Let $M$ be a closed $C^{k+1}$-manifold, $1 \leq k < \infty$. Given a $C^k$-Riemannian metric $g$ on $M$ we know that the space of $C^{k+1}$-diffeomorphisms that preserve $g$ (i.e. the isometries of $(M, g)$) with the $C^0$-topology is compact [8]. In our study of the space of negatively curved metrics ([3], [4], [5], [6]) we have had to deal with the behavior of diffeomorphisms that “almost” preserve a Riemannian metric “up to order $k$.” Even though it has been mentioned sparsely in the literature that “quasi-isometries are pre-compact” (see for instance section 12.55 of Besse’s book [1, p. 253]) we have not found a precise presentation (definitions, statements and proofs) that includes the $C^k$ cases. We believe it is useful to have such a presentation, and in this short paper we aim to give one.

First we introduce some notation and definitions. The space of $C^k$-Riemannian metrics on $M$, with the $C^k$-topology, will be denoted by $\mathcal{MET}^{C^k}(M)$, or simply by $\mathcal{MET}(M)$. The spaces of $C^k$-self maps and $C^k$-self-diffeomorphisms of $M$, with the $C^k$-topology, will be denoted by $C^k(M, M)$ and $DIFF^{k}(M, M)$. Recall that $\mathcal{MET}(M)$ is an open set in the normed vector space of all $C^k$-symmetric two-tensors on $M$. Hence the notion of a set $A \subset \mathcal{MET}(M)$ being $C^k$-bounded is well-defined. In fact it is equivalent to saying the set of all matrix entries of the chart representative (with respect to a fixed finite set of charts) of all elements of $A$ is bounded up to order $k$. Similarly, the notion of $F \subset C^k(M, M)$ being $C^k$-bounded is well-defined (see item 2 below).

Remarks. (1) A technical point here. The fixed set of charts $\mathcal{A} = \{(U, \phi)\}$ mentioned above has to have the following property. Every chart $(U, \phi)$ can be extended to a chart $(V, \phi)$, with $\bar{U} \subset V$, $\bar{U}$ compact. In what follows we assume that a finite set of charts with this property has been given. 

(2) $F \subset C^k(M, M)$ is $C^k$-bounded if the set of representatives of all $f : U_1 \cap f^{-1}(U_2) \to U_2$, $f \in F$, $(U_1, \phi_1), (U_2, \phi_2) \in \mathcal{A}$, is bounded up to order $k$.

(3) The definition of $C^k$-boundedness (in both cases) does not depend on the choice of the set of charts.
We say that $A \subset \mathcal{MET}(M)$ has volume forms away from zero if the set of positive numbers

$$\left\{ \det(a_{ij}^\alpha)(x), a \in A, \alpha = (U, \phi) \in \mathcal{A}, x \in \phi(U) \right\}$$

is bounded away from zero, where $(a_{ij}^\alpha)$ is the local coordinate expression of $a \in A$ with respect to $\alpha$. Note that this implies that the set of entries of all $(a_{ij}^\alpha(x))^{-1}$ is bounded, provided $A$ is $C^0$-bounded. We say that $A \subset \mathcal{MET}(M)$ is $C^k$-well-bounded if it is $C^k$-bounded and has volume forms away from zero.

Let $g, h \in \mathcal{MET}^{C^k}(M)$. A set $F \subset \text{DIFF}^{k+1}(M)$ is a $C^k$-quasi-isometry set with respect to $g, h$ if the sets $\{ f^* h, f \in F \}$ and $\{ f^* g, f \in F \}$ are $C^k$-well-bounded. More generally, a set $F \subset \text{DIFF}^{k+1}(M)$ is a $C^k$-quasi-isometry set if there are $C^k$-well-bounded sets $A, B \subset \mathcal{MET}^{C^k}(M)$, such that for every $f \in F$ there is $a \in A$ with $f_* a \in B$. Clearly $F$ is a $C^k$-quasi-isometry set if it is a $C^k$-quasi-isometry set with respect to some $g$ and $h$. It is a consequence of Theorem A below that the first definition is independent of the metrics $g$ and $h$ and is equivalent to the second definition.

**Remark.** Let $F$ be a $C^k$-quasi-isometry set. Then any subset of $F$ is a $C^k$-quasi-isometry set. Also $F^{-1} = \{ f^{-1}, f \in F \}$ is a $C^k$-quasi-isometry set.

Here is our main theorem.

**Theorem A.** $C^k$-quasi-isometry sets are pre-compact in $\text{DIFF}^k(M)$, $k \geq 0$.

If $F$ is $C^{k+1}$-bounded, then it is also $C^k$-bounded and, since the set of $k$-derivatives of the elements of $f \in F$ is bounded, these elements are Lipschitz (with the same constant); hence this set is equicontinuous. The Arscia-Ascoli theorem implies then the following:

**Lemma.** Let $F \subset C^{k+1}(M, M)$ be $C^{k+1}$-bounded. Then $F$ is pre-compact in $C^k(M, M)$.

Using the lemma above, the proof of Theorem A is reduced to proving:

**Theorem B.** $C^k$-quasi-isometry sets are $C^{k+1}$-bounded.

**Remarks.** (1) Theorem A implies that the definition of $C^k$-quasi-isometry set with respect to two metrics $g, h$ does not depend on $g, h$.

(2) Rigorously, Theorem B implies that $C^k$-quasi-isometry sets are pre-compact in $C^k(M, M)$. But, using the remark before the statement of Theorem A, it is straightforward to check that they are actually pre-compact in $\text{DIFF}^k(M)$.

The proof of Theorem B implies (and, essentially reduces to) the following interesting fact. Let $f$ be a diffeomorphism between open sets of $\mathbb{R}^n$. Let $Df(x) = \left( \frac{\partial f_i}{\partial x_j}(x) \right)$ be its Jacobian matrix at $x$. Assume: (i) the entries of $Df^T(x)Df(x)$ are bounded up to order $k$, and (ii) $\det Df(x)$ is bounded away from zero. Then the entries of $Df(x)$ are bounded up to order $k$. Note that it is important to assume that the map $x \mapsto Df(x)$ has for images Jacobian matrices; otherwise the result fails: let $x \mapsto C(x) \in O(n)$ be any wild function, but we have that $C^T(x)C(x)$ is constant.

**Proof of Theorem B.** Let $F \subset \text{DIFF}^{k+1}(M)$ be a $C^k$-quasi-isometry set. Let $A$ and $B$ be $C^k$-well-bounded sets such that for every $f \in F$, there is $a \in A$ with
$f, a \in B$. In what follows the local representatives of $f \in F$, $a \in A$ and $b \in B$ will be denoted by the same letters. Also, for $f \in F$, $Df(x) = \left( \frac{\partial f}{\partial x_i}(x) \right)$ denotes the Jacobian matrix of (a local representative of) $f$. Then, for $f \in F$ there are $a \in A$, $b \in B$ such that:

$$\begin{align*}
Df(x)^T a(x) Df(x) &= b(x) \\
\text{for every } x \in \phi(U), (U, \phi) \in \mathcal{A}. \text{ We now proceed to verify Theorem B by induction on } k \text{ starting with the special case } k = 0. \text{ Let } c(x) \text{ be the (unique and smoothly defined) square root of } a(x). \text{ Then}
\end{align*}$$

$$\begin{align*}
\left( c(x) Df(x) \right)^T \left( c(x) Df(x) \right) &= b(x)
\end{align*}$$

Since the diagonal entries of this matrix are the norm of the columns of $c(x)Df(x)$ and the entries of $b(x)$ are bounded, we have that the entries of $c(x)Df(x)$ are also bounded. But the entries of $c^{-1}(x)$ are bounded since $\det c(x) = \sqrt{\det a(x)}$ is bounded away from zero. We therefore conclude that the entries of $Df(x)$ are bounded, thus proving the theorem when $k = 0$. Interchanging the sets $A$ and $B$ in the above argument, we obtain also that the $||Df^{-1}||'$'s are bounded as well.

Proceeding with the induction we assume $k > 0$ and that all derivatives of order $\leq k$ of all $f$'s are bounded. We prove that all derivatives of order $k+1$ are also bounded by contradiction. Suppose $F$ is not $C^{k+1}$-bounded. Then there are sequences $(f_m)$ in $F$ and $(x_m)$ in $M$ such that some $k+1$ derivative of $f_m$, at $x_m$, becomes large as $m \to \infty$. Since $M$ is compact and $(f_m)$ is pre-compact in $C^0(M, M)$ we can assume that both sequences converge. It follows that there are two charts, $\psi : V' \to V \subset \mathbb{R}^n$, $\psi' : W' \to W \subset \mathbb{R}^n$, such that $f_m(V) \subset W$ and some $k+1$ derivative of $f_m : V \to W$ becomes large as $m \to \infty$. (The charts $\psi, \psi'$ are not necessarily in $\mathcal{A}$ but can be extended to charts in $\mathcal{A}$.) We have reduced our problem to $\mathbb{R}^n$.

For a j-multi-index $\alpha = (i_1, \ldots, i_j)$, $1 \leq i_t \leq n$ (i.e. $\alpha \in \{1, 2, \ldots, n\}^j$), we write $\partial x_{\alpha} = \partial x_{i_1} \cdots \partial x_{i_j}$. If $\alpha$ and $\beta$ are multi-indices, $\alpha \beta$ denotes the “concatenation” multi-index. Also a 1-multi-index $(i)$ will be denoted just as $i$. Let $\alpha$ be a $k$-multi-index. Applying $\frac{\partial^k}{\partial x_\alpha}$ to equation (1) above we get:

$$\begin{align*}
\left[ Df^T \ a \left( \frac{\partial^k}{\partial x_\alpha}(Df) \right) \right] + \left[ \frac{\partial^k}{\partial x_\alpha}(Df) \right]^T a Df \ + \ d = \frac{\partial^k}{\partial x_\alpha} b,
\end{align*}$$

where $d$ is a sum of products of derivatives of order $\leq k$ of $a$ and $f$. Hence the expression inside the brackets is bounded. Note that the columns of $Df$ are $f_i = \frac{\partial}{\partial x_i} f$, that is, $Df = [f_1 \cdots f_n]$. To simplify our notation, for a multi-index $\beta$ we write $f_\beta = \frac{\partial^k}{\partial x_\beta} f$. Using this notation, the equations above tell us that the following set of expressions is uniformly bounded:

$$\begin{align*}
f_i^T a f_{\alpha j} \ + \ f_j^T a f_{\alpha i}
\end{align*}$$

for all $1 \leq i, j \leq n$ and every $\alpha \in \{1, \ldots, n\}^j$. Note that the second term in (2) is obtained from the first term by interchanging $i$ and $j$.

Claim. The fact that the expressions in (2) are bounded implies that the following expressions are bounded for every $\beta \in \{1, \ldots, n\}^{k+1}$:

$$\begin{align*}
(Df)^T a f_\beta.
\end{align*}$$
Since the $Df^{-1}$'s and $a^{-1}$'s are bounded, we conclude that the expressions $f_\beta = \frac{\partial^{k+1}}{\partial x^\beta} f$ are bounded. This contradicts our assumption that “some $k+1$ derivative of $f_m : V \to W$ becomes large as $m \to \infty$”, and hence proves Theorem B once we verify the claim.

**Proof of the claim.** To simplify our notation we denote the expression $f^T a f_\beta$ as simply $[i, \beta]$. Note that the $i$-th entry of the column vector $Df^T a f_\beta$ is $[i, \beta]$. Therefore to prove the claim we need to show that $[i, \beta]$ is bounded, for every $i$ and every $(k+1)$-multi-index $\beta$. Note that our argument above proving Theorem B for $k = 0$ also proved the claim for $k = 0$. Hence we may assume that $k > 0$. Now, with the bracket notation introduced above, the bounded expression in (2) can be rewritten as:

$$(2') \quad [i, \alpha j] + [j, \alpha i]$$

That is, the expression in (2’) is bounded for every $i, j$ and $k$-multi-index $\alpha$. Since $k+1 \geq 2$, we can write $\beta = \gamma s t$ for some $(k-1)$-multi-index $\gamma$ and $1 \leq s, t \leq n$. By cyclically permuting $i, s, t$ and applying the fact that (2’) is bounded, we obtain that the following three expressions are bounded:

$$[i, \gamma st] + [t, \gamma si]$$
$$[s, \gamma ti] + [i, \gamma ts]$$
$$[t, \gamma is] + [s, \gamma it].$$

Notice that only three numbers occur in these three sums, namely:

$$A = [i, \gamma st] = [i, \gamma ts]$$
$$B = [s, \gamma ti] = [s, \gamma it]$$
$$C = [t, \gamma is] = [t, \gamma si].$$

Therefore the above three bounded terms become:

$$A + C$$
$$A + B$$
$$B + C.$$

Consequently all three terms $A$, $B$, and $C$ are bounded. This completes the proof of the claim because $A = [i, \beta]$.

**Remark.** The above proof is formally similar to that given for the Fundamental Lemma of Riemannian Geometry, i.e. for the construction of the Levi-Civita connection (see, for instance, pp. 48-49 of [7]).

**References**


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