THE HOMOTOPY GROUPS OF $L_2T(1)/(v_1)$ AT AN ODD PRIME

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ABSTRACT. In this paper, all spectra are localized at an odd prime. Let $T(1)$ be the Ravenel spectrum characterized by $BP_*$-homology as $BP_*[t_1]$. $T(1)/(v_1)$ be the cofiber of the self-map $v_1 : \Sigma^{2p-2}T(1) \to T(1)$ and $L_2$ denote the Bousfield localization functor with respect to $v_1^{-1}BP_*$. In this paper, we determine the homotopy groups of $L_2T(1)/(v_1)$.

1. INTRODUCTION

Let $\mathcal{S}_p$ denote the stable homotopy category of spectra localized at the prime number $p$, and let $BP$ denote the Brown-Peterson spectrum characterized by the coefficient ring $BP_* = \pi_*(BP) = \mathbb{Z}_p[v_1, v_2, \cdots]$. Then $(BP_*, BP_*BP)$ is a Hopf algebroid, where $BP_*BP = BP_*[t_1, t_2, \cdots]$. One has the Bousfield localization functor $L_n : \mathcal{S}_p \to \mathcal{S}_p$ with respect to $v_1^{-1}BP_*$, and one denotes the image of it as $L_n$. The homotopy groups $\pi_*(L_nS^0)$ of the sphere spectrum $S^0$ play an important role in understanding $L_n$. The main tool to determine them is the Adams-Novikov spectral sequence. For $n \leq 2$, the Adams-Novikov $E_2$-terms for $\pi_*(L_nS^0)$ were determined in [8], [13], [14], [15] and the homotopy groups of $L_nS^0$ are also determined if $n \leq 2$, except for the case that $n = 2$ and $p = 2$.

Ravenel [7] constructed a spectrum $T(m)$ for each integer $m \geq 0$, whose $BP_*$-homology is the subcomodule algebra $BP_*[t_1, \cdots, t_m]$ of $BP_*BP$. There is a tower of spectra:

$$S^0_{(p)} = T(0) \hookrightarrow T(1) \hookrightarrow T(2) \hookrightarrow \cdots \hookrightarrow T(m) \hookrightarrow \cdots \hookrightarrow T(\infty) = BP.$$

Using the infinite descent method of [11], one could get information of $\pi_*(L_nS^0)$ from $\pi_*(L_nT(m))$.

Consider the Adams-Novikov spectral sequence for $\pi_*(T(m))$:

$$Ext^{*,*}_{BP_*BP}(BP_*, BP_*(T(m))) \Rightarrow \pi_{*-s}(T(m)).$$

It is known that for $i \leq m$, the $v_i$’s are permanent cycles (cf. [7], 6.5.9). Since $T(m)$ is a ring spectrum, the homotopy element $v_i \in \pi_*(T(m))$ extends to the self-map

$v_i : \Sigma^{2(p'-1)}T(m) = S^{2(p'-1)} \wedge T(m) \overset{v_i}{\longrightarrow} T(m) \wedge T(m) \overset{\mu}{\longrightarrow} T(m).$
In this paper, we study the homotopy groups of $L_2T(1)$. Let $T(1)/(v_1)$ be the cofiber of $v_1: \Sigma^{2p-2}T(1) \to T(1)$, let $T(1)/(p)$ be the cofiber of $p: T(1) \to T(1)$ and let $T(1)/(p,v_1)$ be the cofiber of $v_1: \Sigma^{2p-2}T(1)/(p) \to T(1)/(p)$, which is also the cofiber of $p: T(1)/(v_1) \to T(1)/(v_1)$. Consider the localization map $L_n : X \to L_nX$, let $T(1)/(p^\infty,v_1)$ denote the cofiber of $L_0 : T(1)/(v_1) \to L_0T(1)/(v_1)$, let $T(1)/(p,v_1^\infty)$ denote the cofiber of $L_1 : T(1)/(p) \to L_1T(1)/(p)$ and let $T(1)/(p^\infty,v_1^\infty)$ denote the cofiber of $L_1 : T(1)/(p^\infty) \to L_1T(1)/(p^\infty)$, which is the cofiber of $L_0 : T(1)/(v_1^\infty) \to L_0T(1)/(v_1^\infty)$ too. Then by the $3 \times 3$ theorem we have the following cofiber sequences:

$$L_2T(1)/(p^\infty, v_1) \xrightarrow{1/v_1} L_2T(1)/(p^\infty, v_1^\infty) \xrightarrow{v_1} L_2T(1)/(p^\infty, v_1^\infty),$$

$$L_2T(1)/(p, v_1^\infty) \xrightarrow{1/p} L_2T(1)/(p^\infty, v_1^\infty) \xrightarrow{p} L_2T(1)/(p^\infty, v_1^\infty).$$

These give two ways to work out the homotopy groups of $L_2T(1)$, both starting from $L_2T(1)/(p,v_1)$:

1. $\pi_*(L_2T(1)/(p,v_1)) \Rightarrow \pi_*(L_2T(1)/(p))$, then $\Rightarrow \pi_*(L_2T(1)).$
2. $\pi_*(L_2T(1)/(p,v_1)) \Rightarrow \pi_*(L_2T(1)/(v_1))$, then $\Rightarrow \pi_*(L_2T(1)).$

At the prime 2, Shimomura determined the homotopy groups $\pi_*(L_2T(1)/(2))$ in [12]. In [9] and [13], the second author separately with Nakai and Shimomura determined the homotopy groups $\pi_*(L_2T(1)/(v_1))$ and proved that the Adams-Novikov spectral sequence for $\pi_*(L_2T(1))$ has a horizontal vanishing line at the $E_1$-terms. But it seems to be too difficult to work out $\pi_*(L_2T(1))$ from both ways.

In this paper, we will determine the homotopy groups $\pi_*(L_2T(1)/(v_1))$ at odd primes $p$ (Theorem 2.3). This gives a way to approach the homotopy groups of $L_2T(1)$ at odd primes.

2. The Adams spectral sequence for $\pi_*(L_2T(1)/(v_1))$

Let $BP$ denote the Brown-Peterson ring spectrum characterized by

$$\pi_*(BP) = BP_* = \mathbb{Z}(p)[v_1, v_2, \ldots, v_n, \ldots].$$

Here the $v_i$’s are the Hazewinkel generators, and they are assigned the degree $2(p^i - 1)$. Then $(BP_*, BP, BP)$ becomes a Hopf algebroid, where $BP_*BP = BP_*[t_1, \ldots, t_n, \ldots]$ with $|t_n| = 2(p^n - 1)$. For any spectrum $X$, we have the Adams-Novikov spectral sequence

$$E_2^{s,t} = Ext_{BP_*BP}^s(BP_*, BP_*(X)) \Rightarrow \pi_{t-s}(X).$$

Let $E(2)$ be the Johnson-Wilson ring spectrum, which yields the Hopf algebroid

$$(E(2)_*, E(2), E(2)) = (\mathbb{Z}(p)[v_1, v_2^{+1}], E(2)_*[t_1, t_2, \ldots]) \otimes_{BP_*} E(2)_*.$$

The Thom map $f : BP \to E(2)$ induces $f_* : BP_* \to E(2)_*$, which satisfies $f_*(v_i) = v_i$ for $i \leq 2$ and 0 for $i > 2$. Then by the change of rings theorem, we see that for an $I_2$-nil $BP_*BP$-comodule $M$,

$$Ext_{BP_*BP}^{s,t}(BP_*, v_2^{-1}M) = Ext_{E_1^{(2)}}^{s,t}(E(2)_*, M \otimes_{BP_*} E(2)_*).$$

Let $M$ be an $I_2$-nil $BP_*BP$-comodule and $M[t_1]$ be the polynomial ring over $M$. To compute the Ext groups $Ext_{BP_*BP}^{s,t}(BP_*, v_2^{-1}M[t_1])$, consider the $BP_*$-module $E_1(2)_* = E(2)_*[v_3] = \mathbb{Z}(p)[v_1, v_2^{+1}, v_3]$ with action given by sending $v_i$ to
0 for \( i > 3 \). Then \( E_1(2)_* (X) = E_1(2)_* \otimes_{BP_*} BP_* X \) is a homology theory by the Landweber exact functor theorem, and the Hopf algebroid structure

\[
(E_1(2)_*, E_1(2)_*, E_1(2)) = (E_1(2)_*, E_1(2)_*[t_1, t_2, \cdots] \otimes_{BP_*} E_1(2)_*)
\]
is induced from \((BP_*, BP_* BP)\). By the change of rings theorem we have

\[
Ext^{s,t}_{BP_* BP}(BP_* v_2^{-1} M[t_1]) = Ext^{s,t}_{E_1(2)_*, E_1(2)_*}(E_1(2)_*, E_1(2)_* \otimes_{BP_*} M[t_1])
\]
(\text{cf. [2]}).

Let

\[
\Gamma(2, 2) = E_1(2)_*/(t_1) = E_1(2)_*[t_2, t_3, \cdots] \otimes_{BP_*} E_1(2)_*,
\]
and the Hopf algebroid structure of \((E_1(2)_*, \Gamma(2, 2))\) be the one induced from that of \((BP_*, BP_* BP)\). Noting that

\[
E_1(2)_* \otimes_{BP_*} M[t_1] = E_1(2)_* E_1(2)_*/\Gamma(2, 2) E_1(2)_* \otimes_{BP_*} M,
\]
we have the change of rings theorem

\[
Ext^{s,t}_{E_1(2)_*, E_1(2)_*}(E_1(2)_*, E_1(2)_* \otimes_{BP_*} M[t_1]) = Ext^{s,t}_{\Gamma(2, 2)}(E_1(2)_*, E_1(2)_* \otimes_{BP_*} M).
\]

Let \( M_0^2 = E_1(2)_*/(p, v_1) \), \( L_1^1 = E_1(2)_*/(p^{\infty}, v_1) \) and \( Ext^{s,t}_{\Gamma(2, 2)}(E_1(2)_*, M) = H^{*,*}(M) \) for short. We have the short exact sequence of \( \Gamma(2, 2) \)-comodules

\[
0 \rightarrow M_0^2 \xrightarrow{1/p} L_1^1 \xrightarrow{p} L_1^1 \rightarrow 0.
\]

This short exact sequence induces the following long exact sequence:

\[
\cdots \rightarrow H^{*,*} M_0^2 \xrightarrow{1/p} H^{*,*} L_1^1 \xrightarrow{p} H^{*,*} L_1^1 \xrightarrow{\delta} H^{*+1,*} M_0^2 \rightarrow \cdots.
\]

To compute \( H^{*,*} M_0^2 \), let \( K_1(2)_* = E_1(2)_*/(p, v_1) = \mathbb{Z}/p[v_2^{\pm 1}, v_3] \) and

\[
\Sigma(2, 2) = K_1(2)_* \otimes_{BP_*} \Gamma(2, 2) \otimes_{BP_*} K_1(2)_* = K_1(2)[t_2, t_3, \cdots]/(t_i^p - t_i),
\]

\[
S(2, 2) = \mathbb{Z}/p \otimes_{K_1(2)_*} \Gamma(2, 2) \otimes_{K_1(2)_*} \mathbb{Z}/p = \mathbb{Z}/p[t_2, t_3, \cdots]/(t_i^p - t_i),
\]

where \( K_1(2)_* \) acts on \( \mathbb{Z}/p \) by sending \( v_2^{\pm 1} \) and \( v_3 \) to 1. Recalling from [7], 6.3.7, that the Adams-Novikov \( E_2 \)-term for \( \pi_*(L_2 T(1)/(p, v_1)) \) is \( H^{*,*} M_0^2 \), we have

\[
\text{Ext}_{E_1(2)_*, E_1(2)_*}(E_1(2)_*, E_1(2)_*/(p, v_1)) \cong \text{Ext}_{\Gamma(2, 2)}(E_1(2)_*, E_1(2)_*/(p, v_1)) = H^{*,*} M_0^2
\]
\[
\cong \text{Ext}_{S(2, 2)}(K_1(2)_*, K_1(2)_*) \cong \mathbb{Z}/p[v_2^{\pm 1}, v_3] \otimes \text{Ext}_{S(2, 2)}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p[v_2^{\pm 1}, v_3] \otimes \mathbb{Z}[h_2^0, h_1^0, h_0^1, h_1^1],
\]

where the \( h_2^0 \) denotes the cohomology class represented by \( t_i^p \). Let \( \tilde{h}_2^1 = v_2^p h_2^1 + v_2^{-1} h_2^0 \) and \( \tilde{h}_3^1 = h_3^1 + v_2^{-p} v_3 h_3^1 - v_3 h_2^0 \). It is easy to see that

\[
H^{*,*} M_0^2 = \mathbb{Z}/p[v_2^{\pm 1}, v_3] \otimes \mathbb{Z}[h_2^0, h_1^0, h_2^1, h_3^1].
\]

Noticing that the first nontrivial Adams-Novikov differential appears at \( d_{2p-1} \) and

\[
\text{Ext}_{BP_* BP}^{s,t}(BP_*, BP_*(L_2 T(1)/(p, v_1))) = 0
\]
for \( s > 4 \), we see that the Adams-Novikov spectral sequence for \( \pi_*(L_2 T(1)/(p, v_1)) \) collapses and \( H^{*,*} M_0^2 \cong \pi_*(L_2 T(1)/(p, v_1)) \).
In this paper we will compute the Adams-Novikov $E_2$-terms for $\pi_*(L_2T(1)/(p^\infty, v_1))$ by the $p$-Beckstein spectral sequence (the $E_2$-terms collapse also). Then by the long exact sequence

$$\cdots \to \pi_n(L_2T(1)/(v_1)) \to \pi_n(L_0T(1)/(v_1)) \to \pi_n(L_2T(1)/(p^\infty, v_1)) \to \cdots,$$

we could determine the homotopy groups $\pi_*(L_2T(1)/(v_1))$.

To state our results, we decompose the module $H^{**}M_2^0$ into the direct sum of

$$\begin{align*}
(C_0 & \oplus (C_1 \oplus I_1) \oplus I_2) \otimes E[h_2] \quad \text{and} \quad (\widetilde{h_3}C_0 \oplus \widetilde{h_3}h_3C_0) \otimes E[h_2^0, h_3^0],
\end{align*}$$

where

- $C_0 = \mathbb{Z}/p \{ v_2^{mp^n} v_3^{p^m} | 0 \leq m, n \leq \infty \}$,
- $C_1 = C_0^0 \oplus C_1^1$,
- $C_0^0 = \mathbb{Z}/p \{ v_2^{mp^n} v_3^{p^m-1} h_3^0 | 0 \leq n < m \leq \infty \}$,
- $C_1^1 = \mathbb{Z}/p \{ v_2^{s_p^n} v_3^{p^m} h_2^0 | 0 \leq m < n \leq \infty \}$,
- $I_1 = I_1^0 \oplus I_1^1$,
- $I_1^0 = \mathbb{Z}/p \{ v_2^{s_p^n} v_3^{p^m-1} h_3^0 | 0 \leq m \leq n < \infty, m \leq \infty \}$,
- $I_1^1 = \mathbb{Z}/p \{ v_2^{s_p^n-1} v_3^{p^m} h_2^0 | 0 \leq n < m \leq \infty \}$,
- $I_2 = \mathbb{Z}/p \{ v_2^{s_p^n-1} v_3^{p^m-1} h_2^0 h_3^0 | 0 \leq m, n \leq \infty, m < \infty \}$.

**Remark.** For $m = \infty$ or $n = \infty$, the integers $t$ and $s$ will be viewed as 0 separately; otherwise $p \nmid s$ ($s \in \mathbb{Z}$) and $p \nmid t$ ($t > 0$).

Based on these modules, we introduce the submodules of $H^{**}L_1^1$:

- $\widetilde{C}_0 = \mathbb{Z}(p) \{ v_2^{s_p^n} v_3^{p^m} / p^{\min(m,n)+1} | 0 \leq m, n \leq \infty \}$,
- $\widetilde{C}_1 = \widetilde{C}_1^0 \oplus \widetilde{C}_1^1$,
- $\widetilde{C}_1^0 = \mathbb{Z}(p) \{ v_2^{s_p^n} v_3^{p^m-1} h_3^0 / p^{n+m+1} | 0 \leq n < m \leq \infty \}$,
- $\widetilde{C}_1^1 = \mathbb{Z}(p) \{ v_2^{s_p^n-1} v_3^{p^m} h_2^0 / p^{n+m+1} | 0 \leq m \leq n \leq \infty \}$,
- $\widetilde{h_3}C_0 = \mathbb{Z}(p) \{ v_2^{s_p^n} v_3^{p^m} h_3^0 / p | 0 \leq m, n \leq \infty \}$,

where $x/p^k$ is of order $p^k$ in the $\mathbb{Z}(p)$-module $H^{**}L_1^1$ and $x/p$ is the image of $x$ under the map $1/p : H^{**}M_2^0 \to H^{**}L_1^1$. $1/p^\infty$ represents $\{1/p^k | k > 0\}$, where $1/p^k$ is of order $p^k$.

**Theorem 2.2.** $H^{**}L_1^1$ and then the homotopy groups of $L_2T(1)/(p^\infty, v_1)$ are isomorphic to the direct sum of

$$(\widetilde{C}_0 \oplus \widetilde{C}_1) \otimes E[h_2^0] \quad \text{and} \quad \widetilde{h_3}C_0 \otimes E[h_2^0, h_3^0].$$

**Theorem 2.3.** The homotopy groups of $L_2T(1)/(v_1)$ are isomorphic to

$$\pi_*(L_2T(1)/(v_1)) = \mathbb{Z}(p) \oplus (\pi_*(L_2T(1)/(p^\infty, v_1)) - \mathbb{Q}/\mathbb{Z}(p)).$$

**Proof.** From $H^{**}p^{-1}BP_*(t_1)/(v_1) = \mathbb{Q}$, we see that $\pi_*(L_0T(1)/(v_1)) = \mathbb{Q}$ concentrated in degree 0. Considering the long exact sequence of homotopy groups

$$\cdots \to \pi_*(L_2T(1)/(v_1)) \to \pi_*(L_0T(1)/(v_1)) \to \pi_*(L_2T(1)/(p^\infty, v_1)) \to \cdots,$$
induced by the cofiber sequence
\[ L_2 T(1)/(v_1) \to L_0 T(1)/(v_1) \to L_2 T(1)/(p^\infty, v_1), \]
we get the theorem. \[\square\]

3. SOME ELEMENTS IN THE COBAR COMPLEX

Consider the Hopf algebroid \((BP_*, \Gamma(2))\) and \((E_1(2)_*, \Gamma(2, 2))\), where
\[ \Gamma(2) = BP_*BP/(t_1) = BP_*[t_2, t_3, \cdots]. \]
The structure maps \(\eta_R\) and \(\Delta\) are induced from \(\eta_R : BP_* \to BP_*BP = BP_*[t_1, t_2, t_3, \cdots]\) and \(\Delta : BP_*BP \to BP_*BP \otimes_{BP_*} BP_*BP\). Using the Hazewinkel generators \(v_i\) of \(BP_*\) defined by
\[ v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^i, \]
and Quillen’s formulae
\[ \eta_R(m_n) = \sum_{i+j=n} m_i t_j^i, \]
\[ \sum_{i+j=n} m_i \Delta(t_j)^i = \sum_{i+j+k=n} m_i t_j^i \otimes t_k^{i+j}, \]
we have:

Lemma 3.1. In the Hopf algebroid \((BP_*, \Gamma(2))\), the right unit \(\eta_R\) and the coproduct \(\Delta\) act as follows:

\[ \eta_R(v_1) = v_1, \]
\[ \eta_R(v_2) = v_2 + pt_2, \]
\[ \eta_R(v_3) = v_3 + pt_3 \mod (v_1), \]
\[ \eta_R(v_4) = v_4 + v_2 t_2^2 - v_2^2 t_2 + pt_4 \mod (p^2, v_1), \]
\[ \eta_R(v_5) = v_5 + v_3 t_2^3 + v_2 t_2^3 - v_2^3 t_2 - v_2^3 t_3 + pt_5 \mod (p^2, v_1), \]
\[ \Delta(t_i) = t_i \otimes 1 + 1 \otimes t_i \]
\[ \Delta(t_4) = t_4 \otimes 1 + t_2 \otimes t_2^3 + 1 \otimes t_4 - v_2 b_{2.1}, \]
\[ \Delta(t_5) = t_5 \otimes 1 + t_3 \otimes t_2^3 + t_2 \otimes t_3^2 + 1 \otimes t_5 - v_2 b_{3.1} - v_3 b_{2.2}, \]
where \(p \cdot b_{i,j} = \Delta(t_i)^{i+1} - t_i^{i+1} \otimes 1 - 1 \otimes t_i^{i+1} \) for \(i = 2, 3\). Thus in
\[ \Gamma(2, 2) = E_1(2)_* \otimes_{BP_*} \Gamma(2) \otimes_{BP_*} E_1(2)_*, \]

\[ (3.2) \quad v_2 t_2^2 = v_2^2 t_2 - pt_4 \mod (p^2, v_1), \]
\[ v_3 t_3^2 = v_3^2 t_4 + v_3^3 t_2 - v_3 t_3^3 + pt_5 \mod (p^2, v_1). \]

Proof. The formulae follow by sending \(t_1\) and \(v_i\) for \(i \geq 4\) to 0. \[\square\]

Lemma 3.3. In the cobar complex \(\Omega_{\Gamma(2, 2)}^{1,0}(E_1(2)_*/(p^{n+1}, v_1^n))\), there is a cocycle \(\zeta\) for each \(n\), which represents the cohomology class \(h_2^1 = v_2^{-p} h_1^1 + v_2^{-1} h_2^0\) of \(H^{1,0} M_2^1\).
Proof. From \cite{13} and \cite{14}, there is the cocycle $\zeta_n = v_2^{-p} t_2^p + v_2^{-1} t_2 + \cdots$ in the cobar complex $\Omega_{E(2), E(2)}(E(2), (p^{n+1}, t_1^p))$ for each $n$ and prime $p > 2$, which represents the cohomology class $v_2^{-p} h_2^{1} + v_2^{-1} h_2^{0}$ of $\text{Ext}^{1,0}_{K(2), K(2)}(K(2)_*, K(2)_*)$. By the change of rings isomorphism

\[ \text{Ext}^{*,*}_{BP, BP}(BP_*, v_2^{-1} BP_*/(p^{n+1}, t_1^p)) \cong \text{Ext}^{*,*}_{E(2), E(2)}(E_2, E_2)/(p^{n+1}, t_1^p) \]

we get a cocycle $\zeta_n \in \Omega_{BP, BP}(v_2^{-1} BP_*/(p^{n+1}, t_1^p))$. Consider the $BP_*$-comodule homomorphism $K(2)_* \to K(2)_*[t_1]$ and $v_2^{-1} BP_*/(p^{n+1}, t_1^p) \to v_2^{-1} BP_* [t_1]/(p^{n+1}, t_1^p)$. Then we see that $v_2^{-p} h_2^{1} + v_2^{-1} h_2^{0} \in \text{Ext}^{1,0}_{K(2), K(2)}(K(2)_*, K(2)_*)$ is sent to the cohomology class

\[ v_2^{-p} h_2^{1} + v_2^{-1} h_2^{0} \in \text{Ext}^{1,0}_{K(2), K(2)}(K(2)_*, K(2)_*[t_1]). \]

Thus this $\zeta_n$ is sent to $\zeta_n \in \Omega_{BP, BP}(v_2^{-1} BP_*/(p^{n+1}, t_1^p))$, which represents the cohomology class $v_2^{-p} h_2^{1} + v_2^{-1} h_2^{0} \in H^{1,0} M_2^0$. Again by the change of rings isomorphism

\[ \text{Ext}^{*,*}_{BP, BP}(BP_*, v_2^{-1} BP_*[t_1]/(p^{n+1}, t_1^p)) \cong \text{Ext}^{*,*}_{E(1,2), E(1,2)}(E_2, E_2)/(p^{n+1}, t_1^p), \]

we get the cocycle $\zeta \in \Omega_{E(1,2), E(1,2)}(E_2, (p^{n+1}, t_1^p))$.\]

\[ d(t_2^p) \equiv pt_2^p \zeta - (\zeta - 2v_2^{-p} t_2^p) + pv_2^p v_2^{-p} t_2^p \otimes v_2^{-1} t_2 \mod (p^2, v_1). \]

Proof. From (3.2), we see that $v_2^{-p} t_2^p \equiv t_2^p + v_2^{-p} v_2^p t_2^p - v_2^{-1} v_2^p t_2$ mod($p, v_1$).

Then from Lemma 3.1, we compute that, mod($p^2, v_1$),

\[ d(v_2^{-p} t_2^p) = -pv_2^{-p} v_2^p t_2 \]

\[ d(pt_2^p) \equiv pv_2^p v_2^{-p} t_2 \otimes v_2^{-1} t_2 \]

\[ d(-pv_2^{-p} t_2^p) \equiv -pv_2^{-p} v_2^p t_2 \otimes v_2^{-1} t_2. \]

The sum of them gives rise to $\zeta_0$ as desired, where the underlined elements with the same subscripts amount to zero.\]

\[ d \left( \frac{1}{2} pv_2^{-p} t_2 \right) \equiv -pv_2^{-p} v_2^p t_2 \otimes t_2. \]
4. The connecting homomorphisms

Consider the short exact sequence
\[ 0 \to M_2^0 \xrightarrow{1/p} L_1^1 \xrightarrow{p} L_1^1 \to 0 \]
and the induced long exact sequence
\[ \cdots \to H^s M_2^0 \xrightarrow{1/p} H^s L_1^1 \xrightarrow{p} H^s L_1^1 \xrightarrow{\delta_s} H^{s+1} M_2^0 \to \cdots. \]

**Lemma 4.1.** For the connecting homomorphism \( \delta_0 : H^0 L_1^1 \to H^1 M_2^0 \) we have:

1. For \( 0 \leq m \leq n \leq \infty \),
   \[ \delta_0 \left( \frac{v_2^{sp^n} v_3^{tp^m}}{p^{n+1}} \right) = tv_2^{sp^n} v_3^{tp^m} - h_3^0 + sp^{n-m} v_2^{sp^n} - v_3^{tp^m} h_2^0. \]

2. For \( 0 \leq n < m \leq \infty \),
   \[ \delta_0 \left( \frac{v_2^{sp^n} v_3^{tp^m}}{p^{n+1}} \right) = sv_2^{sp^n - h_2^0}. \]

**Proof.** This is a direct computation from
\[ d(v_2^{sp^n}) = sp^{n+1} v_2^{sp^n - t_2} + \cdots, \quad d(v_3^{tp^m}) = tp^{m+1} v_3^{tp^m - t_3} + \cdots \mod (v_1), \]
and \( h_2^0, h_3^0 \) are represented by \( t_2, t_3 \) respectively. \( \square \)

From Lemma 4.1, we see that the cokernel of \( \delta_0 \) is \( C_1^0 \oplus C_1^1 \oplus \widetilde{h_3^0} C_0 \oplus \widetilde{h_2^0} C_0 \).

**Lemma 4.2.** The connecting homomorphism \( \delta_1 : H^1 L_1^1 \to H^2 M_2^0 \) acts on the submodules \( C_1^0 \oplus C_1^1 \) as:

1. For \( 0 \leq n < m < \infty \),
   \[ \delta_1 \left( \frac{v_2^{sp^n} v_3^{tp^m - h_3^0}}{p^{m+1}} \right) = sv_2^{sp^n - h_3^0} v_3^{tp^m - h_2^0 h_3^0}. \]

2. For \( 0 \leq m \leq n \leq \infty \),
   \[ \delta_1 \left( \frac{v_2^{sp^n - h_3^0} v_3^{tp^m}}{p^{n+1}} \right) = tv_2^{sp^n - h_3^0} v_3^{tp^m} - h_2^0 h_3^0. \]

**Proof.** From
\[ d(v_2^{sp^n}) \equiv sp^{n+1} v_2^{sp^n - t_2} + \cdots, \quad d(v_3^{tp^m}) \equiv tp^{m+1} v_3^{tp^m - t_3} + \cdots \mod (v_1), \]
we see that for \( s \neq 0, t \neq 0, \)
\[ d(v_2^{sp^n - t_2} + \cdots) \equiv 0, \quad d(v_3^{tp^m - t_3} + \cdots) \equiv 0 \mod (v_1). \]
For \( n = \infty \), setting
\[ \overline{v_2^{t_2}} = \frac{1}{p} \log(1 + pv_2^{-1} t_2) = \sum_{n>0} (-1)^{n-1} \frac{(pv_2^{-1} t_2)^n}{pn}, \]
one can easily see that \( pv_2^{-1} t_2 = d(\log(v_2)) \). Thus \( d(\overline{v_2^{t_2}}) = 0 \). The lemma follows from
\[ d(v_2^{sp^n} v_3^{tp^m - t_3}) = d(v_2^{sp^n}) \cdot (v_3^{tp^m - t_3}) \]
and
\[ d(v_2^{sp\cdot 1}t_2v_3^{tp\cdot m}) = -(v_2^{sp\cdot 1}t_2) \cdot d(v_3^{tp\cdot m}). \]

**Lemma 4.3.** The connecting homomorphism \( \delta_1 : H^1L_1 \rightarrow H^2M_2^0 \) acts on \( \sim C_0 \)
\[
\delta_1 \left( \frac{v_2^{sp\cdot n}v_3^{tp\cdot m}h_3}{p} \right) = v_2^{sp\cdot n}v_3^{tp\cdot m} \left( h_3^1(h_3^1 - 2v_2^{sp\cdot 1}h_2^0) + v_3^p v_2^{sp\cdot 1}h_2^1h_2^0 \right) + \ldots .
\]

**Proof.** The proof follows from the definition of \( \sim t_0 \) (cf. Lemma 3.4). \( \Box \)

**Proof of Theorem 2.2.** From Lemmas 4.1-4.3 we see that the submodules \( \sim C_0, \sim C_1 \)
and \( h_2^1C_0 \) make the following sequences exact:
\[
0 \rightarrow C_0 \xrightarrow{1/p} C_0 \xrightarrow{p} C_0 \xrightarrow{\delta_0} I_1 \rightarrow 0;
0 \rightarrow C_1 \xrightarrow{1/p} \sim C_1 \xrightarrow{p} \sim C_1 \xrightarrow{\delta_1} I_2 \rightarrow 0,
0 \rightarrow h_3^1C_0 \xrightarrow{1/p} \sim h_3^1C_0 \xrightarrow{p} h_3^1C_0 \xrightarrow{\delta_1} h_3^1h_3^1C_0 \rightarrow 0,
0 \rightarrow h_3^1C_0 \xrightarrow{1/p} \sim h_3^1C_0 \xrightarrow{p} h_3^1C_0 \xrightarrow{\delta_1} h_3^1I_1 \rightarrow 0;
0 \rightarrow \sim h_3^1C_0 \xrightarrow{1/p} \sim h_3^1C_0 \xrightarrow{p} \sim h_3^1C_0 \xrightarrow{\delta_2} \sim h_3^1I_2 \rightarrow 0,
0 \rightarrow h_3^0h_3^0h_3^0C_0 \xrightarrow{1/p} h_3^0h_3^0h_3^0C_0 \xrightarrow{p} h_3^0h_3^0h_3^0C_0 \xrightarrow{\delta_2} h_3^0h_3^0h_3^0C_0 \rightarrow 0,
0 \rightarrow h_3^0h_3^0h_3^0C_0 \xrightarrow{1/p} h_3^0h_3^0h_3^0C_0 \xrightarrow{p} h_3^0h_3^0h_3^0C_0 \xrightarrow{\delta_2} h_3^0h_3^0h_3^0C_0 \rightarrow 0;
0 \rightarrow h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0C_0 \xrightarrow{1/p} h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0C_0 \xrightarrow{p} h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0C_0 \xrightarrow{\delta_3} h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0h_3^0C_0 \rightarrow 0.
\]

It is easy to prove that
\[
\text{coker } \delta_0 = C_0^0 \oplus C_1^1 \oplus \sim h_3^3C_0 \oplus h_3^2C_0,
\text{coker } \delta_1 = \sim h_2^2C_1 \oplus h_2^2h_3^3C_0 \oplus h_2^1h_3^1C_0,
\text{coker } \delta_2 = h_2^0h_3^0h_3^1C_0,
\text{coker } \delta_3 = 0.
\]

So we can construct \( p \)-torsion submodules \( B^* \) of \( H^{*,*}L_1 \):
\[
B^0 = \sim C_0,
B^1 = \sim C_1 \oplus \sim h_3^3C_0 \oplus h_3^2C_0,
B^2 = \sim h_2^2C_1 \oplus h_2^2h_3^3C_0 \oplus h_2^1h_3^1C_0,
B^3 = h_2^0h_3^0h_3^1C_0,
B^k = 0 \ (k \geq 4).
\]
such that the following diagram is commutative:

\[
\cdots \to H^* M_0^1 \to B^* \to B^* \to H^{*+1} M_0^1 \to \cdots
\]

\[
\cdots \to H^* M_0^1 \to H^* L_1^1 \to H^* L_1^1 \to H^{*+1} M_0^1 \to \cdots
\]

Then from [5], Remark 3.11, we see that

\[
H^{*,*} L_1^1 = \left( \widehat{C_0} \oplus \widehat{C_1} \right) \otimes E[\widehat{h_2^1}] \oplus \left( \widehat{h_3^1} C_0 \otimes E[h_2^0, h_3^0] \right).
\]

Similarly, we see that \( H^{*,*} L_1^i = 0 \) for \( s > 3 \) and the Adams-Novikov \( E_2 \)-term for \( \pi_*(L_2 T(1)/(p^\infty, v_1)) \) collapses. This completes the proof of Theorem 2.2. \( \square \)

References


[16] Wang, X., $\pi_*(L_2T(1)/(v_1))$ and its applications in computing $\pi_*(L_2T(1))$ at the prime two, Forum Math. 19 (2007), 127-147. MR2296069 (2008a:55010)

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