DOMINANCE OF A RATIONAL MAP
TO THE COBLE QUARTIC

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Abstract. We show the dominance of the restriction map from a moduli space of stable sheaves on the projective plane to the Coble sixfold quartic. With the dominance and the interpretation of a stable sheaf on the plane in terms of hyperplane arrangements, we expect these tools to reveal the geometry of the Coble quartic.

1. Introduction

Let \( C \) be a smooth non-hyperelliptic curve of genus 3 over complex numbers. Then \( C \) is embedded into \( \mathbb{P}^2 \simeq \mathbb{P}H^0(K_C)^* \) by canonical embedding as a plane quartic curve. The moduli space \( SU_C(2, K_C) \) of semistable vector bundles of rank 2 with canonical determinant over \( C \) is known to be a hypersurface in \( \mathbb{P}^7 \), called the ‘Coble quartic’, [3], [13]. Let \( W_r \) be the closure of the following set

\[
\{ E \in SU_C(2, K_C) \mid h^0(C, E) \geq r + 1 \}
\]

Then we have the following inclusions [14] on the Brill-Noether loci,

\[
SU_C(2, K_C) \supset W \supset W_1 \supset W_2 \supset W_3 = \emptyset,
\]

where \( W = W^0 \). Many properties on the geometry of these Brill-Noether loci have been discovered in [14].

Let \( \overline{M}(c_1, c_2) \) be the moduli space of stable sheaves of rank 2 with the Chern classes \( (c_1, c_2) \) on the projective plane. The dimension of this space is known to be \( 4c_2 - 3 \) if \( c_1 = 0 \) [2] and \( 4c_2 - 4 \) if \( c_1 = -1 \) [9]. Then there exists a rational map [8]

\[
\Phi_k : \overline{M}(1, k) \dashrightarrow SU_C(2, K_C), \quad 1 \leq k \leq 4,
\]

defined by sending \( E \) to \( E|_C \). It is shown in [8] that \( \Phi_k \) is a dominant map to \( W^2 \), \( W^1 \) and \( W \), for \( k = 1, 2, 3 \), respectively. In this article, we give a proof of the dominance of the rational map \( \Phi_4 \). This is equivalent to the dominance of the rational map from \( \overline{M}(3, 6) \) to \( SU_C(2, 3K_C) \) by twisting. For a general bundle \( E \in SU_C(2, 3K_C) \), we embed \( C \) with \( \mathbb{P}^2 \) into a Grassmannian \( Gr(5, 2) \) and take the pull-back of the universal quotient bundle of \( Gr(5, 2) \) to \( \mathbb{P}^2 \). This bundle is shown to be stable and have the Chern classes \( (3, 6) \).

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As a quick consequence, we can obtain the old result that \( SU_C(2, K_C) \) is unirational since \( M(1, 4) \) is rational. The unirationality implies the rationally connectedness. We see how we can obtain a rational curve through two general points of the Coble quartic in terms of hyperplane arrangements.

The restriction of vector bundles on \( \mathbb{P}_2 \) to plane curves was also studied in [7], where the author investigated the restriction of the tangent bundle of \( \mathbb{P}_2 \) to plane curves and gave the conditions for a vector bundle \( E \) on a plane curve to be a pull-back of the tangent bundle of \( \mathbb{P}_2 \), twisted by \( \mathcal{O}_{\mathbb{P}_2}(-1) \).

For the background on vector bundles, we suggest [12] as a good reference.

2. Embedding plane quartics in Grassmannians

Let \( E \) be a semistable vector bundle of rank 2 with the determinant \( 3K_C \) over \( C \), i.e. \( E \in SU_C(2, 3K_C) \). By the following lemma, we can obtain a morphism

\[
\varphi : C \to \text{Gr}(H^0(E), 2)
\]

sending \( p \in C \) to the 2-dimensional quotient space \( E_p \) of \( H^0(E) \).

**Lemma 2.1.** \( H^1(C, E) = 0 \) and \( E \) is globally generated.

**Proof.** \( H^1(E) \cong H^0(E^* \otimes K_C) \neq 0 \) implies the existence of a nonzero homomorphism \( E \to \mathcal{O}_C(K_C) \) which contradicts the semistability of \( E \). Now, by the same argument, we have \( H^1(E(-p)) = 0 \) for all \( p \in C \). From the long exact sequence of the sequence

\[
0 \to E(-p) \to E \to E_p \to 0,
\]

we obtain the surjective evaluation map \( H^0(E) \to E_p \), which implies the global generation of \( E \). \( \square \)

In fact, the morphism \( \varphi \) fits in the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\varphi} & \text{Gr}(H^0(E), 2) \\
\downarrow \quad \mid 3K_C \mid & & \quad \downarrow \vartheta \\
\mathbb{P}^1H^0(3K_C)* & \xrightarrow{\mathbb{P}\lambda^*} & \mathbb{P}(\wedge^2 H^0(E)^*)
\end{array}
\]

where \( \vartheta \) is the Plucker embedding and \( \mathbb{P}\lambda^* \) comes from the dual of the homomorphism

\[
\lambda : \wedge^2 H^0(E) \to H^0(\bigwedge^2 E) \cong H^0(3K_C).
\]

By the following lemma, \( \mathbb{P}\lambda^* \) is an embedding and so is \( \varphi \) for general \( E \).

**Lemma 2.2.** The homomorphism \( \lambda \) is surjective for general \( E \in SU_C(2, 3K_C) \).

**Proof.** If \( E \) is stable, then by the Nagata-Severi theorem [11], we have the exact sequence for \( E(-K_C) \),

\[
0 \to \mathcal{O}(D) \to E(-K_C) \to \mathcal{O}(K_C - D) \to 0,
\]

where \( D \) is a divisor of degree 1. For general \( E \), we have \( H^0(E(-K_C)) = 0 \), i.e. we can assume that \( H^0(\mathcal{O}(D)) = 0 \); i.e. \( D \) is non-effective.

Let \( L = \mathcal{O}(K_C + D) \) and \( F = \mathcal{O}(2K_C - D) \). Then we have

\[
0 \to L \to E \to F \to 0.
\]
Note that $h^0(L) = 3$, $h^0(F) = 5$ and $h^1(L) = h^1(F) = 0$ and, from the long exact sequence of the above sequence, we have

$$H^0(E) \simeq H^0(L) \oplus H^0(F),$$

and hence it is enough to show the surjectivity of the map

$$H^0(L) \otimes H^0(F) \to H^0(L \otimes F) \simeq H^0(3K_C).$$

For every $p \in C$, $h^0(L(-p)) = 2 + h^1(L(-p)) = 2 + h^0(p - D) = 2$ since $D$ is not effective. Hence, we can have a map from $C$ to $Gr(2, H^0(L))$ sending $p$ to $H^0(L(-p))$. Since $Gr(2, H^0(L)) \simeq \mathbb{P}_2$, we can choose $W \in Gr(2, H^0(L))$ which is not the same as $H^0(L(-p))$ for any $p \in C$. Then by the choice of $W$, it does not have base locus on $C$. Now consider the map

$$W \otimes H^0(F) \to H^0(3K_C).$$

By the Base-Point-Free Pencil Trick [1], the kernel of this map is isomorphic to $H^0(C, F \otimes L^{-1})$, and this is isomorphic to $H^0(K_C - 2D)$. Note that $h^0(K_C - 2D) = h^0(2D)$ by the Riemann-Roch theorem. If $h^0(2D) = 0$, then $W \otimes H^0(F)$ is isomorphic to $H^0(3K_C)$ by the counting of the dimensions. Hence, it is enough to show that $H^0(2D) = 0$ for general $E$.

Assume that $h^0(2D) > 0$, and then $O(2D)$ is an element of the theta divisor in $\text{Pic}^1(C)$. The map

$$\text{Pic}^1(C) \to \text{Pic}^2(C),$$

defined by $D \mapsto 2D$, is a finite surjective map of degree 64. Hence the subvariety of $\text{Pic}^1(C)$ whose elements are $D$ such that $h^0(D) = 0$ and $h^0(2D) > 0$ is of 2 dimensions. For these divisors $D$, the extensions of $O(K_C - D)$ by $O(D)$ are parametrized by $\mathbb{P}_3$, which means that the vector bundles that do not satisfy $h^0(2D) = 0$ are of at most 5 dimensions. Hence $h^0(2D) = 0$ in general. \hfill \Box

Now, for the 5-dimensional subspace $V \subset H^0(E)$, we have the following diagram:

(4)

$$\begin{array}{ccc}
C - & \xrightarrow{\varphi_V} & Gr(V, 2) \\
\downarrow{[3K_C]} & & \downarrow{\vartheta} \\
\mathbb{P}H^0(3K_C) & \xrightarrow{p\lambda^*} & \mathbb{P}(\wedge^2 V^*)
\end{array}$$

Consider a natural map

(5)

$$\begin{array}{ccc}
\mathbb{P}(\wedge^2 E) & \xrightarrow{f} & \mathbb{P}(\wedge^2 V_7) \\
\downarrow & & \\
\text{Gr}(5, V_7)
\end{array}$$

where $E$ is the universal subbundle, $V_7$ is a 7-dimensional vector space and $\text{Gr}(5, V_7)$ is the Grassmannian of 5-dimensional subspaces of $V_7$. Over $[V_5] \in \text{Gr}(5, V_7)$, the fibre $\wedge^2 V_5$ is linearly embedded into $\wedge^2 V_7$.

**Lemma 2.3.** The image of $f$ is the secant variety of $\text{Gr}(2, V_7) \subset \mathbb{P}(\wedge^2 V_7)$, and its dimension is equal to 17.
Proof. Let \([x] \in \text{Im}(f)\); i.e. there exists a \(V_5\) such that \(x \in \wedge^2 V_5\). Consider \(G = \text{Gr}(2, V_5) \subset \mathbb{P}(\wedge^2 V_5)\). Since the secant variety of \(G\) is \(\mathbb{P}(\wedge^2 V_5)\), we can express \(x\) by \((v \wedge w)\) or \((v_1 \wedge v_2 + v_3 \wedge v_4)\), which proves that \(\text{Im}(f)\) is contained in the secant variety of \(\text{Gr}(2, V_7)\).

Now we show the inclusion \(\text{Sec}(\text{Gr}(2, V_7)) \hookrightarrow \text{Im}(f)\). Assume that \(x\) is a general point in the secant variety. This means that \(x = v_1 \wedge v_2 + v_3 \wedge v_4\), where \(U = \langle v_1, v_2, v_3, v_4 \rangle\) is a 4-dimensional space. For any \(V_5 \supset U\), we have \(x \in \wedge^2 V_5\). This shows that \(\text{Sec}(\text{Gr}(2, V_7)) = \text{Im}(f)\), since both sides are closed subvarieties of \(\mathbb{P}(\wedge^2 V_7)\). Also the set of such \(V_5\) is 2-dimensional and \(\dim f^{-1}([x]) = 2\). Hence the dimension of \(\text{Im}(f)\) is 17, since \(\dim(\mathbb{P}(\wedge^2 E)) = 19\). □

Remark 2.4. \(\text{Gr}(2, V_2)\) is a Scorza variety of defect \(\delta = 4\) [16]. So, it is known that \(\dim \text{Sec}(\text{Gr}(2, V_7)) = 17\).

Lemma 2.5. For general \(E \in \text{SU}_C(2, 3K_C)\) and general 5-dimensional vector subspace \(V \subset H^0(E)\), the restriction of \(\lambda\) to \(\wedge^2 V\),

\[
\lambda : \wedge^2 V \rightarrow H^0(3K_C),
\]

is an isomorphism.

Proof. In the proof of [22], let 

\[
V_7 := W \oplus V_5,
\]

where \(V_5 \simeq H^0(F)\). In fact, we can take any \(V_5 \subset V_7\) with \(V_5 \cap H^0(L) = 0\). Then, the restriction of \(\lambda\) to \(\wedge^2 V_7\) is also surjective. Let \(K = \ker(\lambda)\) be the 11-dimensional subspace of \(\wedge^2 V_7\). Consider an incidence variety \(\mathcal{R} \subset \text{Gr}(5, V_7) \times \mathbb{P}(K)\),

\[
\mathcal{R} = \{(V_5, [x]) \mid x \in \wedge^2 V_5 \cap K\}.
\]

We have the following diagram:

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{pr_1} & \text{Gr}(5, V_7) \\
\downarrow & & \downarrow \\
\text{pr}_2 & & \mathbb{P}(K).
\end{array}
\]

It is enough to show that the map \(pr_1\) is not dominant, which means that for the general \(V_5 \subset V_7\) not in the image of \(pr_1\), we have the surjection in the assertion. Assume that \(pr_1\) is dominant, then \(\dim(\mathcal{R}) \geq 10\).

If we consider again the map

\[
\mathbb{P}(\wedge^2 E) \xrightarrow{f} \mathbb{P}(\wedge^2 V_7) \\
\downarrow \\
\text{Gr}(5, V_7),
\]

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then the image of $pr_2$ in $\mathbb{P}(K)$ is the intersection of $\text{Im}(f) = \text{Sec}(Gr(2, V_7))$ with $\mathbb{P}(K)$ in $\mathbb{P}(\wedge^2 V_7) \simeq \mathbb{P}_{20}$. Since $\text{Im}(f)$ is 17-dimensional, we have

$$7 \leq \dim \text{Im}(pr_2) \leq 10.$$ 

It is clear that $\mathbb{P}(K)$ contains a point in $\text{Sec}(Gr(2, V_7))$, but not in $Gr(2, V_7)$. The fibre over this point in $\mathcal{R}$ is isomorphic to $Gr(1, 3) \simeq \mathbb{P}_2$. Thus the dimension of $\text{Im}(pr_2)$ is greater than 7.

Now assume that $\dim \mathbb{P}(K) \cap \text{Sec}(Gr(2, 7)) \geq 8$. In the proof of (2.2), we have

$$K \cap (W \wedge V_5) = (0),$$

if $V_5 \cap W = (0)$. If $V_5 \cap W \neq (0)$, the intersection is always $[\wedge^2 W]$. Let us consider the canonical map

$$s : W \otimes V_7/W \to \wedge^2 V_7/W.$$ 

For all $V_5$ with $V_5 \cap W = (0)$, the images in $\wedge^2 V_7/W$ are the same as a 10-dimensional vector space. If we take the preimage of this space in $\wedge^2 V_7$, then it is the union of $W \wedge V_5$ for all $V_5$, which is now an 11-dimensional space. Note that $K \cap (W \wedge V_5) = [\wedge^2 W]$ if $W \cap V_5 \neq (0)$. Let us denote by $D$ the projectivization of the preimage of $s(W \otimes V_7/W)$ in $\wedge^2 V_7$. Then $D$ is a 10-dimensional subvariety of $\mathbb{P}(\wedge^2 V_7)$ and it intersects with $\mathbb{P}(K)$ at the unique point $[\wedge^2 W]$. In fact, $D$ is the projective tangent space $\mathbb{P}T[W]Gr(2, V_7)$ of $Gr(2, V_7)$ at $[W]$ in $\mathbb{P}(\wedge^2 V_7)$. Recall that

$$T[D]Gr(2, V_7) = \text{Hom}(W, V_7/W) \simeq W^* \otimes V_7/W$$

and $T[D][\wedge^2 V_7] = \text{Hom}(\wedge^2 W, \wedge^2 V_7/W)$.

The differential map of the Plücker embedding at $[W]$ is defined as follows: $x = w^* \otimes e \in T[D][Gr(2, V_7)]$ is sent to the map

$$w_1 \wedge w_2 \mapsto s((w^*(w_1)w_2 - w_1 w^*(w_2)) \otimes e),$$

where $W = \langle w_1, w_2 \rangle$. This explains the assertion.

Now since the union of the secant lines of $Gr(2, V_7)$ passing through $[\wedge^2 W]$ is 11-dimensional and $\mathbb{P}(K) \cap \text{Sec}(Gr(2, V_7))$ is of dimension $\geq 8$, we can pick an element $[U] \in \mathbb{P}(K) \cap \text{Sec}(Gr(2, V_7))$ and then the secant line $[U][W]$ lies in $\mathbb{P}(K)$. From the condition on $W$, $U$ and $W$ span a 4-dimensional subspace of $V_7$. In particular, general points on the secant line $[U][W]$ are indecomposable. Let $p$ be such a point. Since $\text{Sing}([\text{Sec}(Gr(2, V_7))]) = Gr(2, V_7)$ [16], the dimension of $T_p([\text{Sec}(Gr(2, V_7))])$ is 17. Note that

$$T_p([\text{Sec}(Gr(2, V_7))]) = \langle T[D]G, T[U]G \rangle.$$ 

Since $T_p(\mathbb{P}(K) \cap \text{Sec}(Gr(2, V_7))) = \mathbb{P}(K) \cap T_p([\text{Sec}(Gr(2, V_7))])$

is at least 8-dimensional, $\mathbb{P}(K)$ intersects $T[D]G$ along at least 1-dimensional subspace, which is a contradiction because $\mathbb{P}(K) \cap D$ is a single point. \qed
From the previous lemma, we have the commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}_2 & \simeq & \mathbb{P}H^0(K_C)^* \\
\searrow & & \searrow \\
\uparrow & & \uparrow \\
C & \rightarrow & \text{Gr}(H^0(E),2) \\
\downarrow & & \downarrow \\
\text{Gr}(V,2) & \rightarrow & \mathbb{P} \binom{\wedge^2 H^0(E)^*}{\mathbf{v}_3}
\end{array}
\]

where the composite of the two vertical maps on the right,

\[
\mathbb{P}H^0(3K_C)^* \hookrightarrow \mathbb{P} \binom{\wedge^2 H^0(E)^*}{\mathbf{v}_3} \rightarrow \mathbb{P} \binom{\wedge^2 V^*}{\mathbf{v}_3},
\]

is an isomorphism and \( \mathbf{v}_3 \) is the 3-tuple Veronese embedding; i.e. \( \mathbf{v}_3 \) is given by the complete linear system \( \mid O_{\mathbb{P}_3}(3) \). In particular, \( C \) is embedded into \( \text{Gr}(V,2) \). Note that \( C \) is non-degenerate in \( \mathbb{P}_9 \simeq \mathbb{P} \binom{\wedge^2 V^*}{\mathbf{v}_3} \) due to the Riemann-Roch theorem and the Noether theorem.

**Corollary 2.6.** General element \( E \) in \( SU_C(2,3K_C) \) is generated by a 5-dimensional subspace of \( H^0(E) \).

### 3. Embedding the projective plane into Grassmannian

In the diagram (10), the projective plane \( \mathbb{P}H^0(K_C)^* \simeq \mathbb{P}_2 \) is embedded into the projective space \( \mathbb{P} \binom{\wedge^2 V^*}{\mathbf{v}_3} \simeq \mathbb{P}_9 \) by the Plücker embedding.

**Lemma 3.1.** For general \( E \in SU_C(2,3K_C) \), there exists a 5-dimensional vector subspace \( V \subset H^0(E) \) such that \( \mathbb{P}H^0(K_C)^* \) is embedded into \( \text{Gr}(V,2) \) in the diagram (10).

**Proof.** Let \( V \subset H^0(E) \) be a 5-dimensional subspace selected in \( \mathbb{P}_9 \) and assume that \( \mathbb{P}H^0(K_C)^* \) is not embedded into \( \text{Gr}(V,2) \). Recall that \( \text{Gr}(V,2) \) is cut out by the 4-dimensional projectively linear family of quadrics of rank 6 in \( \mathbb{P}_9 \) whose singular locus is \( \mathbb{P}_3 \) contained in \( \text{Gr}(V,2) \) as the Schubert variety of lines through a point corresponding to the quadric in \( \mathbb{P}_4 \). Let \( Q(p) \) be one of the quadrics of rank 6 containing \( \text{Gr}(V,2) \) which does not contain \( S \), where \( p \) is a point in \( \mathbb{P}_4 \) and \( S \) is the image of \( \mathbb{P}_2 \) by \( \mathbf{v}_3 \). Since \( \mathbf{v}_3^{-1}(Q(p)) \) is a plane sextic curve, we have

\[ v_3^{-1}(Q(p)) = C + C', \]

where \( C' \) is a conic. First, assume that \( \text{Gr}(V,2) \cap S = C + C' \). If we consider the incidence variety \( Z_C = \{(l,x) | x \in l\} \subset C \times \mathbb{P}_4 \), we have a diagram

\[
\begin{array}{ccc}
& Z_C & \\
\text{Gr}(V,2) & \uparrow & \downarrow \\
C & \leftrightarrow & \mathbb{P}_4.
\end{array}
\]

Let \( S_C \) be the image of \( q \) in \( \mathbb{P}_4 \). If \( S_C \) is degenerate, i.e. there exists a hyperplane \( \mathbb{P}_3 \subset \mathbb{P}_4 \) containing \( S_C \), then \( C \) is contained in some Grassmannian \( \text{Gr}(4,2) \subset \text{Gr}(V,2) \) and, in particular, \( C \) is contained in \( \mathbb{P}_3 \), the Plücker space of \( \text{Gr}(4,2) \),
which is a contradiction to the non-degeneracy of $C$ in $\mathbb{P}_9$. Similarly we can define $Z_{C'}$ and $S_{C'}$. Recall the well known fact that

$$\text{deg}(C) = \text{deg}(S_C) \cdot \text{deg}(q).$$

If $\text{deg}(S_C) = 1$, i.e. $S_C$ is a plane in $\mathbb{P}_4$, then $C$ must be contained in $\mathbb{P}_3(p)$, the singular locus of a quadric $Q(p)$ for $p \in S_C$, which is a contradiction to the fact that $C \subset \mathbb{P}_9$ is nondegenerate. Hence $\text{deg}(S_C) \geq 2$ and so $\text{deg}(q) \leq 6$. This implies that the number of points in $\mathbb{P}_3(p) \cap C$ is less than 7 for $p \in S_C$. Since the intersection of $S_C$ and $S_{C'}$ is at most 1-dimensional in $S_C$, we still have 2-dimensional choices for $p$ for which $\mathbb{P}_3(p) \cap (C + C') = \mathbb{P}_3(p) \cap C$ is less than 7 points. We can also have the same conclusion on the intersection number of $\mathbb{P}_3(p) \cap (C + C')$ in the case when $Gr(V, 2) \cap S$ is the proper subset of $C + C'$ since it still contains $C$. Now choose $p \in \mathbb{P}_4$ such that the singular locus $\mathbb{P}_3(p)$ of $Q(p)$ meets $C + C'$ with $k$ points where $0 < k < 7$. We have the commutative diagram

$$
\begin{array}{cccccc}
\mathbb{P}_3(p) & \xrightarrow{\pi} & Gr(V, 2) & \xrightarrow{\iota} & \mathbb{P}(\wedge^2 V^*) & \xrightarrow{\psi} & C + C' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Gr(4, 2) & \xrightarrow{\pi'} & \mathbb{P}_3 & \xrightarrow{\iota'} & S & \xrightarrow{\psi'} & C + C',
\end{array}
$$

where $\overline{S}$, $\overline{C + C'}$ are the images of $S$, $C + C'$ respectively, via the projection, and the image of $Gr(V, 2)$ lies in the image of the quadric $Q$, i.e. the Grassmannian $Gr(4, 2) \subset \mathbb{P}_5$. Let $Q'$ be another quadric cutting $Gr(V, 2)$ with singular locus $\mathbb{P}_3$. Since $\mathbb{P}_3 \cap \mathbb{P}_3'$ is a single point, the image of $Q'$ by the projection is $\mathbb{P}_5$. Thus the image of $Gr(V, 2)$ is $Gr(4, 2)$. Note that the degree of $C + C'$ is $18 - k$ and the degree of $\overline{S}$ is $9 - k$ since $\mathbb{P}_3(p) \cap S = \mathbb{P}_3(p) \cap (C + C')$. If $Q(p)$ contains $S$ for all such $p \in S_C$, then all quadrics containing $Gr(V, 2)$ of rank 6 should contain $S$ since $S_C$ is nondegenerate in $\mathbb{P}_4$. In particular, $Gr(V, 2)$ should contain $S$, which is against the assumption. So there exists a $p \in S_C$ for which $S$ is not contained in $Q(p)$. Thus the image of $S$ by the projection is also not contained in the image of $Q(p)$, i.e. $Gr(4, 2)$. But the degree of intersection $Gr(4, 2) \cap \overline{S}$ is $2 \times (9 - k) < 18 - k$, which is a contradiction to the fact that this intersection contains $\overline{C + C'}$. □

Let $U_V$ and $\overline{U_V}$ be the universal subbundle and quotient bundle of $Gr(V, 2)$, respectively. With the condition on $V$ in the previous lemma, let

$$E_V := v_3^* U_V,$$

which implies that the restriction of $E_V$ to $C$ is $E$, i.e. $E_V|_C = E$.

**Lemma 3.2.** $E_V$ is stable with the Chern classes $(3, 6)$, i.e. $E_V \in \overline{\mathcal{M}}(3, 6)$.

**Proof.** Since the first Chern class of $\overline{U_V}$ is the hyperplane section of $Gr(V, 2)$ in $\mathbb{P}(\wedge^2 V^*)$ and $v_3$ is the 3-tuple Veronese embedding, we get $c_1(E_V) = 3$.

By the choice of $V$, we have an exact sequence

$$0 \to G \to V \otimes \mathcal{O}_{\mathbb{P}_4} \to E_V \to 0,$$

where $G$ is the kernel of the surjection $V \otimes \mathcal{O}_{\mathbb{P}_4} \to E_V$ and $V$ is a 5-dimensional vector subspace of $H^0(E_V)$. In particular, $h^0(E_V) \geq 5$. By the choice of $E$, we
have \( h^0(E_V(-1)|_C) = 0 \). From the long exact sequence of cohomology of the exact sequence
\[
0 \to E_V(-5) \to E_V(-1) \to E_V(-1)|_C \to 0,
\]
we have
\[
H^0(E_V(-5)) \simeq H^0(E_V(-1)).
\]

For a line \( H \subset \mathbb{P}_2 \), \( E_V|_H \simeq \mathcal{O}_H(a) \oplus \mathcal{O}_H(3-a) \) for \( a = 2 \) or \( 3 \) since \( E_V \) is globally generated. In particular, \( h^0(E_V(-k)|_H) = 0 \) for \( k \geq 4 \). From the long exact sequence of cohomology of the exact sequence
\[
0 \to E_V(-k-1) \to E_V(-k) \to E_V(-k)|_H \to 0,
\]
we have \( h^0(E_V(-k-1)) = h^0(E_V(-k)) \) for all \( k \geq 4 \). Since \( h^0(E_V(-k)) = 0 \) if and only if \( k \) is sufficiently large \( k \), we have \( h^0(E_V(-k)) = 0 \) for \( k \geq 4 \) and in particular, \( h^0(E_V(-1)) = h^0(E_V(-5)) = 0 \); i.e. \( h^0(E_V(-k)) = 0 \) for all \( k \geq 1 \). Hence the vector bundle \( E_V \) is stable.

Again, let \( H \) be a line in \( \mathbb{P}_2 \). From the exact sequence
\[
0 \to E_V(-1) \to E_V \to E_V|_H \to 0,
\]
we get \( h^0(E_V) \leq h^0(E_V|_H) \). Since \( E_V|_H \simeq \mathcal{O}_H(a) \oplus \mathcal{O}_H(3-a) \) for \( a = 2 \) or \( 3 \), \( h^0(E_V|_H) = 5 \) and so \( h^0(E_V) \leq 5 \). Thus we obtain \( h^0(E_V) = \dim V = 5 \).

Now from the long exact sequence of cohomology of \( \text{[13]} \), we have \( h^0(\mathbb{P}_2, G) = 0 \). If we twist \( \text{[13]} \) by \(-1 \), we have \( h^1(\mathbb{P}_2, G(-1)) = 0 \). For any line \( l \subset \mathbb{P}_2 \), consider the exact sequence
\[
0 \to G(-1) \to G \to G|_l \to 0.
\]
From the above statement, we get \( H^0(G|_l) = 0 \). Since \( c_1(G) = -c_1(E_V) = -3 \), we have \( G|_l \simeq \mathcal{O}_l(a) \oplus \mathcal{O}_l(b) \oplus \mathcal{O}_l(c) \) with \( a + b + c = -3 \). The only choice from the vanishing of \( H^0(G|_l) \) is \((a,b,c) = (-1,-1,-1)\). Hence \( G \) is a uniform vector bundle of rank \( 3 \) on \( \mathbb{P}_2 \) with the splitting type \((-1,-1,-1)\). From the classification of such bundles \( \text{[5]} \), we have
\[
G \simeq \mathcal{O}_{\mathbb{P}_2}(-1)^{\oplus 3}.
\]
In particular, \( c_2(G) = 3 \) and so \( c_2(E_V) = 6 \). \( \square \)

Since we can pick an element \( E_V \in \overline{M}(3,6) \) mapping to a general element \( E \in SU_C(2,3K_C) \), the rational map
\[
\overline{M}(3,6) \dashrightarrow SU_C(2,3K_C)
\]
is dominant. By twisting the map \( \text{[14]} \) with \( \mathcal{O}_{\mathbb{P}_2}(-1) \) and \( \mathcal{O}_C(-K_C) \), we have the following main theorem.

**Theorem 3.3.** The restriction map
\[
\Phi_4 : \overline{M}(1,4) \dashrightarrow SU_C(2,K_C)
\]
is dominant.

**Remark 3.4.** Dolgachev and Kapranov \( \text{[4]} \) showed that the logarithmic bundles \( E(\mathcal{H}) \) attached to the general hyperplane arrangement \( \mathcal{H} = (H_1, \cdots, H_6) \) in \( \mathbb{P}_2 \) form an open Zariski subset \( U \subset \overline{M}(3,6) \). For these bundles \( E(\mathcal{H}) \), we have a Steiner resolution
\[
0 \to \mathcal{O}_{\mathbb{P}_2}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}_2}^{\oplus 5} \to E(\mathcal{H}) \to 0.
\]
Let us consider a special type of arrangement of 6 lines. Let

\[ \text{Proposition 3.5.} \]

be 6 lines in general position on the Grassmannian \( \text{Gr}(V,2) \) of 6-dimensional points in \( \mathbb{P}^2 \). From this, we have a 5-dimensional space \( H^0(\mathbb{P}_2, E(H)) \). Tensoring the exact sequence

\[ 0 \to \mathcal{O}_{\mathbb{P}_2}(-4) \to \mathcal{O}_{\mathbb{P}_2} \to \mathcal{O}_C \to 0 \]

by \( E(H) \), we can consider \( V \) as a subspace of \( H^0(C, E(H)|_C) \), which is 8-dimensional. As we have seen already in the proof of \( \text{Lemma 3.2} \), the bundle \( E_V \) has a Steiner resolution, pulled back from the universal exact sequence on the Grassmannian \( \text{Gr}(V,2) \). This motivates the whole argument in this paper.

Since \( \overline{M}(1,4) \) is rational and the map \( \Phi_4 \) is dominant, \( SU_C(2, K_C) \) is unirational. It implies that \( SU_C(2, K_C) \) is rationally connected and so rationally chain-connected. Let \( \mathcal{H} = (H_0, \ldots, H_6) \) be a general arrangement of 6 lines on \( \mathbb{P}_2 \) and then we can associate a logarithmic bundle \( E(H) \in \overline{M}(3,6) \) to \( \mathcal{H} \). It is known that the logarithmic bundles \( E(H) \) form an open Zariski subset of \( \overline{M}(3,6) \) and, after twisting by \( \mathcal{O}_{\mathbb{P}_2}(-1) \), \( \overline{M}(1,4) \). Let \( \mathcal{F} \) be a family of arrangements of 6 lines on \( \mathbb{P}_2 \) and let \( E(F) \) be the closure of the subvariety of \( \overline{M}(1,4) \) whose closed points correspond to \( E(H) \otimes \mathcal{O}_{\mathbb{P}_2}(-1) \) with \( \mathcal{H} \in \mathcal{F} \).

**Proposition 3.5.** \( SU_C(2, K_C) \) is rationally chain-connected. In fact, any two general points in \( SU_C(2, K_C) \) can be connected by at most 6 rational curves which can be described explicitly.

**Proof.** Let us consider a special type of arrangement of 6 lines. Let \( H_0, H_1, \ldots, H_5 \) be 6 lines in general position on \( \mathbb{P}_2 \) and let \( p \) be a fixed point on \( H_0 \) in general position. If we fix \( H_1, \ldots, H_5 \), then we have a 1-dimensional family \( \mathcal{F} \) of 6 lines with \( H_0 \) moving. Consider a map

\[ \Psi : \mathbb{P}_1(\mathcal{F}) \to SU_C(2, K_C), \]

sending \( \mathcal{H} \) to \( E(H)(-1)|_C \). Since \( SU_C(2, K_C) \) is projective, this map is a morphism \( [6] \). Clearly \( \Psi \) is not a constant map; otherwise \( \Phi_4 \) is also a constant map, which is not true. From the fact that logarithmic bundles associated to 6 lines in general position form an open Zariski subset of \( \overline{M}(3,6) \) and \( \Phi_4 \) is dominant, we can find a 1-dimensional family of 6 lines \( \mathcal{F} \) which maps to a rational curve on \( SU_C(2, K_C) \) via \( \Psi \) for a general element of \( SU_C(2, K_C) \). Furthermore, for two general elements \( E_1, E_2 \in \overline{M}(3,6) \), we can find 6 families of 6 lines \( \mathcal{F}_i \), \( 1 \leq i \leq 6 \), as above such that the arrangements corresponding to \( E_1, E_2 \) lie in \( \mathcal{F}_1, \mathcal{F}_6 \) respectively and \( \mathcal{F}_i \cap \mathcal{F}_{i+1} \neq \emptyset \). From this fact with the dominance of \( \Phi_4 \), we can find 6 rational curves passing through two general points on \( SU_C(2, K_C) \).

**Remark 3.6.** Note that we can choose these rational curves not contained in the singular locus of \( SU_C(2, K_C) \) which is the Kummer variety of \( \text{Pic}^2(C) \). Let \( \tilde{S} \) be a desingularization by the blow-up \( [10] \) and consider the proper transform of the previous 6 rational curves on \( SU_C(2, K_C) \). It shows the rationally chain-connectedness of \( \tilde{S} \), and since \( \tilde{S} \) is smooth, it implies the rational connectedness; i.e. the chain of these 6 curves can be deformed to a rational curve and its image on \( SU_C(2, K_C) \) will give us a rational curve through two general points.
References


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