PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 3, March 2010, Pages 827–836
S 0002-9939(09)10116-8
Article electronically published on October 22, 2009

THE LERCH ZETA AND RELATED FUNCTIONS OF NON-POSITIVE INTEGER ORDER

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(Communicated by Walter Van Assche)

Abstract. It is known that the Lerch (or periodic) zeta function of non-positive integer order, $\ell_{-n}(\xi)$, $n \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$, is a polynomial in $\cot(\pi\xi)$ of degree $n+1$. In this paper, a very simple explicit closed-form formula for this polynomial valid for any degree is derived. In addition, novel analogous explicit closed-form formulae for the Legendre chi function, the alternating Lerch zeta function and the alternating Legendre chi function are established. The obtained formulae involve the Carlitz-Scoville tangent and secant numbers of higher order, and the derivative polynomials for tangent and secant are used in their derivation. Several special cases and consequences are pointed out, and some examples are also given.

1. Introduction

The Lerch (or periodic) zeta function of order s and argument ξ, $\ell_s(\xi)$, is defined by the series (see, for instance, [3, p. 257, Eq. 12.7(9)] and [15, p. 122, Eq. 2.5(11)])

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi\imath\xi}}{n^s} \qquad (\imath := \sqrt{-1};\ \xi \in \mathbb{R};\ \Re(s) > 1) \tag{1.1}$$

whenever it converges and by analytic continuation elsewhere. The function $\ell_s(\xi)$ is periodic in ξ with period 1. Moreover, if ξ is not an integer, then $\ell_s(\xi)$ is an entire function in $s \in \mathbb{C}$, and for an integer ξ, $\ell_s(\xi)$ reduces to the Riemann zeta function $\zeta(s)$, which is a meromorphic function in $s \in \mathbb{C}$ with *sole* simple pole at $s = 1$.

The Legendre chi function of order s and argument z, $\chi_s(z)$, is given by the series [8, 9]

$$\chi_s(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s} \qquad (|z| \le 1; \Re(s) > 1) \tag{1.2}$$

and its meromorphic continuations over the whole s-plane.

Received by the editors April 14, 2009, and, in revised form, July 12, 2009.
2000 *Mathematics Subject Classification.* Primary 11M35, 33E20; Secondary 11B83.
Key words and phrases. Lerch zeta function, Legendre chi function, alternating Lerch zeta function, alternating Legendre chi function, derivative polynomials, tangent numbers of order k, secant numbers of order k, higher (generalized) tangent numbers, higher (generalized) secant numbers.

We shall also use the alternating counterparts of the above-introduced functions: the alternating Lerch zeta function (of order s and argument ξ) $\ell_s^*(\xi)$,

$$\ell_s^*(\xi) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{2n\pi\imath\xi}}{n^s} \qquad (\xi \in \mathbb{R};\ \Re(s) > 0), \tag{1.3}$$

and the alternating Legendre chi function (of order s and argument z) $\chi_s^*(z)$,

$$\chi_s^*(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)^s} \qquad \big(|z| \leqq 1;\ \Re(s) > 0\big). \tag{1.4}$$

Apostol [2, pp. 226–228] showed that the Lerch zeta function of non-positive integer order, $\ell_{-n}(\xi)$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N} := \{1, 2, 3, \ldots\}$, is a polynomial in $\cot(\pi\xi)$ of degree $n+1$. In this paper, a very simple explicit closed-form formula for this polynomial valid for any degree is derived. In addition, novel analogous explicit closed-form formulae for $\chi_{-n}(z)$, $\ell_{-n}^*(\xi)$ and $\chi_{-n}^*(z)$, $n \in \mathbb{N}_0$, are established. The derivative polynomials for tangent and secant (for definitions, see Section 3) introduced by Hoffman [10, 11] are used in the derivation, and the obtained formulae involve the Carlitz-Scoville higher order tangent and secant numbers (see, for instance, [5, 6]).

We remark that there exists the relation $\ell_s(\xi) = e^{-2\pi\imath\,\xi}\, L(\xi, 1, s)$ between the Lerch zeta function $\ell_s(\xi)$ and the function of three arguments $L(\xi, a, s)$, also first introduced by Lerch [13], which is given by

$$L(\xi, a, s) := \sum_{k=0}^{\infty} \frac{e^{2k\pi\imath\xi}}{(k+a)^s}.$$

It should be noted that $L(\xi, a, -n)$, $n \in \mathbb{N}_0$, which is clearly the more general expression than $\ell_{-n}(\xi)$ in this study, has been investigated earlier. Apostol [1] derived the formula

$$L(\xi, a, -n) = -\frac{\beta_{n+1}\left(a, e^{2\pi\imath\,\xi}\right)}{n+1}, \tag{1.5}$$

where the $\beta_n(a, \alpha)$ are the generalized Bernoulli polynomials defined by

$$\frac{ze^{\alpha z}}{\alpha\, e^z - 1} = \sum_{n=0}^{\infty} \frac{\beta_n(a, \alpha)}{n!}\, z^n.$$

Recently, the formula (1.5) was further explored by Boyadzhiev [4].

2. Statement of main results

The *tangent numbers (of order k)* $T(n, k)$ and *secant numbers (of order k)* $S(n, k)$ are respectively defined by (see [5, p. 428] and [6, p. 305])

$$\tan^k(t) = \sum_{n=k}^{\infty} T(n, k)\, \frac{t^n}{n!} \qquad (k \in \mathbb{N}) \tag{2.1}$$

and

$$\sec(t) \tan^k(t) = \sum_{n=k}^{\infty} S(n, k)\, \frac{t^n}{n!} \qquad (k \in \mathbb{N}_0). \tag{2.2}$$

Our main results are as follows.

Theorem. *Let $\ell_s(\xi)$, $\chi_s(z)$, $\ell_s^*(\xi)$ and $\chi_s^*(z)$ be the above-defined functions. Then, in terms of the tangent and secant numbers of order k, $T(n,k)$ and $S(n,k)$, we have:*

(a) $$\ell_0(\xi) = -\frac{1}{2} + \frac{\imath}{2}\cot(\pi\xi),$$
$$\ell_{1-n}(\xi) = \left(\frac{\imath}{2}\right)^n \left[T(n-1,1) + \sum_{k=1}^{n} \frac{1}{k}\, T(n,k)\, \cot^k(\pi\xi)\right]$$
$$\Big(\xi \in \mathbb{R}\setminus\mathbb{Z};\ n \in \mathbb{N}\setminus\{1\}\Big);$$

(b) $$\chi_{1-n}(z) = \frac{z}{1-z^2} \sum_{k=0}^{n-1} (-1)^k\, \imath^{n-1+k} S(n-1,k) \left(\frac{1+z^2}{1-z^2}\right)^k$$
$$\Big(z \in \mathbb{C}\setminus\{-1,1\};\ n \in \mathbb{N}\Big);$$

(c) $$\ell_0^*(\xi) = \frac{1}{2} + \frac{\imath}{2}\tan(\pi\xi),$$
$$\ell_{1-n}^*(\xi) = (-1)^{n-1}\left(\frac{\imath}{2}\right)^n \left[T(n-1,1) + \sum_{k=1}^{n} \frac{1}{k}\, T(n,k) \tan^k(\pi\xi)\right]$$
$$\Big(\xi \in \mathbb{R}\setminus\Big\{(2k+1)\tfrac{1}{2}\,|k \in \mathbb{Z}\Big\};\ n \in \mathbb{N}\setminus\{1\}\Big);$$

(d) $$\chi_{1-n}^*(z) = (-1)^{n-1}\, \frac{z}{1+z^2} \sum_{k=0}^{n-1} \imath^{n-1+k} S(n-1,k) \left(\frac{1-z^2}{1+z^2}\right)^k$$
$$\Big(z \in \mathbb{C}\setminus\{-\imath,\imath\};\ n \in \mathbb{N}\Big).$$

Remark 1. We remark that since, as detailed in Section 4, $T(n,k)$ and $S(n,k)$ are nonzero only under certain conditions, the above given formulae can be written in somewhat simplified form. For instance, we have

$$\ell_{1-2m}(\xi) = \frac{(-1)^m}{2^{2m}} \left[T(2m-1,1) + \sum_{r=1}^{m} \frac{1}{2\,r}\, T(2m,2r)\, \cot^{2r}(\pi\xi)\right] \qquad (m \in \mathbb{N})$$

and

$$\ell_{-2m}(\xi) = \imath\, \frac{(-1)^m}{2^{2m+1}} \sum_{r=0}^{m} \frac{1}{2\,r+1}\, T(2m+1,2r+1)\, \cot^{2r+1}(\pi\xi) \qquad (m \in \mathbb{N}).$$

Further, a number of simpler expressions could be obtained by specializing the results. For instance:

$$\ell_{1-2m}\left(\frac{1}{2}\right) = \frac{(-1)^m}{2^{2m}}\, T(2m-1,1), \qquad \ell_{-2m}\left(\frac{1}{2}\right) = 0,$$

and

$$\ell_{1-2m}^*(1) = \frac{(-1)^{m-1}}{2^{2m}}\, T(2m-1,1), \qquad \ell_{-2m}^*(1) = 0.$$

3. Proof of the results

The proof of our Theorem requires the three lemmas below, which might be of independent interest. In addition, we need the following definitions.

Following Hoffman [10, 11] we, by means of the exponential generating functions, define two sequences of polynomials, $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$, $n \in \mathbb{N}_0$, which are here referred to as the *derivative polynomials for tangent* [11, p. 1, Eq. (1)],

$$P(x,t) := \frac{x+\tan(t)}{1-x\tan(t)} = \sum_{n=0}^{\infty} P_n(x)\,\frac{t^n}{n!}, \tag{3.1}$$

and the *derivative polynomials for secant* [11, p. 1, Eq. (1)],

$$Q(x,t) := \frac{\sec(t)}{1-x\tan(t)} = \sum_{n=0}^{\infty} Q_n(x)\,\frac{t^n}{n!}. \tag{3.2}$$

Lemma 1. *Assume that n and k are nonnegative integers, and let $P_n(x)$ and $Q_n(x)$ be the polynomials as defined by* (3.1) *and* (3.2)*. Then, in terms of the tangent numbers of order k, $T(n,k)$, given by* (2.1)*, we have*

$$P_n(x) = T(n,1) + \sum_{k=1}^{n+1} \frac{1}{k}\,T(n+1,k)\,x^k, \tag{3.3}$$

and, in terms of the secant numbers of order k, $S(n,k)$, given by (2.2)*, we have*

$$Q_n(x) = \sum_{k=0}^{n} S(n,k)\,x^k. \tag{3.4}$$

Lemma 2. *In terms of the tangent and secant numbers of order k, $T(n,k)$ and $S(n,k)$, for $n \in \mathbb{N}_0$, we have:*

(a) $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\,\cot(x) = (-1)^n\left[T(n,1) + \displaystyle\sum_{k=1}^{n+1}\frac{1}{k}\,T(n+1,k)\,\cot^k(x)\right];$

(b) $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\,\csc(x) = (-1)^n\,\csc(x)\displaystyle\sum_{k=0}^{n} S(n,k)\,\cot^k(x);$

(c) $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\,\tan(x) = T(n,1) + \displaystyle\sum_{k=1}^{n+1}\frac{1}{k}\,T(n+1,k)\tan^k(x);$

(d) $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\,\sec(x) = \sec(x)\displaystyle\sum_{k=0}^{n} S(n,k)\,\tan^k(x).$

Lemma 3. *Let $\ell_s(\xi)$, $\chi_s(z)$, $\ell_s^*(\xi)$ and $\chi_s^*(z)$ be the above-defined functions. Then:*

(a) $\ell_0(\xi) = -\dfrac{1}{2} + \dfrac{\imath}{2}\cot(\pi\xi)$,

$$\ell_{1-n}(\xi) = \frac{\imath}{2(2\pi\imath)^{n-1}}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}}\cot(\pi\xi) \qquad (\xi \in \mathbb{R}\setminus\mathbb{Z};\ n \in \mathbb{N}\setminus\{1\});$$

(b) $$\chi_{1-n}(e^{\pi\imath\xi}) = \frac{\imath}{2(\pi\imath)^{n-1}}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}}\csc(\pi\xi) \qquad (\xi \in \mathbb{R}\setminus\mathbb{Z};\ n \in \mathbb{N});$$

(c) $\ell_0^*(\xi) = \dfrac{1}{2} + \dfrac{\imath}{2}\tan(\pi\xi)$,

$$\ell_{1-n}^*(\xi) = \frac{\imath}{2(2\pi\imath)^{n-1}}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}}\tan(\pi\xi)$$

$$\Big(\xi \in \mathbb{R}\setminus\Big\{(2k+1)\tfrac{1}{2}\,|k \in \mathbb{Z}\Big\};\ n \in \mathbb{N}\setminus\{1\}\Big);$$

(d) $$\chi_{1-n}^*(e^{\pi\imath\xi}) = \frac{\imath}{2(\pi\imath)^{n-1}}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}}\sec(\pi\xi)$$

$$\Big(\xi \in \mathbb{R}\setminus\Big\{(2k+1)\tfrac{1}{2}\,|k \in \mathbb{Z}\Big\};\ n \in \mathbb{N}\Big).$$

Proof of Lemma 1. First, it should be noted that the polynomials $P_n(x)$ and $Q_n(x)$ may be equivalently defined by the formulae [10, 11]

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\tan(x) = P_n\big(\tan(x)\big) \qquad (n \in \mathbb{N}_0) \tag{3.1*}$$

and

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sec(x) = \sec(x)\,Q_n\big(\tan(x)\big) \qquad (n \in \mathbb{N}_0). \tag{3.2*}$$

Moreover, by making use of the chain rule, it follows from (3.1*) that the polynomials $P_n(x)$ satisfy

$$P_0(x) = x, \qquad P_n(x) = (1+x^2)\,P_{n-1}^{'}(x) \qquad (n \in \mathbb{N}) \tag{3.1**}$$

and, similarly, from (3.2*) that

$$Q_0(x) = 1, \qquad Q_n(x) = (1+x^2)\,Q_{n-1}^{'}(x) + x\,Q_{n-1}(x) \qquad (n \in \mathbb{N}). \tag{3.2**}$$

In order to prove the formula (3.3), note that the generating function of $P_n(x)$ in (3.1) can be rewritten as

$$P(x,t) = \big(x+\tan(t)\big)\sum_{k=0}^{\infty}\big(x\,\tan(t)\big)^k = x + \big(1+x^2\big)\sum_{k=1}^{\infty}x^{k-1}\tan^k(t),$$

which, by making use of the definition of $T(n,k)$ in (2.1) and the elementary double series identities [14, p. 57, Eq. (2)]

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}A(k,n) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}A(k,n+k) = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty}A(k,n),$$

becomes

$$P(x,t) = x + \left(1+x^2\right) \sum_{k=1}^{\infty} x^{k-1} \sum_{n=k}^{\infty} T(n,k)\, \frac{t^n}{n!}$$

$$= x + \sum_{n=1}^{\infty} \left(1+x^2\right) \left(\sum_{k=1}^{n} T(n,k)\, x^{k-1} \right) \frac{t^n}{n!}. \tag{3.5}$$

On the other hand, by (3.1) in conjunction with the recurrence (3.1**), we have

$$P(x,t) = P_0(x) + \sum_{n=1}^{\infty} P_n(x)\, \frac{t^n}{n!} = x + \sum_{n=1}^{\infty} \left(1+x^2\right) P_{n-1}^{'}(x)\, \frac{t^n}{n!}, \tag{3.6}$$

and thus comparing (3.5) with (3.6) clearly yields

$$P_{n-1}^{'}(x) = \sum_{k=1}^{n} T(n,k)\, x^{k-1}$$

so that we find by integration that

$$P_n(x) = P_n(0) + \sum_{k=1}^{n+1} \frac{1}{k}\, T(n+1,k)\, x^k. \tag{3.7}$$

Now, notice that $T(n,1)$ and $S(n,0)$ are, in fact, the well-known tangent and Euler numbers; see, for instance, [12, p. 663, Eqs. (1) and (2)]. More importantly, by the definitions in (2.1) and (3.1) as well as in (2.2) and (3.2), we have

$$P_n(0) = T(n,1) \qquad \text{and} \qquad Q_n(0) = S(n,0). \tag{3.8}$$

Finally, in view of (3.8), the desired result (3.3) follows from (3.7).

Similarly, along the same lines, we have

$$Q(x,t) = \sum_{k=0}^{\infty} \sec(t) \tan^k(t)\, x^k = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} S(n,k)\, \frac{t^n}{n!} \right) x^k$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} S(n,k)\, x^k \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} Q_n(x)\, \frac{t^n}{n!},$$

and in this way we arrive at the second needed result, (3.4). □

Proof of Lemma 2. First, we shall show that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \cot(x) = (-1)^n\, P_n\big(\cot(x)\big) \qquad (n \in \mathbb{N}_0) \tag{3.9}$$

and

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \csc(x) = (-1)^n\, \csc(x)\, Q_n\big(\cot(x)\big) \qquad (n \in \mathbb{N}_0). \tag{3.10}$$

Indeed, it is easily seen that (3.9) and (3.10) follow upon noting that $\tan(x+\frac{\pi}{2}) = -\cot(x)$ and $\sec(x+\frac{\pi}{2}) = -\csc(x)$ and using (3.1*) and (3.2*) in conjunction with the following readily deducible property of $P_n(x)$ and $Q_n(x)$:

$$P_n(-x) = (-1)^{n+1}\, P_n(x) \quad \text{and} \quad Q_n(-x) = (-1)^n Q_n(x) \quad (n \in \mathbb{N}_0).$$

The parts (a)–(d) of Lemma 2, in view of Lemma 1, are direct consequences of, respectively, the formulae (3.9), (3.10), (3.1*) and (3.2*). □

Proof of Lemma 3. To prove part (a) note that

$$\frac{\partial}{\partial \xi}\,\ell_s(\xi) = 2\pi\imath\,\ell_{s-1}(\xi), \tag{3.11}$$

which, in turn, follows from the definition of the Lerch zeta function $\ell_s(\xi)$ in (1.1) for $\Re(s) > 2$ and by analytic continuation for all s. The definition in (1.1) also yields

$$\ell_1(\xi) = -\log\left(1 - e^{2\pi\imath\,\xi}\right) \qquad (\xi \in \mathbb{R} \setminus \mathbb{Z}),$$

and from this we obtain $\ell_0(\xi)$ by (3.11). Using (3.11) repeatedly with initial value $\ell_0(\xi)$ leads to $\ell_{1-n}(\xi)$ given by part (a).

Likewise, starting from the definition of the Legendre chi function $\chi_s(z)$ in (1.2), we have part (b) by making use of

$$\frac{\partial}{\partial \xi}\,\chi_s(e^{\pi\imath\,\xi}) = \pi\imath\,\chi_{s-1}(e^{\pi\imath\,\xi}) \tag{3.12}$$

and

$$\chi_0(e^{\pi\imath\,\xi}) = \frac{\imath}{2}\csc(\pi\xi) \qquad (\xi \in \mathbb{R} \setminus \mathbb{Z}).$$

The proofs of parts (c) and (d) follow precisely along the same lines as those of parts (a) and (b). □

Proof of Theorem. It is straightforward to obtain parts (a) and (c) of the Theorem by combining the corresponding parts in Lemmas 2 and 3.

In the case of parts (b) and (d), by Lemmas 2 and 3, we first find that

$$\chi_{1-n}(e^{\pi\imath\,\xi}) = \frac{1}{2}\,\imath^n\,\csc(\pi\xi)\sum_{k=0}^{n-1} S(n-1,k)\,\cot^k(\pi\xi) \qquad (\xi \in \mathbb{R} \setminus \mathbb{Z};\ n \in \mathbb{N}) \tag{3.13}$$

and

$$\chi^*_{1-n}(e^{\pi\imath\,\xi}) = \frac{(-1)^{n-1}}{2}\,\imath^{n-1}\sec(\pi\xi)\sum_{k=0}^{n-1} S(n-1,k)\,\tan^k(\pi\xi) \qquad \left(\xi \in \mathbb{R} \setminus \left\{(2k+1)\tfrac{1}{2}\,|k \in \mathbb{Z}\right\};\ n \in \mathbb{N}\right). \tag{3.14}$$

However, the required formulae follow at once upon setting $z = e^{\pi\imath\,\xi}$ (*i.e.* $\pi\,\xi = -\imath\,\log z$) in (3.13) and (3.14). All that is needed is to observe that $\csc(-\imath\,\log z) = -2\,\imath z/(1-z^2)$, $\cot(-\imath\,\log z) = -\imath\,(1+z^2)/(1-z^2)$, $\sec(-\imath\,\log z) = 2\,z/(1+z^2)$ and $\tan(-\imath\,\log z) = \imath\,(1-z^2)/(1+z^2)$. □

4. Concluding remarks

We remark that derivative polynomials of Hoffman were recently linked to $\ell_{-n}(\xi)$ by Boyadzhiev, but the coefficients of these polynomials were not identified in terms of the higher tangent and secant numbers [4]. In addition, the connection between derivative polynomials and the higher tangent and secant numbers appears in the work of Chang and Ha [7].

Note that the higher order tangent and secant numbers $T(n,k)$ and $S(n,k)$ appear to be insufficiently investigated, but the simplicity of the above-found formulae suggests that they might be useful and well worthy of further study.

It is obvious, by parity considerations, that $T(n,k) \neq 0$ only when $1 \leq k \leq n$ and either both n and k are even or both n and k are odd (see Equation (2.1) above and Table 1 below). In other words, $T(2m, 2r+1) = 0$ and $T(2m+1, 2r) = 0$ $(m, r \in \mathbb{N}_0)$. The same applies to $S(n,k)$ when $0 \leq k \leq n$ (see Table 2).

TABLE 1. Tangent numbers $T(n,k)$

$n\backslash k$	1	2	3	4	5	6	7	8	9
1	1								
2	0	2							
3	2	0	6						
4	0	16	0	24					
5	16	0	120	0	120				
6	0	272	0	960	0	720			
7	272	0	3 696	0	8 400	0	5 040		
8	0	7 936	0	48 384	0	80 640	0	40 320	
9	7 396	0	168 960	0	645 120	0	846 720	0	362 880

In order to demonstrate the application of the above-derived closed-form formulae, in the examples below we list several first values of $\ell_{-n}(\xi)$ and $\chi_{-n}(z)$.

Example 1. In view of Theorem (a) and the values of $T(n,k)$ which are given in Table 1, for $\xi \in \mathbb{R} \setminus \mathbb{Z}$, we have (*cf.* [2, p. 227]):

$$\ell_{-1}(\xi) = -\tfrac{1}{4}\Big[1 + \cot^2(\pi\xi)\Big],$$

$$\ell_{-2}(\xi) = -\tfrac{i}{8}\Big[2\cot(\pi\xi) + 2\cot^3(\pi\xi)\Big],$$

$$\ell_{-3}(\xi) = \tfrac{1}{16}\Big[2 + 8\cot^2(\pi\xi) + 6\cot^4(\pi\xi)\Big],$$

$$\ell_{-4}(\xi) = \tfrac{i}{32}\Big[16\cot(\pi\xi) + 40\cot^3(\pi\xi) + 24\cot^5(\pi\xi)\Big],$$

$$\ell_{-5}(\xi) = -\tfrac{1}{64}\Big[16 + 136\cot^2(\pi\xi) + 240\cot^4(\pi\xi) + 120\cot^6(\pi\xi)\Big],$$

$$\ell_{-6}(\xi) = -\tfrac{i}{128}\Big[272\cot(\pi\xi) + 1\,232\cot^3(\pi\xi) + 1\,680\cot^5(\pi\xi) + 720\cot^7(\pi\xi)\Big],$$

$$\begin{aligned}\ell_{-7}(\xi) = -\tfrac{1}{256}\Big[&272 - 3\,968\cot^2(\pi\xi) - 12\,096\cot^4(\pi\xi) - 13\,440\cot^6(\pi\xi)\\ &- 5040\cot^8(\pi\xi)\Big],\end{aligned}$$

$$\begin{aligned}\ell_{-8}(\xi) = -\tfrac{i}{512}\Big[&7\,936\cot(\pi\xi) + 56\,320\cot^3(\pi\xi) + 129\,024\cot^5(\pi\xi)\\ &+ 120\,960\cot^7(\pi\xi) + 40\,320\cot^9(\pi\xi)\Big].\end{aligned}$$

TABLE 2. Secant numbers $S(n,k)$

$n\backslash k$	1	2	3	4	5	6	7	8	9
1	1								
2	0	2							
3	5	0	6						
4	0	28	0	24					
5	61	0	180	0	120				
6	0	662	0	1 320	0	720			
7	1 385	0	7 266	0	10 920	0	5 040		
8	0	24 568	0	83 664	0	100 800	0	40 320	
9	50 521	0	408 360	0	1 023 120	0	1 028 160	0	362 880

Example 2. In view of Theorem (b) and the values of $S(n,k)$ which are given in Table 2, for $z \in \mathbb{C} \setminus \{-1,1\}$, we have:

$$\chi_0(z) = \frac{1}{1-z^2}\, z,$$
$$\chi_{-1}(z) = \frac{1}{(1-z^2)^2}\left[z + z^3\right],$$
$$\chi_{-2}(z) = \frac{1}{(1-z^2)^3}\left[z + 6z^3 + z^5\right],$$
$$\chi_{-3}(z) = \frac{1}{(1-z^2)^4}\left[z + 23z^3 + 23z^5 + z^7\right],$$
$$\chi_{-4}(z) = \frac{1}{(1-z^2)^5}\left[z + 76z^3 + 230z^5 + 76z^7 + z^9\right],$$
$$\chi_{-5}(z) = \frac{1}{(1-z^2)^6}\left[z + 237z^3 + 1\,682z^5 + 1\,682z^7 + 237z^9 + z^{11}\right],$$
$$\chi_{-6}(z) = \frac{1}{(1-z^2)^7}\left[z + 722z^3 + 10\,543z^5 + 23\,548z^7 + 10\,543z^9 + 722z^{11} + z^{13}\right],$$
$$\chi_{-7}(z) = \frac{1}{(1-z^2)^8}\left[z + 2\,179z^3 + 60\,657z^5 + 259\,723z^7 + 259\,723z^9 + 60\,657z^{11} + 2\,179z^{13} + z^{15}\right],$$
$$\chi_{-8}(z) = \frac{1}{(1-z^2)^9}\left[z + 6\,552z^3 + 331\,612z^5 + 2\,485288z^7 + 4\,675014z^9 + 2\,485288z^{11} + 331\,612z^{13} + 6\,552z^{15} + z^{17}\right].$$

To conclude, we have considered the Lerch zeta function, the Legendre chi function and their alternative counterparts of non-positive integer order. It is shown that $\ell_{-n}(\xi)$, $\chi_{-n}(z)$, $\ell^*_{-n}(\xi)$ and $\chi^*_{-n}(z)$, $n \in \mathbb{N}_0$, are elementary functions, and very simple explicit closed-form formulae for these functions are derived.

Note added in proof. Since writing this paper, the author has become aware of another paper of K. N. Boyadzhiev ("Derivative polynomials for tanh, tan, sech and sec in explicit form", *Fibonacci Quart.* **45** (2007), 291–303) also dealing with derivative polynomials.

Acknowledgements

The author is very grateful to the anonymous referee for a careful and thorough reading and for detailed, insightful, very knowledgeable and constructive analysis of this paper. Valuable comments and suggestions have led to a greatly improved presentation of the results. The author acknowledges financial support from the Ministry of Science of the Republic of Serbia under Research Projects 142025 and 144004.

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