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ON GENERIC ROTATIONLESS DIFFEOMORPHISMS OF THE ANNULUS

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Abstract. Let f be a C^r-diffeomorphism of the closed annulus A that preserves the orientation, the boundary components and the Lebesgue measure. Suppose that f has a lift $\tilde{f}$ to the infinite strip $\tilde{A}$ which has zero Lebesgue measure rotation number. If the rotation number of $\tilde{f}$ restricted to both boundary components of A is positive, then for such a generic f ($r \geq 16$), zero is an interior point of its rotation set. This is a partial solution to a conjecture of P. Boyland.

1. Introduction and statement of the main result

In this paper we consider diffeomorphisms f of the closed annulus $A = S^1 \times [0,1]$, which satisfy certain special conditions, namely:

(1) f preserves orientation and the boundary components;
(2) f preserves the Lebesgue measure of A;
(3) there exists a special lift $\widetilde{f}$ of f to the universal cover of the annulus $\widetilde{A} = \mathbb{R} \times [0,1]$, satisfying the following: If $p_1 : \widetilde{A} \to \mathbb{R}$ is the projection on the first coordinate and $p : \widetilde{A} \to A$ is the covering mapping, we can define the displacement function $\phi : A \to \mathbb{R}$ as

$$\phi(x,y) = p_1 \circ \widetilde{f}(\widetilde{x},\widetilde{y}) - \widetilde{x}, \tag{1.1}$$

for any $(\widetilde{x},\widetilde{y}) \in p^{-1}(x,y)$. Then the rotation number of the Lebesgue measure λ satisfies

$$\rho(\lambda) \stackrel{def.}{=} \int_A \phi d\lambda = 0.$$

Following the usual definition (see [2]), we refer to such mappings as rotationless diffeomorphisms. Every time we say that f is a rotationless diffeomorphism, a special lift $\widetilde{f}$ is fixed and used to define ϕ.

Our objective here is to study a problem posed by P. Boyland, which will be explained below.

Given a rotationless homeomorphism of the annulus f, by a result of Franks (see [6]), if there are two f-invariant Borel probability measures μ_1 and μ_2 with

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$\rho(\mu_1) < \rho(\mu_2)$, then for every rational $\rho(\mu_1) < \frac{p}{q} < \rho(\mu_2)$, there exists a q-periodic orbit for f with this rotation number. So, suppose there exists a measure with positive rotation number. By a classical result (a version of the Conley-Zehnder theorem to the annulus) there must be fixed points with zero rotation number, so Boyland's question is: Is it true that in the above situation there must be orbits with negative rotation number? This is a very difficult problem, which we did not solve in full generality. We considered a generic approach:

Theorem 1.1. *For $r \geq 16$, there exists a residual subset V of*

$$Diff^{r+}_{rot}(A) = \left\{ \begin{array}{c} C^r \text{ rotationless diffeomorphisms of } A, \text{ such that} \\ p_1 \circ \widetilde{f}(\widetilde{x}, i) - \widetilde{x} > 0 \text{ for all } \widetilde{x} \in \mathbb{R} \text{ and } i = 0, 1 \end{array} \right\}, \tag{1.2}$$

such that if $f \in V$, then 0 is contained in the interior of the rotation set of $\widetilde{f}$, $\rho(\widetilde{f}) = \{\omega \in \mathbb{R}\colon \omega = \int_A \phi d\mu$ for some Borel probability f-invariant measure $\mu\}$.

Remarks. (1) Our proof will show that V contains the subset of Moser generic diffeomorphisms of $Diff^{r+}_{rot}(A)$; that is, all periodic points of $f \in V$ not in the boundary of A are either hyperbolic saddles or Moser stable elliptic points. By Moser stable, we mean the usual: $z \in A$ is a Moser stable elliptic periodic point for f (of period n) if z is accumulated by homotopically trivial simple closed f^n-invariant curves, the dynamics of f^n restricted to each of these curves is minimal and the rotation numbers of f^n on these curves are not constant in any neighborhood of z. Moreover, there are no saddle connections between invariant manifolds of hyperbolic periodic saddles in $interior(A)$; and if $z \in interior(A)$ is a hyperbolic periodic saddle, then any two branches of z, one stable and one unstable, have non-empty intersection.

(2) There are two main restrictions in our result, namely:
   (a) The rotation number in the boundaries must be positive. It is much harder (at least following our approach to the problem) to consider the case when some boundary (or both) has a fixed point for $\widetilde{f}$. We are considering this case in an ongoing work.
   (b) Our proof holds only for a residual subset of $Diff^{r+}_{rot}(A)$, and we need $r \geq 16$ in order to generically have Moser stable elliptic periodic points; see subsection 2.3.

2. Basic tools

2.1. **The set B^-.** In this subsection we define the set B^-, introduced in [1], that will play an important role in the proof of our theorem. Although much of what is done in this subsection can be found in [1], for completeness sake we present all results needed with proofs.

To this purpose, we will sometimes make use of the left and right compactification of $\widetilde{A} = \mathbb{R} \times [0, 1]$, denoted L, R-compactification; that is, we compactify the infinite strip adding two points, L (left end) and R (right end), getting a closed disk, denoted $\widehat{A}$. Clearly $\widetilde{f}$ induces a homeomorphism $\widehat{f} : \widehat{A} \to \widehat{A}$ such that $\widehat{f}(L) = L$ and $\widehat{f}(R) = R$.

Given a real number a, let

$$V_a = \{a\} \times [0, 1],$$
$$V_a^- =]-\infty, a] \times [0, 1] \text{ and } V_a^+ = [a, +\infty[\times[0, 1].$$

Denote the corresponding sets on $\widehat{A}$ by $\widehat{V}_a$, $\widehat{V}_a^-$ and $\widehat{V}_a^+$. We will also denote the sets V_0, V_0^- and V_0^+ simply by V, V^- and V^+ respectively.

If we consider the closed set

$$\widehat{B} = \bigcap_{n\leq 0} \widehat{f}^n(\widehat{V}^-),$$

we get that $\widehat{f}(\widehat{B}) \subset \widehat{B}$ and $L \in \widehat{B}$. Denote by $\widehat{B}^-$ the connected component of $\widehat{B}$ which contains L, and denote by B^- the corresponding set on the strip, that is, $B^- = \widehat{B}^- \backslash \{L\}$.

Lemma 2.1. *If $(f, \widetilde{f})$ is a rotationless homeomorphism, then $\widehat{B}^- \cap \widehat{V} \neq \emptyset$ (equivalently for the strip: $B^- \cap V \neq \emptyset$).*

Proof. See [3], [9] and even Birkhoff's paper [4]. □

So, we know that $B^- \subset \widetilde{A}$ is a closed non-empty set, limited to the right ($B^- \subset V^-$), whose connected components (which may be unique) are all unbounded to the left, and at least one connected component of B^- intersects V.

An important point here is that, as the rotation numbers in the boundaries of the annulus are both positive, B and thus B^- do not intersect $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ (because $\widetilde{f}(B) \subset B \subset V^-$).

2.2. **The limit set of B^-.** Here we examine some properties of the set $\omega(\widehat{B}^-) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{i=n}^{\infty} \widehat{f}^i(\widehat{B}^-)}$, a subset of $\widehat{A}$, and the corresponding set $\omega(B^-) \subset \widetilde{A}$.

Since $\widehat{f}(\widehat{B}^-) \subset \widehat{B}^-$, and since $\widehat{B}^-$ is closed, we have

$$\omega(\widehat{B}^-) = \bigcap_{n=0}^{\infty} \widehat{f}^n(\widehat{B}^-);$$

therefore $\omega(\widehat{B}^-)$ is the intersection of a nested sequence of compact connected sets, and so it is also a compact connected set. Moreover, the following lemma holds:

Lemma 2.2. *If $\omega(B^-)$ is not empty, then it is a closed, $\widetilde{f}$-invariant set whose connected components are all unbounded.*

Proof. Since $L \in \widehat{B}^-$ and $\widehat{f}(L) = L$, we get that $L \in \omega(\widehat{B}^-)$. This implies, since $\omega(\widehat{B}^-)$ is connected, that each connected component of $\omega(B^-)$ is unbounded. The other properties follow directly from the previous considerations. □

Of course, since B^- is closed and positively invariant, we also have that $\omega(B^-) \subset B^-$, and as such, $\omega(B^-) \cap \mathbb{R} \times \{i\} = \emptyset$, $i \in \{0, 1\}$, and $\omega(B^-) \subset V^-$. Note that it is still possible that $\omega(B^-) = \emptyset$, and later (see Lemma 3.1) it will be shown that if this is the case, then Theorem 1.1 is proved.

2.3. **Generic properties of diffeomorphisms of the annulus.** In this subsection we state a result which describes the set V of Moser generic elements of

$$Diff^r_{rot}(A) = \{C^r \text{ rotationless diffeomorphisms of } A\},$$

which appears in the statement of Theorem 1.1.

Theorem 2.3. *For all $r \geq 16$, the subset of Moser generic diffeomorphisms $V \subset Diff^r_{rot}(A)$ (see Remark 1, right after the statement of Theorem 1.1) is residual.*

Proof. If we had no restriction on the rotation number of the Lebesgue measure, this result would be standard: the Kupka-Smale theorem + a result by Douady on genericity of Moser stable elliptic points (this is the part where $r \geq 16$ is needed) + theorems due to Pixton [13], Oliveira [12] and Robinson [16]. See for instance Theorem 6.3 of [8].

To see that the residual subset we want exists, proceed as follows:

Given $g \in Diff^r_{rot}(A)$, if we follow the main ideas in the proof of the Kupka-Smale theorem, as in chapter 10 of [14], the following steps appear naturally:

1) We must perturb g so that all periodic points for the perturbed mapping are non-elementary; that is, 1 is not allowed as an eigenvalue at a periodic point. This is achieved by a series of perturbations which rely on the transversality theorem of Thom (for a parametric version, see Theorem 2.3 of chapter 10 of [14]), each supported in a small disk of A. After this step, we end with a rotationless diffeomorphism g_1 arbitrarily C^r-close to g. We do not lose the rotationless property because it is preserved by area preserving perturbations supported on disks of A.

2) Here, g_1 must be perturbed so that all its periodic points become either hyperbolic or elliptic, with no root of the unity up to order 5 as an eigenvalue, and there shall not be any saddle connections. As seen in Robinson's book [14] and in [15], this is also achieved by a series of local perturbations (again supported on small disks of A); that is, the perturbed mapping g_2 still belongs to $Diff^r_{rot}(A)$.

3) Here, g_2 must be perturbed so that all its elliptic periodic points become Moser stable. As seen in [5], this is also achieved by a series of small local perturbations supported on disks of A. As above, the perturbed mapping g_3 still belongs to $Diff^r_{rot}(A)$.

4) Finally g_3 must be perturbed so that for any hyperbolic periodic saddle, any two branches of it, one stable and one unstable, have non-empty intersection. As seen in [12] this can be achieved by a series of perturbations again supported on disks, which implies, as in all the previous steps, that we do not lose the rotationless property.

So after these four steps, we end with an element g_4 of $Diff^r_{rot}(A)$. An important remark here is that Robinson's proof in [14] is not for the conservative world, but the main ideas in this case are the same; the major difference is that the types of local perturbations applied in steps 1 and 2 are much more delicate. See [15] for the conservative version of these perturbations. □

2.4. **Some consequences of prime ends theory.** Here we state two theorems we use. The first is contained in Proposition 5.2 of [7], and the second is Corollary 8.2 of [7]:

Theorem 2.4. *Let $f : A \to A$ be an orientation and area preserving homeomorphism of A and let $K \subset interior(A)$ be a compact connected f-invariant set such that $S^1 \times \{0\}$ and $S^1 \times \{1\}$ are in different connected components of K^c. If K has no periodic points, then there exists an irrational α such that for all $z \in K$ and $\widetilde{z} \in p^{-1}(z)$,*

$$\lim_{n\to\infty} \frac{p_1 \circ \widetilde{f}^n(\widetilde{z}) - \widetilde{z}}{n} = \alpha.$$

Theorem 2.5. *If f is a Moser generic diffeomorphism of A (see Theorem 2.3) and $K \subset interior(A)$ is a boundary component of some f-invariant annulus of A, then there are no periodic points in K.*

2.5. **A general result for area preserving homeomorphisms of the annulus.** In this section we prove a lemma that will be useful in the proof of our main result.

Lemma 2.6. *Let f be a homeomorphism of A that preserves the orientation, the boundary components of A and the Lebesgue measure λ, and let $\tilde{f}$ be a lift of f to $\tilde{A}$. Let Ω be an open subset of the strip, $\Omega \subset (-\infty, 0] \times [0,1]$, such that $\Omega \subset \tilde{f}(\Omega)$ and $\lambda(\tilde{f}(\Omega) \setminus \Omega) > 0$. Clearly Ω is unbounded. Then there is an open f-invariant set $E \subset A$ such that*

$$\int_E \phi d\lambda > 0.$$

Proof. Since Ω is open, $p(\Omega)$ is also open, as is $f(p(\Omega))$. Also, since $\Omega \subset \tilde{f}(\Omega)$, $p(\Omega) \subset f(p(\Omega))$. Therefore, the set

$$E = \bigcup_{i=0}^{\infty} f^i(p(\Omega))$$

is an f-invariant open set. But f is measure preserving, and since $\lambda(f(p(\Omega)))$ and $\lambda(p(\Omega))$ are equal, we have $\lambda(f(p(\Omega)) \setminus p(\Omega)) = 0$, and thus $\lambda(E \setminus (p(\Omega)) = 0$.

Let $C = \cup_{i=1}^{\infty}(\Omega - (i,0))$, and let $D = \Omega \setminus C$. From $\Omega \subset \tilde{f}(\Omega)$, it follows that $C \subset \tilde{f}(C)$.

Let (x,y) be a point in $p(\Omega)$, and let $(\tilde{x}, \tilde{y}) \in \Omega \cap p^{-1}(x,y)$. Since Ω is limited to the right, there must exist a positive integer k such that $(\tilde{x}+k, \tilde{y}) \in \Omega$ and, for all $j > k$, $(\tilde{x}+j, \tilde{y}) \notin \Omega$. But then the point $(\tilde{x}+k, \tilde{y})$ must belong to D. This shows that $p(D) = p(\Omega)$ and that

$$\Omega \subset \bigcup_{i=0}^{\infty}(D - (i,0)). \tag{2.1}$$

It also follows that

$$\int_E \phi d\lambda = \int_{p(D)} \phi d\lambda.$$

Also, since $\lambda(f(p(D)) \setminus p(D)) = 0$, we get that

$$\lambda\left[\tilde{f}(D) \backslash (p^{-1}(p(D)))\right] = 0. \tag{2.2}$$

As $D \subset \Omega$, from the definition of C we get that $C \supset \cup_{i=1}^{\infty}(D - (i,0))$. On the other hand, from (2.1) we have $C \subset \cup_{i=1}^{\infty}(D - (i,0))$ and so

$$C = \bigcup_{i=1}^{\infty}(D - (i,0)).$$

From the fact that $C \subset \tilde{f}(C)$ and $C \cap D = \emptyset$, we obtain

$$\tilde{f}(D) \cap (\bigcup_{i=1}^{\infty}(D - (i,0))) = \emptyset. \tag{2.3}$$

This and (2.2) imply that

$$\lambda\left[\tilde{f}(D) \backslash (\bigcup_{i=0}^{\infty}(D + (i,0)))\right] = 0, \text{ since } p^{-1}(p(D)) = \bigcup_{i=-\infty}^{\infty}(D + (i,0)).$$

Now denote for every integer $i \geq 0$,

$$D_i = \tilde{f}(D) \cap (D + (i,0)).$$

From (2.2) and (2.3), we have that $f(p(D)) \cap p(D) = \bigcup_{i=0}^{\infty} p(D_i)$. Note that, since the covering mapping p restricted to D is injective, it is also injective when restricted to $\tilde{f}(D)$, and so $p(D_i) \cap p(D_j)$ is empty if $i \neq j$. Finally, from $\lambda(f(p(D)) \setminus p(D)) = \lambda(f(p(D)) \setminus (\cup_{i=0}^{\infty} p(D_i))) = 0$, we obtain

$$\lambda\left[D \setminus (\bigcup_{i=0}^{\infty}(D_i - (i,0)))\right] = 0. \tag{2.4}$$

We claim that $\lambda(\cup_{i=1}^{\infty} D_i) \neq 0$. If this were not so, we would have $\lambda(\tilde{f}(D) \setminus D) = 0$. Let $F_i = \Omega \cap (D - (i,0))$, and so $\Omega = \cup_{i=0}^{\infty} F_i$; see (2.1). Of course, since the covering mapping p is injective on D, $\lambda(D) \leq 1$, and so $\lambda(F_i) \leq 1$. Now $\tilde{f}(F_i) = \tilde{f}(D - (i,0)) \cap \tilde{f}(\Omega) \supset \tilde{f}(D - (i,0)) \cap \Omega$. This implies that $\lambda(\tilde{f}(F_i) \setminus F_i) = 0$, and so $\lambda(\tilde{f}(\Omega) \setminus \Omega) = 0$, which contradicts our hypothesis.

Now

$$\int_E \phi d\lambda = \int_{p(D)} \phi d\lambda = \int_D p_1(\tilde{f}(x,y)) d\lambda - \int_D x d\lambda,$$

but

$$\int_D p_1(\tilde{f}(x,y)) d\lambda = \int_{\tilde{f}(D)} x d\lambda = \sum_{i=0}^{\infty} \int_{D_i} x d\lambda > \sum_{i=0}^{\infty} \int_{D_i - (i,0)} x d\lambda,$$

where the last equality comes from (2.4) and from $(D_i - (i,0)) \cap (D_j - (j,0)) = \emptyset$ if $i \neq j$, and the strict inequality comes from $\lambda(\cup_{i=1}^{\infty} D_i) > 0$.

Since $\int_D x d\lambda = \sum_{i=0}^{\infty} \int_{D_i - (i,0)} x d\lambda$, we have the result. □

An immediate corollary is

Corollary 2.7. *Let $(f, \tilde{f})$ be a rotationless homeomorphism and let Ω be an open subset of the strip, $\Omega \subset (-\infty, 0] \times [0,1]$, such that $\Omega \subset \tilde{f}(\Omega)$ and $\lambda(\tilde{f}(\Omega) \setminus \Omega) > 0$. Then 0 is an interior point of the rotation set of $\tilde{f}$.*

Proof. From the previous lemma we know that there is an invariant set E in the annulus such that $\int_E \phi d\lambda > 0$. Therefore there is a point in E with a strictly positive rotation number. On the other hand, since $\int_A \phi d\lambda = 0$, we have $\int_{E^C} \phi d\lambda < 0$, and since E^C is invariant, there must be a point in E^C with strictly negative rotation number. □

3. Proof of the main theorem

First, let us suppose that $\omega(B^-) = \emptyset$. In this case, as in [1], we can prove the following:

Lemma 3.1. *There exists an integer $N_1 > 0$ such that $\tilde{f}^{N_1}(B^-) \subset B^- - (1,0)$.*

Proof. There is an integer $N_1 > 0$ such that, for all $n \geq N_1$, $\tilde{f}^n(B^-) \subset V_{-2}^-$. So, $\tilde{f}^n(B^-) + (1,0) \subset V^-$, and as each connected component of $\tilde{f}^n(B^-) + (1,0)$ is unbounded and this set is positively invariant, it must be the case that $\tilde{f}^n(B^-) + (1,0) \subset B^-$. □

As $\tilde{f}^{N_1}(B^-) \subset B^- - (1,0)$, for any positive integer k,

$$\tilde{f}^{kN_1}(B^-) \subset B^- - (k,0) \subset V_{-k}^-,$$

and so it follows that, for any point $\widetilde{z} \in B^-$,

$$\limsup_{n\to\infty} \frac{p_1 \circ \tilde{f}^n(\widetilde{z}) - p_1(\widetilde{z})}{n} \leq -\frac{1}{N_1},$$

and this proves our theorem. So, we can suppose that $\omega(B^-) \neq \emptyset$.

Since the rotation number of $\widetilde{f}$ restricted to $S^1 \times \{0\}$ and $S^1 \times \{1\}$ is strictly positive, there exists $\sigma > 0$ such that $p_1(\widetilde{f}(\widetilde{x}, i)) > \widetilde{x} + 2\sigma$ for all $\widetilde{x} \in \mathbb{R}$ and $i = 0, 1$. Let $\epsilon > 0$ be sufficiently small such that for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R} \times \{[0, \epsilon] \cup [1 - \epsilon, 1]\}$, $p_1 \circ \tilde{f}(\tilde{x}, \widetilde{y}) > \tilde{x} + \sigma$.

Let us first consider the case when $S^1 \times \{[0, \epsilon/2] \cup [1-\epsilon/2, 1]\}$ intersects $\overline{p(\omega(B^-))}$. Then there is a real a such that

$$\omega(B^-) \cap \{a\} \times [0, \epsilon] \neq \emptyset \text{ or } \omega(B^-) \cap \{a\} \times [1 - \epsilon, 1] \neq \emptyset. \tag{3.1}$$

Without loss of generality, we can suppose that the first intersection in expression (3.1) is non-empty. The fact that $\omega(B^-) \subset B^-$ is closed implies that there must be a $\delta \leq \epsilon$ such that $(a, \delta) \in \omega(B^-)$ and such that for all $0 \leq \tilde{y} < \delta, (a, \tilde{y}) \notin \omega(B^-)$ (remember that $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ do not intersect B^-). In other words, (a, δ) is the "lowest" point of $\omega(B^-)$ in $\{a\} \times [0, \epsilon]$. We denote by v the interval $\{a\} \times [0, \delta[$.

Let Ω be the connected component of $(\omega(B^-) \cup v)^c$ that contains $]-\infty, a[\times\{0\}$. In this case, our main theorem follows from Corollary 2.7 and the next proposition:

Proposition 3.2. *The following holds:* $\Omega \subset \tilde{f}(\Omega)$ *and* $\lambda(\tilde{f}(\Omega) \setminus \Omega) > 0$.

Proof. First, note that the boundary of $\tilde{f}(\Omega)$ is contained in $\omega(B^-) \cup \widetilde{f}(v)$. We claim that $\partial\tilde{f}(\Omega) \cap \Omega = \emptyset$. This follows from the two conditions below:

1) By the choice of $\epsilon > 0, \widetilde{f}(v) \cap v = \emptyset$.

2) As $\omega(B^-)$ is $\widetilde{f}$-invariant, $\omega(B^-) \cap \widetilde{f}(v) = \emptyset$.

So, as $\Omega \cap \tilde{f}(\Omega) \neq \emptyset$, we get that $\Omega \subset \tilde{f}(\Omega)$. In order to see that $\lambda(\tilde{f}(\Omega) \setminus \Omega) > 0$, we note that for sufficiently small $\xi > 0$, $B_\xi(\widetilde{f}(a, 0)) \cap \widetilde{f}(\Omega)$ is non-empty and contained in $closure(\Omega)^c$. □

Let us deal with the remaining case, when $S^1 \times \{[0, \epsilon/2] \cup [1-\epsilon/2, 1]\} \cap \overline{p(\omega(B^-))} = \emptyset$, but $\omega(B^-)$ is not empty. In this case, let A_* be the connected component of $\left(\overline{p(\omega(B^-))}\right)^c$ which contains $S^1 \times \{0\}$.

Lemma 3.3. *The set* A_* *is an* f*-invariant open sub-annulus.*

Proof. This follows from the fact that each connected component of the complement of a compact connected subset of a sphere is an open disk; see for instance [11]. □

The boundary of A_* has two connected components: one is $S^1 \times \{0\}$ and the other is denoted by K. Clearly, $K \subset S^1 \times [\epsilon/2, 1 - \epsilon/2]$, and as A_* is an f-invariant annulus, we can compute the rotation number of the Lebesgue measure restricted to A_*. If it is non-zero, then the proof is over. So, suppose it is zero. First, note that Theorems 2.5 and 2.4 imply that there exists an irrational α such that for all $z \in K$ and $\widetilde{z} \in p^{-1}(z)$,

$$\lim_{n\to\infty} \frac{p_1 \circ \widetilde{f}^n(\widetilde{z}) - \widetilde{z}}{n} = \alpha. \tag{3.2}$$

If $\alpha < 0$, then the proof is over. If it is positive, then using prime end theory (see for instance Proposition 5.2 of [7]), we know that there exists a homeomorphism g of

the open annulus $S^1\times]0,1[$ which is conjugate to $f\mid_{A_*}$ (by an orientation preserving conformal homeomorphism), preserves a measure equivalent to Lebesgue and has a rotationless lift $\widetilde{g}$. Moreover, there is a rotationless homeomorphism of the closed annulus which extends $(g,\widetilde{g})$, denoted $(g',\widetilde{g}')$, such that:

$$\begin{aligned}\rho(\widetilde{g}')\mid_{S^1\times\{0\}}&=\rho(\widetilde{f})\mid_{S^1\times\{0\}}>0,\\ \rho(\widetilde{g}')\mid_{S^1\times\{1\}}&=\rho(\widetilde{f})\mid_{K}=\alpha>0.\end{aligned}\tag{3.3}$$

To conclude our proof, let us consider the set $(B^-)'$ for $\widetilde{g}'$. It is not empty, does not contain any point of $\mathbb{R}\times\{0,1\}$ (because of the two conditions in (3.3)) and may or may not have an ω-limit. If $\omega((B^-)')=\emptyset$, then Lemma 3.1 and the argument right after it imply that g has points with negative rotation number. This finishes the proof, because g is conjugate to $f\mid_{A_*}$. If $\omega((B^-)')\neq\emptyset$, then we get that $p(\omega(B^-))$ intersects A_*, again because $g'\mid_{S^1\times]0,1[}$ is conjugate to $f\mid_{A_*}$. This contradicts the definition of A_* and proves our main theorem. □

The proof of the main theorem can be adapted to obtain an interesting byproduct. If A is a region of instability for f, i.e., A has no f-invariant proper sub-annulus, then Lemma 3.3 implies that it is not possible that

$$S^1\times\{[0,\epsilon/2]\cup[1-\epsilon/2,1]\}\cap\overline{p(\omega(B^-))}=\emptyset,$$
$$\text{but } \omega(B^-) \text{ is not empty.}$$

This was the only case where we needed the genericity hypothesis. Therefore, the following result is true:

Theorem 3.4. *If A is a region of instability for $f\in Hom^+_{rot}(A)=\{$rotationless homeomorphisms of A such that $p_1\circ\widetilde{f}(\widetilde{x},i)-\widetilde{x}>0$ for all $\widetilde{x}\in\mathbb{R}$ and $i=0,1\}$, then 0 is an interior point of $\rho(\widetilde{f})$.*

This result is a particular case of the main theorem in [17], where we replace the hypothesis of f being rotationless with: For every $M<0$ there exist a point $\tilde{z}$ in $[0,\infty[\times[0,1]$ and a positive integer n such that $\tilde{f}^n(\tilde{z})\in]-\infty,M]\times[0,1]$.

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