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ON EMBEDDING THE INFINITE CYCLIC COVERINGS OF KNOT COMPLEMENTS INTO THREE SPHERE

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Abstract. We construct a class of knots with the CI* property, that is, $\pi_1(M(n) \mid \partial M(n)) \neq \{e\}$ for some $n > 0$. It follows that the infinite cyclic covering of such a knot cannot be embedded in any compact 3-manifold.

1. Introduction

A (tame) knot is a smooth (or PL) embedding $f : S^1 \to S^3$. Let $E(k) = S^3 - N(k)$ denote the compact complement of a knot k. Let $\widetilde{E}(k)$ denote the infinite cyclic covering of $E(k)$ [4]. A natural question is:

Question

When does $\widetilde{E}(k)$ embed in S^3?

Professor Robert D. Edwards attributes this problem to J. Stallings. In [2], Jiang, Ni, Wang and Zhou studied the above problem. They gave a necessary condition for the existence of such an embedding when the genus equals one. Later, C. McA. Gordon [1] showed that the infinite cyclic cover of the exterior of the untwisted Whitehead double of a non-trivial knot does not embed in any compact 3-manifold. In this paper, we describe a class of knots whose infinite cyclic covers do not embed in any compact 3-manifold. We approach the question from a group-theoretical point of view and construct a class of knots with the CI* property (section 2). The infinite cyclic covering of a CI* knot does not embed in any compact 3-manifold. On the other hand, 9_{46} is a non-fibered, non-CI* knot by [2].

2. The main example

Let k be a knot in S^3. A Seifert surface of k is a once punctured orientable surface of genus g, which we regard as a disk with $2g$ bands glued to it. Figure 1 is a punctured torus. Figure 2 is another way to denote it. We use lines to indicate the bands. If there are some twists in a band, we will indicate the twists in the text.

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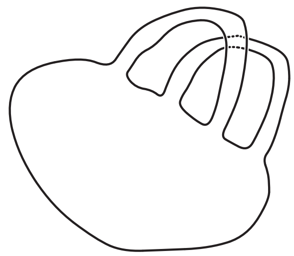

FIGURE 1. A punctured torus

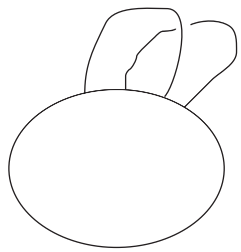

FIGURE 2. Another picture of the punctured torus

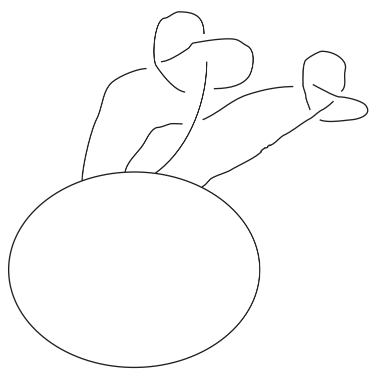

FIGURE 3. The main example

In this paper, from now on, we do not draw those bands; we just use the second type figures. Figure 3 is our primary example. Here, the core of each band forms a part of a trefoil knot, and along each band there are three positive full twists. The picture shows a genus 1 Seifert surface F with boundary a knot k. We orient the surface F in such a way that the up side of the disk is positive. Later, we shall tell how this can be generalized.

Lemma 2.1. *The fundamental group G of $S^3 - F$ is a free product $G = H * H$, where H is the fundamental group of the trefoil knot complement.*

Proof. If we regard the disk as a 3-ball, then the space $S^3 - F$ can be deformed into a union of two trefoil knot complements attached along a 2 dimensional disk. $\square$

Remark 2.2. H has the presentation: $\langle x_1, x_2, x_3 \mid x_1 x_2 = x_2 x_3 = x_3 x_1 \rangle$.

Definition 2.3. Let N be a manifold with boundary, and let A be a subset of N. We use $\pi_1(N \mid A)$ to denote the group $\pi_1(N)/\langle i_*(\pi_1(A))\rangle^n$, where $\langle i_*(\pi(A))\rangle^n$ denotes the normal closure of the group $i_*(\pi(A))$ in $\pi_1(N)$, and $i : A \to N$ is the inclusion map.

Let F be a Seifert surface of a knot k. If we cut $E(k)$ along F, we will get a 3-manifold M with a connected boundary which is a union of two copies F^+, F^- of F and an annulus A. The infinite cyclic covering $\widetilde{E}(k)$ can be constructed by taking a countably infinite collection of copies of M and identifying F^+ in the ith copy with F^- in the $(i+1)$st copy for each i. Let $M(n)$ be the submanifold of $\widetilde{E}(k)$ consisting of n consecutive copies of M. Thus the boundary of $M(n)$ is also a union of two copies $F^+(n), F^-(n)$ of F and an annulus A.

Lemma 2.4. *For the knot k illustrated above, the image of $i_* : \pi_1(F^-) \to \pi_1(M \mid F^+)$ is a rank two free abelian group.*

Proof. The fundamental group G of M is a free product $H * H$, and H has the presentation: $\langle x_1, x_2, x_3 \mid x_1x_2 = x_2x_3 = x_3x_1\rangle$. We denote the two copies of H by H_1, H_2. Hence G has the presentation:

$$H_1 * H_2 = \langle x_1, x_2, x_3, y_1, y_2, y_3 \mid x_1x_2 = x_2x_3 = x_3x_1, y_1y_2 = y_2y_3 = y_3y_1\rangle.$$

The group $i_*(\pi_1(F_1^+))$ is normally generated by a_1^+, a_2^+ which are represented by the simple closed curves that are the cores of two bands. The group $i_*(\pi_1(F_1^-))$ is generated by the same curves a_1^-, a_2^- pushed to the negative side of the Seifert surface. We can check the following relations:

$a_1^+y_1^{-1} = a_1^-$, $a_2^-x_1^{-1} = a_2^+$.

Since we only work on the normal closure of $i_*(\pi_1(F_1^+))$, then, up to some base point change, we can take $a_1^+ = x_2x_1x_3 = x_2x_1x_2^{-1}x_1x_2x_1^{-3}$, $a_2^+ = y_2y_1y_2^{-1}y_1y_2y_1^{-3}x_1^{-1}$. Let $\alpha : H_1 \to Z$ denote the abelianization map, then $\alpha(x_i) = 1$, and $\alpha(a_1^+) = 0$. Hence in the group $H_1/\langle a_1^+\rangle^n$, each x_i is of infinite order. $\pi_1(M \mid F^+) = G/\langle a_1^+, a_2^+\rangle^n$ can be rewritten as a free product with amalgamation

$$(H_1/\langle a_1^+\rangle^n) *_{x_1 = y_2y_1y_2^{-1}y_1y_2y_1^{-3}} H_2.$$

The image $i_*(\pi_1(F_1^-))$ is generated by $\{a_1^-, a_2^-\} \sim \{y_1^{-1}, x_1\} \sim \{x_1, y_1\} \sim \{y_2y_1y_2^{-1}y_1y_2y_1^{-3}, y_1\}$. This is the meridian and longitude pair in a trefoil knot group H_2. Hence the subgroup generated by $\{y_2y_1y_2^{-1}y_1y_2y_1^{-3}, y_1\}$ in H_2 is a rank two free abelian group.

By the theorems on group amalgamation [3], H_2 embeds in $\pi_1(M \mid F^+)$; hence the image of $i_* : \pi_1(F^-) \to \pi_1(M \mid F^+)$ is a rank two free abelian group. □

Remark 2.5. Likewise, the image of $i_* : \pi_1(F^+) \to \pi_1(M \mid F^-)$ is a rank two free abelian group.

Lemma 2.6. $\pi_1(M(2) \mid \partial M(2)) \neq \{e\}$.

Proof. $M(2)$ is a union of 2 copies of M, say M_1 and M_2, glued along F_1^+ and F_2^-. By van Kampen's theorem, and by the previous lemma, $\pi_1(M(2) \mid \partial M(2))$ is a free product of $\pi_1(M \mid F^-)$ and $\pi_1(M \mid F^+)$ amalgamated along a rank two free abelian group. Hence it is non-trivial. □

Proposition 2.7. (1) *$M(2)$ does not embed in S^3.*

(2) *For any compact 3-manifold N, if n is sufficiently large, $M(2n)$ does not embed in N.*

Proof. (1) If $M(2)$ embeds in S^3, let A denote $S^3 - M(2)$. Then, by van Kampen's theorem, $\{e\} = \pi_1(S^3 \mid A) = \pi_1(M(2) \mid \partial M(2))$. This contradicts Lemma 3.

(2) There is a surjective homomorphism from $\pi_1(M(2n) \mid \partial M(2n))$ onto $*_n(\pi_1(M(2) \mid \partial M(2)))$. For any compact 3-manifold N, if $M(2n)$ embeds into N, then by van Kampen's theorem, there is a surjective homomorphism from $\pi_1(N)$ onto $\pi_1(N \mid N - M(2n)) \cong \pi_1(M(2n) \mid \partial M(2n))$, and hence onto $*_n(\pi_1(M(2) \mid \partial M(2)))$. For a fixed N, n can't be arbitrarily large. This is a contradiction. $\square$

Definition 2.8. 1. We say that a knot has property CI*, or that it is a CI* knot, if $\pi_1(M(n) \mid \partial M(n)) \neq \{e\}$ for some $n > 0$.

2. We say that a knot has property CI, or that it is a CI knot, if the commutator subgroup of the knot group has K^∞ (the free product of countable many copies of some group $K \neq \{e\}$) as a quotient.

Denote the homomorphism as $p_1 : G'(k) \to K^\infty$, where $G'(k)$ is the commutator subgroup of the knot group.

Theorem 2.9. (1) *There exist CI* knots.*

(2) *The infinite cyclic covering of a CI* knot does not embed in any compact 3-manifolds.*

Proposition 2.10. 1. *The CI* property does not depend on the choice of the Seifert surface.*

2. *A CI* knot k is clearly a CI knot.*

3. *If we assume the following hypothesis, then any CI knot is a CI* knot too.*

Hypothesis. Let g be a positive integer, and let K^l be the free product of l copies of any non-trivial group K. Then if l is big enough, any quotient group of K^l by adding g relations is a non-trivial group.

Proof of Proposition 2.10(1). Choose any two Seifert surfaces F, F' for a knot k and construct the two spaces $M(n)$ and $M'(m)$. For any $m > 0$, if n is big enough, then $M'(m)$ embeds in $M(n)$. It follows that if $\pi_1(M'(m) \mid \partial M'(m)) \neq \{e\}$, then $\pi_1(M(n) \mid \partial M(n)) \neq \{e\}$. $\square$

Proof of Proposition 2.10(3). Suppose k is a CI knot. We have the following commutative diagram:

$$\begin{tikzcd}
\pi_1(M(n)) \arrow[d, "i"] \arrow[drr, "f"] & & \\
G'(k) \arrow[r, "p_1"] & K^\infty \arrow[r, "p_2"] & K^l
\end{tikzcd}$$

Here, $G'(k) \cong \pi_1(\widetilde{E}(k))$, i is induced by the inclusion map, p_2 is a projection to finitely many factors, f is the composition of those homomorphisms. p_1 and p_2 are surjective, and for any l, if we choose n big enough, f will be surjective too.

$\partial M(n)$ is a surface of fixed genus, say g. If $\pi_1(M(n) \mid \partial M(n)) = \{e\}$ for any $n > 0$, then $\pi_1(M(n))$ is normally generated by $2g$ elements. This contradicts our hypothesis on groups if l is big enough. $\square$

3. Discussion and generalizations

The reader might already have noticed that our example can be extensively generalized. First of all, the core of the two bands can be other knots, and the twisting of the bands can vary too. Secondly, one can add more bands to construct higher genus knots.

On the other hand, fibered knots are clearly not CI*. Their infinite cyclic coverings embed in S^3. There are other knots, like 9_{46} [2], which are non-fibered but whose infinite cyclic coverings embed in S^3. Those knots are called IE knots (infinite cyclic covering embeds in S^3). Therefore

$$\{\text{fibered knots}\} \subset \{\text{IE knots}\} \subset \{\text{non-CI}^* \text{ knots}\}.$$

If k is a non-CI* knot, then $\pi_1(M(n) \mid \partial M(n)) = \{e\}$ for any $n > 0$. This suggests that $M(n)$ most likely embeds in S^3. We propose the following conjectures.

Conjecture. 1. *Non-CI* knots are IE knots.*

2. *Property CI and property CI* are equivalent.*

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