The last statement holds for the higher Euler-Poincaré characteristics, too:

**Theorem 3.3.** For any $q \geq 0$ and any $t \geq -1$, we have

$$\chi_q(K_\bullet(\tilde{M};t)) \geq 0.$$  

**Proof.** We use ideas from [6]. Let $\tilde{M} = (c_{ij}) \in \text{Mat}_{r \times n}(A)$ be the matrix of $M$, and let $X = (X_{ij})$ be the generic matrix of the same size $r \times n$. Let $A[X] = A[X_{ij} | 1 \leq i \leq r, 1 \leq j \leq n]$ be a polynomial ring over $A$, and let $B = A[X]_{(m,X)}$. We will consider the ring $A$ as a $B$-algebra via the substitution homomorphism $\phi : B \to A ; X_{ij} \mapsto c_{ij}$. Let

$$b = \text{Ker } \phi = (X_{ij} - c_{ij} | 1 \leq i \leq r, 1 \leq j \leq n)B.$$  

We note here that $K_\bullet(X,t) \otimes_B A \cong K_\bullet(\tilde{M};t)$, because the generalized Koszul complex is compatible with the base change. Let $C_t(X) := H_0(K_\bullet(X,t))$. By Proposition 2.1(2), the complex $K_\bullet(X,t)$ is a $B$-free resolution of the $B$-module $C_t(X)$ for any $t \geq -1$. By tensoring with $A$ and taking the homology, we have that

$$H_p(K_\bullet(\tilde{M};t)) \cong H_p(K_\bullet(X,t) \otimes_B A) \cong \text{Tor}^B_p(C_t(X), A)$$  

for all $p \geq 0$. On the other hand, since the ideal $b$ in $B$ is generated by a regular sequence of length $rn$, the ordinary Koszul complex $K_\bullet(b)$ associated to the sequence $b$ is a $B$-free resolution of $A$. Hence, by tensoring with $C_t(X)$, we can compute the Tor as follows:

$$\text{Tor}^B_p(C_t(X), A) \cong H_p(K_\bullet(b) \otimes_B C_t(X)).$$

Therefore, for any $p \geq 0$, we have

$$H_p(K_\bullet(\tilde{M};t)) \cong H_p(K_\bullet(b) \otimes_B C_t(X)).$$
It follows that for any \( t \geq -1 \) and any \( q \geq 0 \) we have the equality
\[
\chi_q(K_*(\tilde{M}; t)) = \chi_q(K_*(b) \otimes_B C_t(X)).
\]

Here the right hand side is non-negative by Serre’s Theorem ([12, Ch. IV, Appendix II]). Therefore \( \chi_q(K_*(\tilde{M}; t)) \geq 0 \).

\[\square\]

4. Proof of Theorem 1.3

Theorem 1.3 will be a consequence of the following more general result:

**Theorem 4.1.** Let \((A, \mathfrak{m})\) be a Noetherian local ring of dimension \( d \geq 0 \).

1. For any rank \( r > 0 \), the inequality \( \ell_A(H_0(K_*(\tilde{N}; t))) \geq e(F/N) \) holds true for any integer \( t \geq -1 \) and any parameter module \( N \) in \( F = A^r \).

2. The following statements are equivalent:
   (i) \( A \) is a Cohen-Macaulay local ring.
   (ii) For some rank \( r > 0 \), there exists an integer \(-1 \leq t \leq d\) and a parameter module \( N \) in \( F = A^r \) such that the equality \( \ell_A(H_0(K_*(\tilde{N}; t))) = e(F/N) \) holds true.

When this is the case, the equality \( \ell_A(H_0(K_*(\tilde{N}; t))) = e(F/N) \) holds true for any integer \(-1 \leq t \leq d\) and any parameter module \( N \) in \( F = A^r \) of any rank \( r > 0 \).

**Proof.** (1): Let \( N \) be a parameter module in \( F = A^r \), and let \( t \geq -1 \). By Theorem 3.2 we obtain that
\[
e(F/N) = \chi(K_*(\tilde{N}; t)) = \ell_A(H_0(K_*(\tilde{N}; t))) - \chi_1(K_*(\tilde{N}; t)).
\]

Since \( \chi_1(K_*(\tilde{N}; t)) \geq 0 \) by Theorem 1.3, the desired inequality follows.

(2): Assume that \( A \) is Cohen-Macaulay. Let \( N \) be any parameter module in \( F = A^r \) of any rank \( r > 0 \). Let \( n = \mu_A(N) = d + r - 1 \). Then grade \( I(N) = \text{ht} I(N) = d = n - r + 1 \). Hence, by Proposition 2.1(2), \( K_*(\tilde{N}; t) \) is acyclic for all \(-1 \leq t \leq n - r + 1 = d\). Therefore, by Theorem 3.2 we have \( e(F/N) = \chi(K_*(\tilde{N}; t)) = \ell_A(H_0(K_*(\tilde{N}; t))) \). This proves the implication (i) \(\Rightarrow\) (ii) and also the last assertion.

It remains to show the implication (ii) \(\Rightarrow\) (i). Assume that there exist integers \( r > 0 \), \(-1 \leq t \leq d\), and a parameter module \( N \) in \( F = A^r \) such that \( \ell_A(H_0(K_*(\tilde{N}; t))) = e(F/N) \). Arguing as in the proof of Theorem 1.3 and using the same notation, we get
\[
\chi_1(K_*(b) \otimes_B C_t(X)) = \chi_1(K_*(\tilde{N}; t)) = \ell_A(H_0(K_*(\tilde{N}; t))) - e(F/N) = 0.
\]

We observe here that \( \sqrt{Ann_B C_t(X)} = \sqrt{I_t(X)} \) (see [14, Lemma 2.7]). Thus \( \dim_B C_t(X) = \dim B/I_t(X) = d + (n + 1)(r - 1) = rn \) (see [2] (5.12), Corollary). Therefore \( b \) is a parameter ideal of \( C_t(X) \). Hence we have the equality
\[
\ell_B(C_t(X)/bC_t(X)) - e(b; C_t(X)) = \chi_1(K_*(b) \otimes_B C_t(X)) = 0,
\]

where \( e(b; C_t(X)) \) is the multiplicity of the module \( C_t(X) \) with respect to an ideal \( b \). Since \( \ell_B(C_t(X)/bC_t(X)) = e(b; C_t(X)) \), this implies that \( C_t(X) \) is a Cohen-Macaulay \( B \)-module. On the other hand, \( \text{pd}_B C_t(X) = d \), because the complex
$K_\bullet (X; t)$ is a minimal $B$-free resolution of $C_t (X)$ of length $n - r + 1 = d$. Hence, by the Auslander-Buchsbaum formula, we have

$$d + rn = \text{pd}_B C_t (X) + \text{depth}_B C_t (X)$$

$$= \text{depth}_B$$

$$\leq \dim B$$

$$= d + rn.$$ 

Thus $\text{depth} B = \dim B$ so that $B$ is Cohen-Macaulay. Therefore $A$ is also a Cohen-Macaulay local ring. □

Taking $t = 0, 1$ in Theorem $4.1$ now readily gives Theorem $1.3$.

We want to close this article with a question. For that, let us first recall the notion of a Buchsbaum local ring, which was introduced by Stückrad and Vogel (for more details on Buchsbaum rings, we refer the reader to [13]). Let $A$ be a Noetherian local ring. Then $A$ is said to be a Buchsbaum ring if the difference

$$\ell_A (A/Q) - e(A/Q)$$

between the colength and multiplicity of a parameter ideal $Q$ in $A$ is independent of the choice of $Q$. This difference, which is an invariant of a Buchsbaum ring $A$, is denoted by $I(A)$. The ring $A$ is Cohen-Macaulay if and only if it is Buchsbaum and $I(A) = 0$. In this sense, the notion of a Buchsbaum ring is a natural generalization of that of a Cohen-Macaulay ring. In Theorem $1.3$ the inequality $\ell_A (F/N) \geq e(F/N)$, for any parameter module $N$ in $F$, is an analogue of the well-known inequality $\ell_A (A/Q) \geq e(A/Q)$ for any parameter ideal $Q$ in $A$. Also, the characterization of the Cohen-Macaulay property of $A$ based on the equality $\ell_A (F/N) = e(F/N)$ generalizes the usual one using parameter ideals. With these remarks in mind, it is natural to ask the following question:

**Question 4.2.** Let $F$ be a fixed free module of rank $r > 0$. Is it then true that the difference

$$\ell_A (F/N) - e(F/N)$$

between the colength and multiplicity of a parameter module $N$ in $F$ is independent of the choice of $N$ if the ring $A$ is Buchsbaum?

**References**


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