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THE HOMOTOPY GROUPS OF $L_2T(1)/(v_1)$ AT AN ODD PRIME

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Abstract. In this paper, all spectra are localized at an odd prime. Let $T(1)$ be the Ravenel spectrum characterized by BP_*-homology as $BP_*[t_1]$, $T(1)/(v_1)$ be the cofiber of the self-map $v_1 : \Sigma^{2p-2}T(1) \to T(1)$ and L_2 denote the Bousfield localization functor with respect to $v_2^{-1}BP_*$. In this paper, we determine the homotopy groups of $L_2T(1)/(v_1)$.

1. Introduction

Let $\mathcal{S}_{(p)}$ denote the stable homotopy category of spectra localized at the prime number p, and let BP denote the Brown-Peterson spectrum characterized by the coefficient ring $BP_* = \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$. Then (BP_*, BP_*BP) is a Hopf algebroid, where $BP_*BP = BP_*[t_1, t_2, \cdots]$. One has the Bousfield localization functor $L_n : \mathcal{S}_{(p)} \to \mathcal{S}_{(p)}$ with respect to $v_n^{-1}BP$, and one denotes the image of it as $\mathcal{L}_n$. The homotopy groups $\pi_*(L_nS^0)$ of the sphere spectrum S^0 play an important role in understanding $\mathcal{L}_n$. The main tool to determine them is the Adams-Novikov spectral sequence. For $n \leqslant 2$, the Adams-Novikov E_2-terms for $\pi_*(L_nS^0)$ were determined in [8], [13], [14], [15] and the homotopy groups of L_nS^0 are also determined if $n \leqslant 2$, except for the case that $n = 2$ and $p = 2$.

Ravenel [7] constructed a spectrum $T(m)$ for each integer $m \geqslant 0$, whose BP_*-homology is the subcomodule algebra $BP_*[t_1, \cdots, t_m]$ of BP_*BP. There is a tower of spectra:

$$S^0_{(p)} = T(0) \hookrightarrow T(1) \hookrightarrow T(2) \hookrightarrow \cdots \hookrightarrow T(m) \hookrightarrow \cdots \hookrightarrow T(\infty) = BP.$$

Using the infinite descent method of [11], one could get information of $\pi_*(L_nS^0)$ from $\pi_*(L_nT(m))$.

Consider the Adams-Novikov spectral sequence for $\pi_*(T(m))$:

$$Ext^{s,t}_{BP_*BP}(BP_*, BP_*(T(m))) \Longrightarrow \pi_{t-s}(T(m)).$$

It is known that for $i \leqslant m$, the v_i's are permanent cycles (cf. [7], 6.5.9). Since $T(m)$ is a ring spectrum, the homotopy element $v_i \in \pi_*(T(m))$ extends to the self-map

$$v_i : \Sigma^{2(p^i-1)}T(m) = S^{2(p^i-1)} \wedge T(m) \xrightarrow{v_i \wedge 1} T(m) \wedge T(m) \xrightarrow{\mu} T(m).$$

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In this paper, we study the homotopy groups of $L_2T(1)$. Let $T(1)/(v_1)$ be the cofiber of $v_1 : \Sigma^{2p-2}T(1) \longrightarrow T(1)$, let $T(1)/(p)$ be the cofiber of $p : T(1) \to T(1)$ and let $T(1)/(p, v_1)$ be the cofiber of $v_1 : \Sigma^{2p-2}T(1)/(p) \to T(1)/(p)$, which is also the cofiber of $p : T(1)/(v_1) \to T(1)/(v_1)$. Consider the localization map $L_n : X \to L_nX$, let $T(1)/(p^\infty, v_1)$ denote the cofiber of $L_0 : T(1)/(v_1) \to L_0T(1)/(v_1)$, let $T(1)/(p, v_1^\infty)$ denote the cofiber of $L_1 : T(1)/(p) \to L_1T(1)/(p)$ and let $T(1)/(p^\infty, v_1^\infty)$ denote the cofiber of $L_1 : T(1)/(p^\infty) \to L_1T(1)/(p^\infty)$, which is the cofiber of $L_0 : T(1)/(v_1^\infty) \to L_0T(1)/(v_1^\infty)$ too. Then by the 3×3 theorem we have the following cofiber sequences:

$$L_2T(1)/(p^\infty, v_1) \xrightarrow{1/v_1} L_2T(1)/(p^\infty, v_1^\infty) \xrightarrow{v_1} L_2T(1)/(p^\infty, v_1^\infty),$$

$$L_2T(1)/(p, v_1^\infty) \xrightarrow{1/p} L_2T(1)/(p^\infty, v_1^\infty) \xrightarrow{p} L_2T(1)/(p^\infty, v_1^\infty).$$

These give two ways to work out the homotopy groups of $L_2T(1)$, both starting from $L_2T(1)/(p, v_1)$:

(1) $\pi_*(L_2T(1)/(p, v_1)) \Rightarrow \pi_*(L_2T(1)/(p))$, then $\Rightarrow \pi_*(L_2T(1))$.
(2) $\pi_*(L_2T(1)/(p, v_1)) \Rightarrow \pi_*(L_2T(1)/(v_1)$, then $\Rightarrow \pi_*(L_2T(1))$.

At the prime 2, Shimomura determined the homotopy groups $\pi_*(L_2T(1)/(2))$ in [12]. In [6] and [16], the second author separately with Nakai and Shimomura determined the homotopy groups $\pi_*(L_2T(1)/(v_1))$ and proved that the Adams-Novikov spectral sequence for $\pi_*(L_2T(1))$ has a horizontal vanishing line at the E_4-terms. But it seems to be too difficult to work out $\pi_*(L_2T(1))$ from both ways.

In this paper, we will determine the homotopy groups $\pi_*(L_2T(1)/(v_1))$ at odd primes p (Theorem 2.3). This gives a way to approach the homotopy groups of $L_2T(1)$ at odd primes.

2. The Adams spectral sequence for $\pi_*(L_2T(1)/(v_1))$

Let BP denote the Brown-Peterson ring spectrum characterized by

$$\pi_*(BP) = BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n, \cdots].$$

Here the v_i's are the Hazewinkel generators, and they are assigned the degree $2(p^i - 1)$. Then (BP_*, BP_*BP) becomes a Hopf algebroid, where $BP_*BP = BP_*[t_1, \cdots, t_n, \cdots]$ with $|t_n| = 2(p^n - 1)$. For any spectrum X, we have the Adams-Novikov spectral sequence

$$E_2^{s,t} = Ext^{s,t}_{BP_*BP}(BP_*, BP_*(X)) \Rightarrow \pi_{t-s}(X).$$

Let $E(2)$ be the Johnson-Wilson ring spectrum, which yields the Hopf algebroid

$$(E(2)_*, E(2)_*E(2)) = (\mathbb{Z}_{(p)}[v_1, v_2^{\pm 1}], E(2)_*[t_1, t_2, \cdots] \otimes_{BP_*} E(2)_*).$$

The Thom map $f : BP \longrightarrow E(2)$ induces $f_* : BP_* \to E(2)_*$, which satisfies $f_*(v_i) = v_i$ for $i \leqslant 2$ and 0 for $i > 2$. Then by the change of rings theorem, we see that for an I_2-nil BP_*BP-comodule M,

$$Ext^{s,t}_{BP_*BP}(BP_*, v_2^{-1}M) = Ext^{s,t}_{E(2)_*E(2)}(E(2)_*, M \otimes_{BP_*} E(2)_*).$$

Let M be an I_2-nil BP_*BP-comodule and $M[t_1]$ be the polynomial ring over M. To compute the Ext groups $Ext^{s,t}_{BP_*BP}(BP_*, v_2^{-1}M[t_1])$, consider the BP_*-module $E_1(2)_* = E(2)_*[v_3] = \mathbb{Z}_{(p)}[v_1, v_2^{\pm 1}, v_3]$ with action given by sending v_i to

0 for $i > 3$. Then $E_1(2)_*(X) = E_1(2)_* \otimes_{BP_*} BP_*X$ is a homology theory by the Landweber exact functor theorem, and the Hopf algebroid structure

$$(E_1(2)_*, E_1(2)_*E_1(2)) = (E_1(2)_*, E_1(2)_*[t_1, t_2, \cdots] \otimes_{BP_*} E_1(2)_*)$$

is induced from (BP_*, BP_*BP). By the change of rings theorem we have

$$Ext^{s,t}_{BP_*BP}(BP_*, v_2^{-1}M[t_1]) = Ext^{s,t}_{E_1(2)_*E_1(2)}(E_1(2)_*, E_1(2)_* \otimes_{BP_*} M[t_1])$$

(cf. [2]).

Let

$$\Gamma(2,2) = E_1(2)_*E_1(2)/(t_1) = E_1(2)_*[t_2, t_3, \cdots] \otimes_{BP_*} E_1(2)_*$$

and the Hopf algebroid structure of $(E_1(2)_*, \Gamma(2,2))$ be the one induced from that of (BP_*, BP_*BP). Noting that

$$E_1(2)_* \otimes_{BP_*} M[t_1] = E_1(2)_*E_1(2)\Box_{\Gamma(2,2)}E_1(2)_* \otimes_{BP_*} M,$$

we have the change of rings theorem

$$Ext^{s,t}_{E_1(2)_*E_1(2)}(E_1(2)_*, E_1(2)_* \otimes_{BP_*} M[t_1]) = Ext^{s,t}_{\Gamma(2,2)}(E_1(2)_*, E_1(2)_* \otimes_{BP_*} M).$$

Let $M_2^0 = E_1(2)_*/(p, v_1)$, $L_1^1 = E_1(2)_*/(p^\infty, v_1)$ and $Ext^{s,t}_{\Gamma(2,2)}(E_1(2)_*, M) = H^{s,t}(M)$ for short. We have the short exact sequence of $\Gamma(2,2)$-comodules

$$0 \to M_2^0 \xrightarrow{1/p} L_1^1 \xrightarrow{p} L_1^1 \to 0.$$

This short exact sequence induces the following long exact sequence:

$$\cdots \to H^{s,*}M_2^0 \xrightarrow{1/p} H^{s,*}L_1^1 \xrightarrow{p} H^{s,*}L_1^1 \xrightarrow{\delta} H^{s+1,*}M_2^0 \to \cdots.$$

To compute $H^{*,*}M_2^0$, let $K_1(2)_* = E_1(2)_*/(p, v_1) = \mathbb{Z}/p[v_2^{\pm 1}, v_3]$ and

$$\Sigma(2,2) = K_1(2)_* \otimes_{BP_*} \Gamma(2,2) \otimes_{BP_*} K_1(2)_* = K_1(2)[t_2, t_3, \cdots]/(t_i^{p^2} - t_i),$$

$$S(2,2) = \mathbb{Z}/p \otimes_{K_1(2)_*} \Gamma(2,2) \otimes_{K_1(2)_*} \mathbb{Z}/p = \mathbb{Z}/p[t_2, t_3, \cdots]/(t_i^{p^2} - t_i),$$

where $K_1(2)$ acts on $\mathbb{Z}/p$ by sending $v_2^{\pm 1}$ and v_3 to 1. Recalling from [7], 6.3.7, that the Adams-Novikov E_2-term for $\pi_*(L_2T(1)/(p, v_1))$ is $H^{*,*}M_2^0$, we have

$$\begin{aligned}
&Ext_{E_1(2)_*E_1(2)}(E_1(2)_*, E_1(2)_*T(1)/(p, v_1)) \\
\cong &Ext_{\Gamma(2,2)}(E_1(2)_*, E_1(2)_*/(p, v_1)) = H^{*,*}M_2^0 \\
\cong &Ext_{\Sigma(2,2)}(K_1(2)_*, K_1(2)_*) \\
\cong &\mathbb{Z}/p[v_2^{\pm 1}, v_3] \otimes Ext_{S(2,2)}(\mathbb{Z}/p, \mathbb{Z}/p) \\
\cong &\mathbb{Z}/p[v_2^{\pm 1}, v_3] \otimes E[h_2^0, h_2^1, h_3^0, h_3^1],
\end{aligned}$$

where the h_i^j denotes the cohomology class represented by $t_i^{p^j}$. Let $\widetilde{h_2^1} = v_2^{-p}h_2^1 + v_2^{-1}h_2^0$ and $\widetilde{h_3^1} = h_3^1 + v_2^{-p^4}v_3^{p^3}h_2^1 - v_3^p v_2^{-1}h_2^0$. It is easy to see that

$$H^{*,*}M_2^0 = \mathbb{Z}/p[v_2^{\pm 1}, v_3] \otimes E[h_2^0, h_3^0, \widetilde{h_2^1}, \widetilde{h_3^1}].$$

Noticing that the first nontrivial Adams-Novikov differential appears at d_{2p-1} and

$$Ext^{s,*}_{BP_*BP}(BP_*, BP_*(L_2T(1)/(p, v_1))) = 0$$

for $s > 4$, we see that the Adams-Novikov spectral sequence for $\pi_*(L_2T(1)/(p, v_1))$ collapses and $H^{*,*}M_2^0 \cong \pi_*(L_2T(1)/(p, v_1))$.

In this paper we will compute the Adams-Novikov E_2-terms for $\pi_*(L_2T(1)/(p^\infty, v_1))$ by the p-Bockstein spectral sequence (the E_2-terms collapse also). Then by the long exact sequence

$$\cdots \to \pi_n(L_2T(1)/(v_1)) \to \pi_n(L_0T(1)/(v_1)) \to \pi_n(L_2T(1)/(p^\infty, v_1)) \to \cdots,$$

we could determine the homotopy groups $\pi_*(L_2T(1)/(v_1))$.

To state our results, we decompose the module $H^{*,*}M_2^0$ into the direct sum of

$$(C_0 \oplus (C_1 \oplus I_1) \oplus I_2) \otimes E[\widetilde{h_2^1}] \quad \text{and} \quad (\widetilde{h_3^1}C_0 \oplus \widetilde{h_3^1}\widetilde{h_2^1}C_0) \otimes E[h_2^0, h_3^0], \tag{2.1}$$

where

$$\begin{aligned}
C_0 &= \mathbb{Z}/p\{v_2^{sp^n} v_3^{tp^m} | 0 \leqslant m, n \leqslant \infty\},\\
C_1 &= C_1^0 \oplus C_1^1,\\
&\quad C_1^0 = \mathbb{Z}/p\{v_2^{sp^n} v_3^{tp^m-1} h_3^0 | 0 \leqslant n < m < \infty\},\\
&\quad C_1^1 = \mathbb{Z}/p\{v_2^{sp^n-1} v_3^{tp^m} h_2^0 | 0 \leqslant m \leqslant n \leqslant \infty\},\\
I_1 &= I_1^0 \oplus I_1^1,\\
&\quad I_1^0 = \mathbb{Z}/p\{v_2^{sp^n} v_3^{tp^m-1} h_3^0 | 0 \leqslant m \leqslant n \leqslant \infty, m < \infty\},\\
&\quad I_1^1 = \mathbb{Z}/p\{v_2^{sp^n-1} v_3^{tp^m} h_2^0 | 0 \leqslant n < m \leqslant \infty\},\\
I_2 &= \mathbb{Z}/p\{v_2^{sp^n-1} v_3^{tp^m-1} h_2^0 h_3^0 | 0 \leqslant m, n \leqslant \infty, m < \infty\}.
\end{aligned}$$

Remark. For $m = \infty$ or $n = \infty$, the integers t and s will be viewed as 0 separately; otherwise $p \nmid s$ ($s \in \mathbb{Z}$) and $p \nmid t$ ($t > 0$).

Based on these modules, we introduce the submodules of $H^{*,*}L_1^1$:

$$\begin{aligned}
\widetilde{C_0} &= \mathbb{Z}_{(p)}\{v_2^{sp^n} v_3^{tp^m} / p^{min\{m,n\}+1} | 0 \leqslant m, n \leqslant \infty\},\\
\widetilde{C_1} &= \widetilde{C_1^0} \oplus \widetilde{C_1^1},\\
&\quad \widetilde{C_1^0} = \mathbb{Z}_{(p)}\{v_2^{sp^n} v_3^{tp^m-1} h_3^0 / p^{n+1} | 0 \leqslant n < m < \infty\},\\
&\quad \widetilde{C_1^1} = \mathbb{Z}_{(p)}\{v_2^{sp^n-1} v_3^{tp^m} h_2^0 / p^{m+1} | 0 \leqslant m \leqslant n \leqslant \infty\},\\
\widetilde{h_3^1 C_0} &= \mathbb{Z}_{(p)}\{v_2^{sp^n} v_3^{tp^m} \widetilde{h_3^1}/p | 0 \leqslant m, n \leqslant \infty\},
\end{aligned}$$

where x/p^k is of order p^k in the $\mathbb{Z}_{(p)}$-module $H^{*,*}L_1^1$ and x/p is the image of x under the map $1/p : H^*M_2^0 \to H^*L_1^1$. $1/p^\infty$ represents $\{1/p^k | k > 0\}$, where $1/p^k \in H^{*,*}L_1^1$ is of order p^k.

Theorem 2.2. *$H^{*,*}L_1^1$ and then the homotopy groups of $L_2T(1)/(p^\infty, v_1)$ are isomorphic to the direct sum of*

$$(\widetilde{C_0} \oplus \widetilde{C_1}) \otimes E[\widetilde{h_2^1}] \quad \textit{and} \quad \widetilde{h_3^1 C_0} \otimes E[h_2^0, h_3^0].$$

Theorem 2.3. *The homotopy groups of $L_2T(1)/(v_1)$ are isomorphic to*

$$\pi_*(L_2T(1)/(v_1)) = \mathbb{Z}_{(p)} \oplus (\pi_*(L_2T(1)/(p^\infty, v_1)) - \mathbf{Q}/\mathbb{Z}_{(p)}).$$

Proof. From $H^*p^{-1}BP_*[t_1]/(v_1) = \mathbf{Q}$, we see that $\pi_*(L_0T(1)/(v_1)) = \mathbf{Q}$ concentrated in degree 0. Considering the long exact sequence of homotopy groups

$$\cdots \to \pi_t(L_2T(1)/(v_1)) \to \pi_t(L_0T(1)/(v_1)) \to \pi_t(L_2T(1)/(p^\infty, v_1)) \to \cdots$$

induced by the cofiber sequence

$$L_2T(1)/(v_1) \to L_0T(1)/(v_1) \to L_2T(1)/(p^\infty, v_1),$$

we get the theorem. □

3. Some elements in the cobar complex

Consider the Hopf algebroid $(BP_*, \Gamma(2))$ and $(E_1(2)_*, \Gamma(2,2))$, where

$$\Gamma(2) = BP_*BP/(t_1) = BP_*[t_2, t_3, \cdots].$$

The structure maps η_R and Δ are induced from $\eta_R : BP_* \to BP_*BP = BP_*[t_1, t_2, t_3, \cdots]$ and $\Delta : BP_*BP \to BP_*BP \otimes_{BP_*} BP_*BP$. Using the Hazewinkel generators v_i of BP_* defined by

$$v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i}$$

and Quillen's formulae

$$\eta_R(m_n) = \sum_{i+j=n} m_i t_j^{p^i},$$

$$\sum_{i+j=n} m_i \Delta(t_j)^{p^i} = \sum_{i+j+k=n} m_i t_j^{p^i} \otimes t_k^{p^{i+j}},$$

we have:

Lemma 3.1. *In the Hopf algebroid $(BP_*, \Gamma(2))$, the right unit η_R and the coproduct Δ act as follows:*

$$\begin{aligned}
\eta_R(v_1) &= v_1, \\
\eta_R(v_2) &= v_2 + pt_2, \\
\eta_R(v_3) &\equiv v_3 + pt_3 && \mod (v_1), \\
\eta_R(v_4) &\equiv v_4 + v_2 t_2^{p^2} - v_2^{p^2} t_2 + pt_4 && \mod (p^2, v_1), \\
\eta_R(v_5) &\equiv v_5 + v_3 t_2^{p^3} + v_2 t_3^{p^2} - v_3^{p^2} t_2 - v_2^{p^3} t_3 + pt_5 && \mod (p^2, v_1), \\
\Delta(t_i) &= t_i \otimes 1 + 1 \otimes t_i && \textit{for } i = 2, 3, \\
\Delta(t_4) &= t_4 \otimes 1 + t_2 \otimes t_2^{p^2} + 1 \otimes t_4 - v_2 b_{2,1}, \\
\Delta(t_5) &= t_5 \otimes 1 + t_3 \otimes t_2^{p^3} + t_2 \otimes t_3^{p^2} + 1 \otimes t_5 - v_2 b_{3,1} - v_3 b_{2,2},
\end{aligned}$$

where $p \cdot b_{i,j} = \Delta(t_i)^{p^{j+1}} - t_i^{p^{j+1}} \otimes 1 - 1 \otimes t_i^{p^{j+1}}$ for $i = 2, 3$. Thus in

$$\Gamma(2,2) = E_1(2)_* \otimes_{BP_*} \Gamma(2) \otimes_{BP_*} E_1(2)_*,$$

$$\begin{aligned}
v_2 t_2^{p^2} &= v_2^{p^2} t_2 - pt_4 && \mod (p^2, v_1), \\
v_2 t_3^{p^2} &= v_2^{p^3} t_3 + v_3^{p^2} t_2 - v_3 t_2^{p^3} + pt_5 && \mod (p^2, v_1).
\end{aligned} \tag{3.2}$$

Proof. The formulae follow by sending t_1 and v_i for $i \geqslant 4$ to 0. □

Lemma 3.3. *In the cobar complex $\Omega^{1,0}_{\Gamma(2,2)}(E_1(2)_*/(p^{n+1}, v_1^{p^n}))$, there is a cocycle ζ for each n, which represents the cohomology class $\widetilde{h_2^1} = v_2^{-p} h_2^1 + v_2^{-1} h_2^0$ of $H^{1,0}M_2^0$.*

Proof. From [13] and [14], there is the cocycle $\zeta_n = v_2^{-p}t_2^p + v_2^{-1}t_2 + \cdots$ in the cobar complex $\Omega^1_{E(2)_*E(2)}(E(2)_*/(p^{n+1}, v_1^{p^n})$ for each n and prime $p > 2$, which represents the cohomology class $v_2^{-p}h_2^1 + v_2^{-1}h_2^0$ of $Ext^{1,0}_{K(2)_*K(2)}(K(2)_*, K(2)_*)$. By the change of rings isomorphism

$$Ext^{*,*}_{BP_*BP}(BP_*, v_2^{-1}BP_*/(p^{n+1}, v_1^{p^n})) \cong Ext^{*,*}_{E(2)_*E(2)}(E(2)_*, E(2)_*/(p^{n+1}, v_1^{p^n})),$$

we get a cocycle $\zeta_n \in \Omega^{1,0}_{BP_*BP}(v_2^{-1}BP_*/(p^{n+1}, v_1^{p^n}))$. Consider the BP_*BP-comodule homomorphism $K(2)_* \to K(2)_*[t_1]$ and $v_2^{-1}BP_*/(p^{n+1}, v_1^{p^n}) \to v_2^{-1}BP_*[t_1]/(p^{n+1}, v_1^{p^n})$. Then we see that $v_2^{-p}h_2^1 + v_2^{-1}h_2^0 \in Ext^{1,0}_{K(2)_*K(2)}(K(2)_*, K(2)_*)$ is sent to the cohomology class

$$v_2^{-p}h_2^1 + v_2^{-1}h_2^0 \in Ext^{1,0}_{K(2)_*K(2)}(K(2)_*, K(2)_*[t_1]).$$

Thus this ζ_n is sent to $\zeta_n \in \Omega^{1,0}_{BP_*BP}(v_2^{-1}BP_*[t_1]/(p^{n+1}, v_1^{p^n}))$, which represents the cohomology class $v_2^{-p}h_2^1 + v_2^{-1}h_2^0 \in H^{1,0}M_2^0$. Again by the change of rings isomorphism

$$Ext^{*,*}_{BP_*BP}(BP_*, v_2^{-1}BP_*[t_1]/(p^{n+1}, v_1^{p^n})) \cong Ext^{*,*}_{\Gamma(2,2)}(E_1(2)_*, E_1(2)_*/(p^{n+1}, v_1^{p^n})),$$

we get the cocycle $\zeta \in \Omega^{1,0}_{\Gamma(2,2)}(E_1(2)_*/(p^{n+1}, v_1^{p^n}))$. □

Lemma 3.4. *In the cobar complex $\Omega^1_{\Gamma(2,2)}(E_1(2)_*)$, there is a cochain $\widetilde{t_3^p}$, which represents the cohomology class $\widetilde{h_3^1} = h_3^1 + v_2^{-p^4}v_3^{p^3}h_2^1 - v_3^p v_2^{-1}h_2^0$ of $H^{1,*}M_2^0$ and*

$$d(\widetilde{t_3^p}) \equiv pt_3^p \otimes (\zeta - 2v_2^{-1}t_2) + pv_3^p v_2^{-p}t_2^p \otimes v_2^{-1}t_2 \mod (p^2, v_1).$$

Proof. From (3.2), we see that $v_2^{p-p^4}t_3^{p^3} \equiv t_3^p + v_2^{-p^4}v_3^{p^3}t_2^p - v_3^p v_2^{-1}t_2 \bmod (p, v_1)$. Then from Lemma 3.1, we compute that, $\bmod (p^2, v_1)$,

$$\begin{aligned}
d(v_2^{-p^4+p}t_3^{p^3}) &\equiv -\underline{pv_2^{-p^4+p}b_{3,2}}_1,\\
d(pv_2^{-p^4}t_5^p) &\equiv pv_2^{-p^4+p}b_{3,2} + pv_2^{-p^4}v_3^p b_{2,3} - pv_2^{-p^4}(t_3^p \otimes t_2^{p^4} + t_2^p \otimes t_3^{p^3})\\
&\equiv \underline{pv_2^{-p^4+p}b_{3,2}}_1 + \underline{pv_2^{-p^4}v_3^p b_{2,3}}_2 - pt_3^p \otimes v_2^{-1}t_2\\
&\quad - \underline{pv_2^{-p}t_2^p \otimes t_3^p}_3 - \underline{pv_2^{-p^4-p}v_3^{p^3}t_2^p \otimes t_2^p}_4 + pv_2^{-p-1}v_3^p t_2^p \otimes t_2,\\
d(-pv_2^{-p^4-p^2}v_3^p t_4^{p^2}) &\equiv -\underline{pv_2^{-p^4}v_3^p b_{2,3}}_2 + \underline{pv_2^{-2}v_3^p t_2 \otimes t_2}_5,\\
d(-pv_2^{-p}t_2^p \cdot t_3^p) &\equiv \underline{pv_2^{-p}t_2^p \otimes t_3^p}_3 + pt_3^p \otimes v_2^{-p}t_2^p,\\
d\left(-\frac{1}{2}pv_2^{-p^4-p}v_3^{p^3}t_2^{2p}\right) &\equiv \underline{pv_2^{-p^4-p}v_3^{p^3}t_2^p \otimes t_2^p}_4,\\
d\left(\frac{1}{2}pv_2^{-2}v_3^p t_2^2\right) &\equiv \underline{-pv_2^{-2}v_3^p t_2 \otimes t_2}_5.
\end{aligned}$$

The sum of them gives rise to $\widetilde{t_3^p}$ as desired, where the underlined elements with the same subscripts amount to zero. □

4. The connecting homomorphisms

Consider the short exact sequence

$$0 \longrightarrow M_2^0 \xrightarrow{1/p} L_1^1 \xrightarrow{p} L_1^1 \to 0$$

and the induced long exact sequence

$$\cdots \longrightarrow H^s M_2^0 \xrightarrow{1/p} H^s L_1^1 \xrightarrow{p} H^s L_1^1 \xrightarrow{\delta_s} H^{s+1} M_2^0 \longrightarrow \cdots .$$

Lemma 4.1. *For the connecting homomorphism $\delta_0 : H^0 L_1^1 \longrightarrow H^1 M_2^0$ we have:*

(1) *For $0 \leqslant m \leqslant n \leqslant \infty$,*

$$\delta_0\left(\frac{v_2^{sp^n} v_3^{tp^m}}{p^{m+1}}\right) = t v_2^{sp^n} v_3^{tp^m-1} h_3^0 + s p^{n-m} v_2^{sp^n-1} v_3^{tp^m} h_2^0.$$

(2) *For $0 \leqslant n < m \leqslant \infty$,*

$$\delta_0\left(\frac{v_2^{sp^n} v_3^{tp^m}}{p^{n+1}}\right) = s v_2^{sp^n-1} h_2^0.$$

Proof. This is a direct computation from

$$d(v_2^{sp^n}) = s p^{n+1} v_2^{sp^n-1} t_2 + \cdots , \quad d(v_3^{tp^m}) = t p^{m+1} v_3^{tp^m-1} t_3 + \cdots \mod (v_1),$$

and h_2^0, h_3^0 are represented by t_2, t_3 respectively. □

From Lemma 4.1, we see that the cokernel of δ_0 is $C_1^0 \oplus C_1^1 \oplus \widetilde{h_3^1} C_0 \oplus \widetilde{h_2^1} C_0$.

Lemma 4.2. *The connecting homomorphism $\delta_1 : H^1 L_1^1 \longrightarrow H^2 M_2^0$ acts on the submodules $\widetilde{C_1^0} \oplus \widetilde{C_1^1}$ as:*

(1) *For $0 \leqslant n < m < \infty$,*

$$\delta_1\left(\frac{v_2^{sp^n} v_3^{tp^m-1} h_3^0}{p^{n+1}}\right) = s v_2^{sp^n-1} v_3^{tp^m-1} h_2^0 h_3^0.$$

(2) *For $0 \leqslant m \leqslant n \leqslant \infty$,*

$$\delta_1\left(\frac{v_2^{sp^n-1} v_3^{tp^m} h_2^0}{p^{m+1}}\right) = t v_2^{sp^n-1} v_3^{tp^m-1} h_3^0 h_2^0.$$

Proof. From

$$d(v_2^{sp^n}) \equiv s p^{n+1} v_2^{sp^n-1} t_2 + \cdots , \quad d(v_3^{tp^m}) \equiv t p^{m+1} v_3^{tp^m-1} t_3 + \cdots \mod (v_1),$$

we see that for $s \neq 0$, $t \neq 0$,

$$d(v_2^{sp^n-1} t_2 + \cdots) \equiv 0, \qquad d(v_3^{tp^m-1} t_3 + \cdots) \equiv 0 \mod (v_1).$$

For $n = \infty$, setting

$$\widetilde{v_2^{-1} t_2} = \frac{1}{p} \log(1 + p v_2^{-1} t_2) = \sum_{n>0} (-1)^{n-1} \frac{(p v_2^{-1} t_2)^n}{pn},$$

one can easily see that $\widetilde{p v_2^{-1} t_2} = d(\log(v_2))$. Thus $d(\widetilde{v_2^{-1} t_2}) = 0$. The lemma follows from

$$d(v_2^{sp^n} v_3^{tp^m-1} t_3) = d(v_2^{sp^n}) \cdot (v_3^{tp^m-1} t_3)$$

and

$$d(v_2^{sp^n-1}t_2v_3^{tp^m}) = -(v_2^{sp^n-1}t_2)\cdot d(v_3^{tp^m}).$$

□

Lemma 4.3. *The connecting homomorphism* $\delta_1 : H^1L_1^1 \longrightarrow H^2M_2^0$ *acts on* $\widetilde{h_3^1C_0}$ *as*

$$\delta_1\left(\frac{v_2^{sp^n}v_3^{tp^m}\widetilde{h_3^1}}{p}\right) = v_2^{sp^n}v_3^{tp^m}\left(h_3^1(\widetilde{h_2^1} - 2v_2^{-1}h_2^0) + v_3^pv_2^{-p-1}h_2^1h_2^0\right) + \cdots .$$

Proof. The proof follows from the definition of $\widetilde{t_3^p}$ (cf. Lemma 3.4). □

Proof of Theorem 2.2. From Lemmas 4.1-4.3 we see that the submodules $\widetilde{C_0}$, $\widetilde{C_1}$ and $\widetilde{h_3^1C_0}$ make the following sequences exact:

$$0 \to C_0 \xrightarrow{1/p} \widetilde{C_0} \xrightarrow{p} \widetilde{C_0} \xrightarrow{\delta_0} I_1 \to 0;$$

$$\begin{aligned}
&0 \to C_1 \xrightarrow{1/p} \widetilde{C_1} \xrightarrow{p} \widetilde{C_1} \xrightarrow{\delta_1} I_2 \to 0,\\
&0 \to \widetilde{h_3^1}C_0 \xrightarrow{1/p} \widetilde{h_3^1C_0} \xrightarrow{p} \widetilde{h_3^1C_0} \xrightarrow{\delta_1} \widetilde{h_3^1}\widetilde{h_2^1}C_0 \to 0,\\
&0 \to \widetilde{h_2^1}C_0 \xrightarrow{1/p} \widetilde{h_2^1}\widetilde{C_0} \xrightarrow{p} \widetilde{h_2^1}\widetilde{C_0} \xrightarrow{\delta_1} \widetilde{h_2^1}I_1 \to 0;
\end{aligned}$$

$$\begin{aligned}
&0 \to \widetilde{h_2^1}C_1 \xrightarrow{1/p} \widetilde{h_2^1}\widetilde{C_1} \xrightarrow{p} \widetilde{h_2^1}\widetilde{C_1} \xrightarrow{\delta_2} \widetilde{h_2^1}I_2 \to 0,\\
&0 \to h_2^0\widetilde{h_3^1}C_0 \xrightarrow{1/p} h_2^0\widetilde{h_3^1C_0} \xrightarrow{p} h_2^0\widetilde{h_3^1C_0} \xrightarrow{\delta_2} h_2^0\widetilde{h_3^1}\widetilde{h_2^1}C_0 \to 0,\\
&0 \to h_3^0\widetilde{h_3^1}C_0 \xrightarrow{1/p} h_3^0\widetilde{h_3^1C_0} \xrightarrow{p} h_3^0\widetilde{h_3^1C_0} \xrightarrow{\delta_2} h_3^0\widetilde{h_3^1}\widetilde{h_2^1}C_0 \to 0;
\end{aligned}$$

$$0 \to h_2^0h_3^0\widetilde{h_3^1}C_0 \xrightarrow{1/p} h_2^0h_3^0\widetilde{h_3^1C_0} \xrightarrow{p} h_2^0h_3^0\widetilde{h_3^1C_0} \xrightarrow{\delta_3} h_2^0h_3^0\widetilde{h_3^1}\widetilde{h_2^1}C_0 \to 0.$$

It is easy to prove that

$$\begin{aligned}
\operatorname{coker}\delta_0 &= C_1^0 \oplus C_1^1 \oplus \widetilde{h_3^1}C_0 \oplus \widetilde{h_2^1}C_0,\\
\operatorname{coker}\delta_1 &= \widetilde{h_2^1}C_1 \oplus h_2^0\widetilde{h_3^1}C_0 \oplus h_3^0\widetilde{h_3^1}C_0,\\
\operatorname{coker}\delta_2 &= h_2^0h_3^0\widetilde{h_3^1}C_0,\\
\operatorname{coker}\delta_3 &= 0.
\end{aligned}$$

So we can construct p-torsion submodules B^* of $H^{*,*}L_1^1$:

$$\begin{aligned}
B^0 &= \widetilde{C_0},\\
B^1 &= \widetilde{C_1} \oplus \widetilde{h_3^1C_0} \oplus \widetilde{h_2^1}\widetilde{C_0},\\
B^2 &= \widetilde{h_2^1}\widetilde{C_1} \oplus h_2^0\widetilde{h_3^1C_0} \oplus h_3^0\widetilde{h_3^1C_0},\\
B^3 &= h_2^0h_3^0\widetilde{h_3^1C_0},\\
B^k &= 0 \ (k \geqslant 4)
\end{aligned}$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & H^sM_2^0 \arrow[r, "1/p"] \arrow[d, equal] & B^s \arrow[r, "p"] \arrow[d] & B^s \arrow[r, "\delta_s"] \arrow[d] & H^{s+1}M_2^0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H^sM_2^0 \arrow[r, "1/p"] & H^sL_1^1 \arrow[r, "p"] & H^sL_1^1 \arrow[r, "\delta_s"] & H^{s+1}M_2^0 \arrow[r] & \cdots
\end{tikzcd}$$

Then from [5], Remark 3.11, we see that

$$H^{*,*}L_1^1 = \left((\widetilde{C_0} \oplus \widetilde{C_1}) \otimes E[\widetilde{h_2^1}]\right) \oplus \left(\widetilde{h_3^1 C_0} \otimes E[h_2^0, h_3^0]\right).$$

Similarly, we see that $H^{s,*}L_1^1 = 0$ for $s > 3$ and the Adams-Novikov E_2-item for $\pi_*(L_2T(1)/(p^\infty, v_1))$ collapses. This completes the proof of Theorem 2.2. $\square$

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