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GLOBAL STABILITY OF A CLASS OF NON-AUTONOMOUS DELAY DIFFERENTIAL SYSTEMS

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ABSTRACT. This paper is concerned with a class of systems of non-autonomous delay differential equations which are defined on the non-negative function space. Under proper conditions, we employ a novel proof to establish several criteria of the global stability of a positive equilibrium. Moreover, we give two examples to illustrate our main results.

1. Introduction

In this paper, we consider the following system of delay differential equations:

$$x'(t) = f(t, x_t), \tag{1.1}$$

where $f : R^1 \times \prod_{i=1}^{n} C([-r_i, 0], R^1) \to R^n$ is completely continuous, $f(t, \varphi)$ is locally Lipschitzian in φ, $f(t, \widehat{0}) \equiv f(t, \widehat{x^*}) \equiv 0$ with $x^* \triangleq (x_1^*, x_2^*, \cdots, x_n^*) \in IntR_+^n$, and r_i is a positive constant for $i = 1, 2, \cdots, n$.

To further our discussion, we also assume that the functional $f(t, \varphi)$ is almost periodic in t for fixed $\varphi \in \prod_{i=1}^{n} C([-r_i, 0], R^1)$.

When $n = 1$, system (1.1) is a class of general scalar delay differential equations, including the Nicholson blowflies equation and the logistic-type delay equation. Recently, the Nicholson blowflies equation and the logistic-type delay equation have served as the models for some population dynamics and ecology problems which have been extensively studied by many authors, and many interesting results have been obtained; see for example [1, 4, 7, 8]. Here we shall investigate the global stability of the positive equilibrium of (1.1). For the system with a large n, several methods have been used, such as the Liapunov functional method, the invariance principle of Liapunov-Rarumikhin type and the monotone method, to investigate the global dynamics of (1.1) (see [2, 3, 5, 6, 9]). Moreover, in the application of the Liapunov functional method, the authors often require that the Liapunov functional's derivative along (1.1) is less than or equal to 0 (see [3] and the references cited therein). Before illustrating our study method, we shall recall one

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recent paper [8] on the global attractivity of the positive steady state of the diffusive Nicholson equation. By combining a dynamical systems argument with the maximum principle and some subtle inequalities, the authors of [8] have obtained the global attractivity of the positive steady state of the diffusive Nicholson equation with homogeneous Neumann boundary value under a condition that makes the equation a non-monotone dynamical system. This synthetic method motivates us to integrate the Liapunov functional with other methods to analyze the global stability of the positive equilibrium of (1.1).

In fact, we first construct a "proper" Liapunov functional under some appropriate conditions. However, we usually cannot deduce the Liapunov functional's derivatives along (1.1). Therefore, some alternative methods are required. Then, by the contrary argument and the comparison technique, we show that the Liapunov functional along every non-trivial solution of (1.1) is either strictly decreasing or eventually 0. Finally, by applying the basic theory of the Liapunov direct method and the properties of the Liapunov functional, we obtain the global stability of (1.1).

As some applications, we also consider the following non-autonomous Nicholson blowflies equation with patch structure:

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + \beta x_i(t-\tau_i(t))e^{-x_i(t-\tau_i(t))} - dx_i(t),\ i = 1, 2, \cdots, n, \tag{1.2}$$

and non-autonomous delay logistic equation with patch structure:

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + k_i x_i(t)[\alpha_i - b_{i0}x_i(t) - \sum_{j=1}^{m} b_{ij}x_i(t-\tau_j(t))],\ i = 1, 2, \cdots, n. \tag{1.3}$$

The remaining part of this paper is organized as follows. In Section 2 we give some basic definitions and preliminary results. In Section 3, by applying the Liapunov direct method, we establish several criteria of the global stability of (1.1). In Section 4, we make some applications to the non-autonomous Nicholson blowflies equation and/or non-autonomous logistic equation with patch structure.

2. Definitions and preliminary results

In this paper, we denote by $R^n(R^n_+)$ the set of all (non-negative) real vectors. We will use $x = (x_1,\ x_2,\ \cdots, x_n) \in R^n$ to denote a column vector and define $|x| = \max_{1\le i\le n} |x_i|$. Let $(r_1, r_2, \cdots, r_n) \in IntR^n_+$ be given and $C = \prod_{i=1}^{n} C([-r_i, 0], R^1)$ be the continuous functions space equipped with the usual supremun norm $||\cdot||$. We also let

$$r = \max_{1\le i\le n} r_i,\ I = \{1, 2, \cdots, n\},\ C_+ = \prod_{i=1}^{n} C([-r_i, 0], R^1_+),\ C_H = \{\varphi \in C : ||\varphi|| \le H\}, \tag{2.1}$$

and

$$C_{H(L)} = \{\varphi \in C_H : |\varphi_i(t_1) - \varphi_i(t_2)| \le L|t_1 - t_2| \text{ for each } t_1, t_2 \in [-r_i, 0], i \in I\}, \tag{2.2}$$

where $H > 0$ and $L > 0$ are constants.

If $x_i(t)$ is defined on $[t_0 - r_i, \sigma)$ with $t_0, \sigma \in R^1$ and $i \in I$, then we define $x_t \in C$ as $x_t = (x^1_t, x^2_t, \cdots, x^n_t)$, where $x^i_t(\theta) = x_i(t + \theta)$ for all $\theta \in [-r_i, 0]$ and $i \in I$. For any $\varphi, \psi \in C$, we write $\varphi \leq \psi$ if $\psi - \varphi \in C_+$, $\varphi < \psi$ if $\varphi \leq \psi$ and $\varphi \neq \psi$. For any $x, y \in R^n$, we write $x \leq y$ if $x - y \in R^n_+$, $x < y$ if $x \leq y$ and $x \neq y$. Similarly, we can define "$\geq$" and "$>$". For $x = (x_1, x_2, \cdots, x_n) \in R^n$, we write $\widehat{x}$ for the element of C satisfying $\widehat{x} = (\widehat{x_1}, \widehat{x_2}, \cdots, \widehat{x_n})$, where $(\widehat{x_i})(\theta_i) = x_i$ for all $\theta_i \in [-r_i, 0]$ and $i \in I$.

We write $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ for a solution of the initial value problem (1.1) with $x_{t_0}(t_0, \varphi) = \varphi \in C$ and $t_0 \in R^1$. Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of existence of $x_t(t_0, \varphi)$.

Definition 2.1 (3)**.** A continuous function $F(t) : R^1 \to R^n$ is called almost periodic if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that any segment $[\alpha, \alpha + l], \alpha \in R^1$, contains at least one number τ such that $|F(t+\tau) - F(t)| < \varepsilon$ for every $t \in R^1$. A number τ is called an ε-almost period of F.

Definition 2.2 (3)**.** A continuous functional $F(t, \varphi) : R^1 \times C_{H(L)} \to R^n$ is called uniformly almost periodic in t if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that any segment $[\alpha, \alpha + l], \alpha \in R^1$, contains at least one number τ such that $|F(t + \tau, \varphi) - F(t, \varphi)| < \varepsilon$ for all $t \in R^1,\ \varphi \in C_{H(L)}$.

Lemma 2.1 (3)**.** *Let $F_1(t), \cdots, F_N(t) : R^1 \to R^n$ be almost periodic functions. Then for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that any segment $[\alpha, \alpha + l], \alpha \in R^1$, contains a number τ such that*

$$|F_i(t+\tau) - F_i(t)| < \varepsilon,\ i = 1, 2, \cdots, N, t \in R^1. \tag{2.3}$$

Lemma 2.2. *If L is a positive constant and if the functional $F(t, \varphi) : R^1 \times C_{H(L)} \to R^n$ is Lipschitzian in φ and almost periodic in t for every fixed $\varphi \in C_{H(L)}$, then it is uniformly almost periodic in t.*

Proof. Since the functional $F(t, \varphi)$ satisfies Lipschitz conditions in φ, then there exists a constant $L_1 > 0$ such that

$$|F(t, \varphi) - F(t, \varphi)| \leq L_1 \|\varphi - \psi\|, \quad \text{for all} \quad \varphi, \psi \in C_{H(L)}. \tag{2.4}$$

Let $\varepsilon > 0$ be any real number. From the definition of $C_{H(L)}$ and the Ascoli-Arzelà Theorem, we have that $C_{H(L)}$ is a compact set. Hence there exist $\varphi^1, \varphi^2, \cdots, \varphi^N \in C_{H(L)}$ such that

$$\min_{i \in I_N} \|\varphi - \varphi^i\| < \frac{\varepsilon}{3L_1} \quad \text{for all} \quad \varphi \in C_{H(L)}, \tag{2.5}$$

where $I_N \triangleq \{1, 2, \cdots, N\}$. By Lemma 2.1, it follows that there exists $l > 0$ such that in any segment $[\alpha, \alpha + l]$ there exists a number τ such that

$$|F(t, \varphi^i) - F(t + \tau, \varphi^i)| < \frac{\varepsilon}{3} \quad \text{for all} \quad t \in R^1,\ i \in I_N. \tag{2.6}$$

Then, for every $\varphi \in C_{H(L)}$, there exists $k \in I_N$ such that

$$\|\varphi - \varphi^k\| < \varepsilon / 3L_1.$$

Thus, from (2.4)$-$(2.6), we obtain

$$\begin{aligned} &|F(t+\tau, \varphi) - F(t, \varphi)| \leq |F(t+\tau, \varphi) - F(t+\tau, \varphi^k)| \\ &+|F(t+\tau, \varphi^k) - F(t, \varphi^k)| + |F(t, \varphi^k) - F(t, \varphi)| \\ &< \tfrac{\varepsilon}{3} + 2L_1 \cdot \tfrac{\varepsilon}{3L_1} = \varepsilon. \end{aligned} \tag{2.7}$$

The inequality (2.7) implies Lemma 2.2. □

Remark 2.1. By locally Lipschitzian conditions and the compactness of $C_{H(L)}$, we obtain that the functional $f(t,\varphi)$ is Lipschitzian in $C_{H(L)}$. Again by Lemma 2.2, $F(t,\varphi) = f(t,\varphi)|_{R\times C_{H(L)}}$ is uniformly almost periodic in t.

Lemma 2.3 ([3, Lemma 3]). *For a fixed $\varphi^0 \in C$, we suppose that $x_t(t_0,\varphi^0)$ belongs to C_H for all $t \geq t_0$. Then there exist constants $L > 0$ and $r^* > r$ such that*

(i) $x_t(t_0;\varphi^0) \in C_{H(L)}$ for all $t \geq t_0 + r^*$.

(ii) Let $\{\varepsilon_k\}_{k=1}^{+\infty}$ be a monotonically approaching zero sequence of positive numbers and $\{\tau_k : \tau_k > r^*\}_{k=1}^{+\infty}$ be a sequence of ε_k-almost periods of $f(t,\varphi)$ for all $\varphi \in C_{H(L)}$(for every ε_k there corresponds an ε_k-almost period τ_k). Then, for a given $t^* > t_0$, we get

$$\lim_{k\to+\infty} \|x_{t^*}(t_0,\varphi^k) - x_{t^*+\tau_k}(t_0,\varphi^0)\| = 0 \tag{2.8}$$

with $\varphi^k = x_{t_0+\tau_k}(t_0,\varphi^0)$.

Remark 2.2. From Definition 2.1, we can choose a monotonically approaching $+\infty$ sequence $\{\tau_k : \tau_k > r^*\}_{k=1}^{+\infty}$ satisfying Lemma 2.3.

3. Main results

We first introduce the following proposition:

Proposition 3.1. *Suppose that the system* (1.1) *has a unique solution $x_t(t_0,\varphi)$ on $[t_0,\ +\infty)$ with $x_{t_0}(t_0,\varphi) = \varphi \in C_+$. We also assume that there exists a positive constant m such that*

$$\|x_t(t_0,\varphi)\| \leq ||\varphi|| + m||x^*|| \text{ for all } s \geq t_0, \varphi \in C_+. \tag{3.1}$$

Let $V(\varphi) = ||\varphi - x^*|| \overset{\Delta}{=} \sup\{|\varphi_i(\theta) - x_i^*| : i \in I \text{ and } \theta \in [-r_i, 0]\}$ for all $\varphi \in C_+$. Moreover, there exist $T_1 \geq 0$ and $M \subseteq \{\varphi \in C_+ : \varphi(0) = 0\}$ with $0 \in M$ such that the following conditions hold:

(i) $x_t(t_0,\varphi) \in Int(C_+)$ for $\varphi \in C_+ \setminus M$ and $t \geq T_1 + t_0$.

(ii) If $\varphi \in C_+ \setminus M$ and $x_t(t_0,\varphi) \neq x^*$ for all $t \in [t_0,\ +\infty)$, then $V(x_t(t_0,\varphi))$ is strictly decreasing on $[T_1 + t_0, +\infty)$.

(iii) $\varphi \in C_+ \setminus M$ and $V(x_t(t_0,\varphi))$ is non-increasing on $[T_1 + t_0, +\infty)$.

Then x^* is a globally stable equilibrium on $C_+ \setminus M$.

Proof. We now prove that x^* is a stable equilibrium. For any $\epsilon > 0$, it follows from the continuous dependence of the solution of (1.1) that there is $\delta \in [0, \min\{\epsilon, \min\{x_i^* : i \in I\}\})$ such that $V(x_t(t_0,\varphi)) < \epsilon$ for all $t \in [t_0, T_1 + t_0]$ and $\varphi \in C_+ \setminus M$ with $V(\varphi) < \delta$. On the other hand, the assumption *(iii)* implies $V(x_t(t_0,\varphi)) \leq V(x_{T_1+t_0}(t_0,\varphi)) < \epsilon$ for all $t \in [T_1 + t_0, +\infty)$ and φ with $V(\varphi) < \delta$. Hence, $V(x_t(t_0,\varphi)) < \epsilon$ for all $t \in [t_0,\ +\infty)$ and φ with $V(\varphi) < \delta$. So, x^* is a stable equilibrium.

We next show that x^* attracts $C_+ \setminus M$. Given $\varphi \in C_+ \setminus M$, from *(iii)*, we have that $\lim\limits_{t\to+\infty} V(x_t(t_0,\varphi)) = V_0$ for some $V_0 \geq 0$. We shall show $V_0 = 0$; otherwise, $V_0 > 0$, and it is easy to see that $V(x_t(t_0,\varphi)) \geq V_0$ for all $t \in [T_1 + t_0, +\infty)$.

Set $H = ||\varphi|| + 2m||x^*||$. Then, from (3.1), we have

$$x_t(t_0,\varphi) \in C_H \quad \text{for all} \quad t \geq t_0.$$

By Lemma 2.3, we can choose positive constants L and $r^* > r$ such that

$$x_t(t_0,\varphi) \in C_{H(L)} \quad \text{for all} \quad t \geq t_0 + r^*. \tag{3.2}$$

By Remark 2.2, let $\{\varepsilon_k\}_{k=1}^{+\infty}$ be a monotonic sequence of positive numbers and $\{\tau_k : \tau_k \geq r^*\}_{k=1}^{+\infty}$ be a monotonic sequence of ε_k-almost periods of $f(t,\varphi)$ for all $\varphi \in C_{H(L)}$ (for every ε_k there corresponds an ε_k-almost period τ_k) such that $\lim_{k\to+\infty}\varepsilon_k = 0$ and $\lim_{k\to+\infty}\tau_k = +\infty$. Then, again by Lemma 2.3, for a given $t^* \geq t_0$, we get

$$\lim_{k\to+\infty} \|x_{t^*}(t_0,\varphi^k) - x_{t^*+\tau_k}(t_0,\varphi)\| = 0 \tag{3.3}$$

with $\varphi^k = x_{t_0+\tau_k}(\varphi)$.

Without loss of generality, we may assume that

$$\lim_{k\to+\infty} \varphi^k = \varphi^* \text{ for some } \ \varphi^* \in C_+.$$

From (3.3) and the continuity of $V(\cdot)$ and $x_t(t_0,\cdot)$, we obtain

$$V(x_t(t_0,\varphi^*)) = \lim_{k\to+\infty} V(x_t(t_0,\varphi^k)) = \lim_{k\to+\infty} V(x_{t+\tau_k}(t_0,\varphi)) = V_0 \ \text{ for all } \ t \geq t_0. \tag{3.4}$$

We shall prove that

$$x_{t_1}(t_0,\varphi^*) \notin M \text{ for some } \ t_1 \in [t_0, t_0+r].$$

Otherwise,

$$x_{t_0+r}(t_0,\varphi^*) = 0 \text{ and } \ V_0 = \|x^*\|,$$

which, together with (3.3), imply that

$$\lim_{k\to+\infty} \|x_{t_0+r+\tau_k}(t_0,\varphi)\| = 0.$$

On the other hand, by the assumptions $(i), (iii)$ and $\varphi \in C_+ \setminus M$, we have

$$x_t(t_0,\varphi) \in Int(C_+) \text{ for all } \ t \geq T_1 + t_0$$

and

$$V(x_t(t_0,\varphi)) \ \text{ is non-increasing on } \ [T_1+t_0,+\infty).$$

Obviously, there exists a positive integer k_0 such that the following statements hold:

(1) $x_{t_0+r+\tau_k}(t_0,\varphi) \in Int(C_+)$ for $k \geq k_0$;

(2) $V(x_{t_0+r+\tau_k}(t_0,\varphi))$ is non-increasing for $k \geq k_0$;

(3) $x_{t_0+r+\tau_k}(t_0,\varphi) \in C_{H^*}$ for $k \geq k_0$, where $H^* = \min_{i\in I}\{\frac{x_i^*}{5}\}$.

It follows from (1), (3) and the definition of $V(\cdot)$ that

$$\begin{aligned} V(x_{t_0+r+\tau_k}(t_0,\varphi)) &= \sup\{x_i^* - x_i(t_0+r+\tau_k+\theta_i; t_0,\varphi) : i \in I \text{ and } \theta_i \in [-r_i,0]\} \\ &< \|x^*\| \text{ for all } \ k \geq k_0. \end{aligned}$$

This and (2) imply that

$$V(x_{t_0+r+\tau_k}(t_0,\varphi)) \leq V(x_{t_0+r+\tau_{k_0}}(t_0,\varphi)) < \|x^*\| \text{ for all } \ k \geq k_0.$$

Hence, by (2) and (3.4), we obtain $\lim_{k\to+\infty} V(x_{t_0+r+\tau_k}(t_0,\varphi)) < \|x^*\|$, a contradiction to $V_0 = \|x^*\|$.

So, there exists $t_1 \in [t_0, t_0+r]$ such that

$$x_{t_1}(t_0,\varphi^*) \in C_+ \setminus M.$$

Let $\varphi^{**} = x_{t_1}(t_0,\varphi^*)$. Then,

$$\varphi^{**} \in C_+ \setminus M \ \text{ and } \ x_t(t_0,\varphi^*) = x_t(t_1, x_{t_1}(t_0,\varphi^*)) = x_t(t_1,\varphi^{**}) \text{ for all } t \geq t_1.$$

By applying the conditions $(i)-(iii)$ of Proposition 3.1 to φ^{**}, we have

$$V(x_t(t_0,\varphi^*)) \ \text{ is strictly decreasing for all } \ t \geq T_1 + t_1,$$

a contradiction to (3.4). This completes the proof Proposition 3.1. □

In the remainder of this paper, we shall introduce the following assumptions to guarantee the global stability of system (1.1):

(A1) If $\varphi \in C_+$ and $t \in [t_0, \eta(\varphi))$, then $x_t(t_0, \varphi) \in C_+$.

(A2) If $\varphi \in C_+ \setminus \{0\}$ and $t \in [t_0 + r,\ \eta(\varphi))$, then $x_t(t_0, \varphi) \in IntC_+$.

(A3) $f(t, \widehat{0}) = f(t, \widehat{x^*}) \equiv 0$, where $x^* \triangleq (x_1^*, x_2^*, \cdots, x_n^*) \in IntR_+^n, t \in R^1$.

(A4) For some i and $\varphi \in C_+ \setminus \{x^*\}$ such that $\varphi_i(0) - x_i^* \geq ||\varphi - x^*||$, we have $f_i(t, \varphi) < 0$ for $t \in R^1$.

(A5) For some i and $\varphi \in IntC_+ \setminus \{x^*\}$ such that $x_i^* - \varphi_i(0) \geq ||\varphi - x^*||$, we have $f_i(t, \varphi) > 0$ for $t \in R^1$.

Proposition 3.2. *Let* (A1) *and* (A4) *hold. If* $\varphi \in C_+ \setminus \{0\}$*, then the set of* $\{x_t(t_0; \varphi) : t \in [t_0, \eta(\varphi))\}$ *is bounded and hence* $\eta(\varphi) = +\infty$.

Proof. For $\varphi \in C_+ \setminus \{0\}$, we claim that $||x_t(t_0, \varphi) - x^*|| \leq ||\varphi|| + 2||x^*||$ for all $t \in [t_0, \eta(\varphi))$. Otherwise, there exists $t_1 \in (t_0, \eta(\varphi))$ such that

$$||x_{t_1}(t_0, \varphi) - x^*|| > ||\varphi|| + 2||x^*||.$$

Let

$$t^* = \inf\{t : t \in [t_0, \eta(\varphi)), ||x_t(t_0, \varphi) - x^*|| > ||\varphi|| + 2||x^*||\}.$$

Then $t^* \in (t_0, t_1)$ and

$$||\varphi|| + 2||x^*|| = ||x_{t^*}(t_0, \varphi) - x^*|| \geq ||x_t(t_0, \varphi) - x^*|| \quad \text{for all } t \in [t_0, t^*]. \tag{3.5}$$

In view of (3.5) and $x_t(t_0, \varphi) \in C_+$, there exist $t^{**} \in [t_0, t^*]$ and $i \in I$ such that

$$x_i(t^{**}; t_0, \varphi) - x_i^* = |x_i(t^{**}; t_0, \varphi) - x_i^*| = ||\varphi|| + 2||x^*|| \geq ||x_{t^{**}}(t_0, \varphi) - x^*||.$$

By (A4), we have $f_i(t^{**}, x_{t^{**}}(t_0, \varphi)) < 0$. On the other hand, it follows from system (1.1) and the choice of t^{**} that $f_i(t^{**}, x_{t^{**}}(t_0, \varphi)) = x_i'(t^{**}; t_0, \varphi) \geq 0$, a contradiction. This implies that the claim holds. Thus,

$$||x_t(t_0, \varphi)|| \leq ||x_t(t_0, \varphi) - x^*|| + ||x^*|| \leq ||\varphi|| + 3||x^*||, \text{ for all } t \in [t_0, \eta(\varphi)).$$

Finally, according to Theorem 3.1 in [4, p. 45], we easily obtain $\eta(\varphi) = +\infty$. □

Theorem 3.1. *Let* (A1)–(A5) *hold. Then* x^* *is a globally stable equilibrium on* $C_+ \setminus \{0\}$.

Proof. Define $V : C_+ \to C_+$ such that $V(\varphi) = ||\varphi - x^*|| \triangleq \sup\{||\varphi_i(\theta) - x_i^*|| : i \in I \text{ and } \theta \in [-r_i, 0]\}$. Given $\varphi \in C_+ \setminus \{0\}$, obviously, by (A2), we have

$$x_t(t_0, \varphi) \in IntC_+ \quad \text{for all} \quad t \geq t_0 + r.$$

We next show that the following claims are true:

Claim (i). If there exists $T_0 \geq t_0$ such that $x_{T_0}(t_0, \varphi) = x^*$, then $x_t(t_0, \varphi) = x^*$ for all $t \geq T_0$.

Claim (ii). If $x_t(t_0, \varphi) \neq x^*$ for all $t \in [t_0, +\infty)$, then $V(x_t(t_0, \varphi))$ is strictly decreasing on $[t_0 + r, +\infty)$.

Claim (iii). $V(x_t(t_0, \varphi))$ is non-increasing on $[t_0 + r, +\infty)$.

(i) Obviously, Claim (i) follows from (A3).

(ii) If Claim (ii) does not hold, there exist $t_1, t_2 \in [r + t_0, +\infty)$ such that $t_1 < t_2$ and $V(x_{t_2}(t_0, \varphi)) \geq V(x_{t_1}(t_0, \varphi))$. By the choice of t_1 and t_2, we can deduce that there exists $t_3 \in [t_1, t_2]$ such that $||x(t_3; t_0, \varphi) - x^*|| = V(x_{t_3}(t_0, \varphi))$. Thus, there exists $i \in I$ such that either $x_i(t_3; t_0, \varphi) - x_i^* = V(x_{t_3}(t_0, \varphi))$ or $x_i(t_3; t_0, \varphi) - x_i^* = -V(x_{t_3}(t_0, \varphi))$. Note that $x_{t_3}(t_0, \varphi) \in Int(C_+) \setminus \{x^*\}$. If the former holds, then by (A4) we have $f_i(t_3, x_{t_3}(t_0, \varphi)) < 0$. On the other hand, it follows from (1.1) that $f_i(t_3, x_{t_3}(t_0, \varphi)) = x_i'(t_3; t_0, \varphi) \geq 0$, a contradiction. If the latter holds, then by (A5), (1.1) and a similar discussion, we can deduce a contradiction. Therefore, Claim (ii) follows.

(iii) Obviously, Claims (i) and (ii) imply Claim (iii).

Therefore, Theorem 3.1 follows from Proposition 3.1.

Theorem 3.1 is not completely satisfactory since it does not characterize the global stability of (1.1) solely in terms of properties of f. In the following, we give sufficient conditions on f to guarantee assumptions (A1) and (A2).

(A11) There exist $k = (k_1, k_2, \cdots, k_n) \in R_+^n$ and a non-negative map $g : R^1 \times C_+ \to R_+^n$ such that for all $\varphi \in C_+$, $f(t, \varphi) = -\text{diag}(k_1, k_2, \cdots, k_n)\varphi(0) + g(t, \varphi)$.

(A12) For each $\varphi \in C_+$, denote $\bar{I} = \{i : \varphi_i(\theta_i) = 0 \text{ for all } \theta_i \in [-r_i, 0], i \in I\}$ and $\bar{J} = \{i : \varphi_i(\theta_i) > 0 \text{ for all } \theta_i \in [-r_i, 0], i \in I\}$. If $\bar{I} \neq \emptyset$, $\bar{J} \neq \emptyset$ and $\bar{I}^\sharp + \bar{J}^\sharp = n$, then there exists $i \in \bar{I}$ such that $f_i(t, \varphi) > 0$ for all $t \in R^1$. ☐

Lemma 3.1. *Let* (A11)*,* (A12) *and* (A4) *hold. Then for any $\varphi \in C_+$, we have either $x_t(t_0, \varphi) = 0$ for all $t \geq t_0$ or $x_t(t_0, \varphi) \in IntC_+$ for all $t \geq t_0 + (n+2)r$.*

Proof. For $\varphi \in C_+$, using the assumption (A11) and Theorem 5.2.1 in [6, p. 81], we have $x_t(t_0, \varphi) \in C_+$ for all $t \in [t_0, \eta(\varphi))$, which, together with (A4), implies that the conclusion of Proposition 3.2 holds. Moreover, if $\varphi_i(0) > 0$ for some i, then $x_i(t; t_0, \varphi) > 0$ for all $t \in [t_0, +\infty)$. Hence for any $i \in I$, either $x_i(t; t_0, \varphi) > 0$ for all $t \geq t_0 + r$ or $x_i(t; t_0, \varphi) = 0$ for all $t \in [t_0, t_0 + r]$. Assume, by way of a contradiction, that the conclusion of Lemma 3.1 does not hold. Then there exists $t_1 \in [t_0, t_0 + r]$ such that $x_i(t_1; t_0, \varphi) > 0$ for some i.

Let $M_t = \{i \in I : x_i(t; t_0, \varphi) > 0\}$, where $t \geq t_0$. It follows that $M_{t_1} \neq \phi$ and $M_s \subseteq M_t$ when $t_0 \leq s \leq t$.

Claim. If $t^* \in R_+^1$ and $M_{t^*} \notin \{\phi, I\}$, then $M_{t^*} \neq M_{t^*+r}$.

If the claim is not true, then $M_t = M_{t^*}, \;\; t \in [t^*, t^* + r]$. Thus, it follows from (A12) that there exists $i \in I \setminus M_{t^*+r}$ such that $f_i(t^* + r, x_{t^*+r}(t_0, \varphi)) > 0$. Hence, from (1.1), we obtain

$$x_i{}'(t^* + r; t_0, \varphi) = f_i(t^* + r, x_{t^*+r}(t_0, \varphi)) > 0.$$

Thus, there exists $\varepsilon > 0$ such that

$$\frac{\mathrm{d}(x_i(t; t_0, \varphi))}{\mathrm{d}t} > 0, \quad t \in [t^* + r - \varepsilon, t^* + r].$$

Since $x_t(t_0, \varphi) \geq 0, \;\; t \geq t_0$, we have $x_i(t^* + r; t_0, \varphi) > 0$. So, it follows that $i \in M_{t^*+r}$, which yields a contradiction. This completes the proof of the claim.

Now, we will show that $M_{t_1+(n-1)r} = I$. Otherwise, by the above claim, we have

$$\phi \neq M_{t_1} \subseteq M_{t_1+r} \subseteq \ldots \subseteq M_{t_1+(n-1)r} \subseteq M_{t_1+nr},$$

and $M_{t_1+ir} \neq M_{t_1+(i-1)r}, \;\; i \in I$. But this contradicts $M_t \subseteq I$. This completes the proof of Lemma 3.1. ☐

Remark 3.1. For $\varphi \in \{\varphi \in C_+ : \varphi(0) > 0\}$, using the proof used in Lemma 3.1, we have that $x_t(t_0;\varphi) \in IntC_+$ for all $t \geq t_0 + (n+2)r$.

Theorem 3.2. *Let* (A11), (A12) *and* (A3)–(A5) *hold. Then x^* is a globally stable equilibrium in the set of* $\{\varphi \in C_+ : \varphi(0) > 0\}$.

Proof. Suppose $\varphi \in C_+$ with $\varphi(0) > 0$. Then by Remark 3.1, we have $x_t(t_0,\varphi) \in IntC_+$ for all $t \geq (n+2)r + t_0$. Define $V : C_+ \to C_+$ such that $V(\varphi) = ||\varphi - x^*|| \triangleq \sup\{||\varphi_i(\theta) - x_i^*|| : i \in I \text{ and } \theta \in [-r_i, 0]\}$. Using a similar argument as in Theorem 3.1, we may deduce that the following statements are true:

(i) If there exists $T_0 \geq t_0$ such that $x_{T_0}(t_0,\varphi) = x^*$, then $x_t(t_0,\varphi) = x^*$ for all $t \geq T_0$.

(ii) If $x_t(t_0,\varphi) \neq x^*$ for all $t \in R_+^1$, then $V(x_t(t_0,\varphi))$ is strictly decreasing on $[t_0 + (n+2)r,\ +\infty)$.

(iii) $V(x_t(t_0,\varphi))$ is non-increasing on $[t_0 + (n+2)r,\ +\infty)$. □

Consequently, Theorem 3.2 follows from Proposition 3.1. This completes the proof of Theorem 3.2.

Next, consider the following systems of delay differential equations:

$$x_i'(t) = f_i(t,x_t),\ f_i(t,x_t) = L_i x_t + x_i(t) g_i(t,x_t), i \in I, \tag{3.6}$$

where $\bar{L} \triangleq (L_1, L_2, \cdots, L_n) : C_+ \to R^n$ is a bounded linear operator and $g \triangleq (g_1, g_2, \cdots, g_n) : R^1 \times C_+ \to R^n$.

Here, we give the following assumptions:

(C1) If $\varphi \in C_+$ and $\varphi_i(0) = 0$ for some i, then $L_i(\varphi) \geq 0$.

(C2) Denote $f(t,\varphi) = \bar{L}\varphi + \text{diag}\varphi(0) g(t,\varphi)$ for all $\varphi \in C_+$. There exists $x^* \in IntR_+^n$ such that $f(t,x^*) = 0$ and f satisfies assumptions (A4) and (A5).

(C3) f satisfies assumption (A12).

In view of (C1), it follows from Proposition 1.2 of [6] that system (3.6) satisfies condition (A1). Moreover, together with (A12), we can show that Lemma 3.1 also holds. Hence, by using a similar argument as in the proof of Theorem 3.2, we obtain the following result.

Theorem 3.3. *Let* (C1)–(C3) *hold. Then x^* is a global stability equilibrium in the set of* $\{\varphi \in C_+ : \varphi(0) > 0\}$.

4. Applications

In this section, we give several examples to illustrate the applications of the main results in Section 2.

Example 4.1. Consider the following non-autonomous Nicholson blowflies models with multi-patches:

$$x_i'(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \beta x_i(t - \tau_i(t)) e^{-x_i(t-\tau_i(t))} - d x_i(t),\ \text{for } i = 1, 2, \cdots, n, \tag{4.1}$$

where $\tau_i(t) : R^1 \to (0, +\infty)$ is almost periodic, $\beta > 0$, $A = (a_{ij})_{n\times n}$ is a cooperative and irreducible matrix, $r_i = \sup_{t\in R^1} \tau_i(t)$ and $(r_1, r_2, \cdots, r_n) \in IntR_+^n$. In what follows, we always assume that $\frac{\beta}{d} \in [e, e^2]$ and $\sum_{j\neq i} a_{ij} = -a_{ii}$ for all i.

Lemma 4.1 ([8, Lemma 2.3])**.** *Let* $\bar{\beta} = \frac{\beta}{d}$*. If* $a \geq 0$ *and* $b \geq 0$*, then we have the following results:*

(i) *If* $a - \ln \bar{\beta} \geq |b - \ln \bar{\beta}|$*, then* $-a + \bar{\beta} b e^{-b} \leq 0$*. Moreover,*$-a + \bar{\beta} b e^{-b} = 0$ *if and only if* $a = b = \ln \bar{\beta}$*.*

(ii) *If* $\ln \bar{\beta} - a \geq |b - \ln \bar{\beta}|$*, then* $-a + \bar{\beta} b e^{-b} \geq 0$*. Moreover,*$-a + \bar{\beta} b e^{-b} = 0$ *if and only if either* $a = b = \ln \bar{\beta}$ *or* $a = b = 0$*.*

In Example 4.1*, let* $x^* = (\ln \frac{\beta}{d}, \ln \frac{\beta}{d}, \cdots, \ln \frac{\beta}{d})$*,* $D = \mathrm{diag}\{d, d, \cdots, d\}$ *and* $f(t, \varphi) = A\varphi(0) - D\varphi(0) + \beta h(t, \varphi)$*, where*

$$h(t, \varphi) = \mathrm{diag}\{\varphi_1(-\tau_1(t)) e^{\varphi_1(-\tau_1(t))}, \varphi_2(-\tau_2(t)) e^{\varphi_2(-\tau_2(t))}, \cdots, \varphi_n(-\tau_n(t)) e^{\varphi_n(-\tau_n(t))}\}.$$

Remark 4.1. (i) For some i and $\varphi \in C_+ \setminus \{x^*\}$ such that $\varphi_i(0) - x_i^* \geq ||\varphi - x^*||$, by Lemma 4.1 (i), we have

$$\begin{aligned} f_i(t, \varphi) &= \sum_{j=1}^{n} a_{ij} \varphi_j(0) + \beta \varphi_i(-\tau_i(t)) e^{-\varphi_i(-\tau_i(t))} - \varphi_i(0) \\ &\leq \sum_{j=1}^{n} a_{ij} \varphi_i(0) + d(-\varphi_i(0) + \frac{\beta}{d} \varphi_i(-\tau_i(t)) e^{-\varphi_i(-\tau_i(t))}) < 0, \ t \in R^1. \end{aligned}$$

(ii) For some i and $\varphi \in IntC_+ \setminus \{x^*\}$ such that $x_i^* - \varphi_i(0) \geq ||\varphi - x^*||$, by Lemma 4.1 (ii), we obtain

$$\begin{aligned} f_i(t, \varphi) &= \sum_{j=1}^{n} a_{ij} \varphi_j(0) + \beta \varphi_i(-\tau_i(t)) e^{-\varphi_i(-\tau_i(t))} - \varphi_i(0) \\ &\geq \sum_{j=1}^{n} a_{ij} \varphi_i(0) + d(-\varphi_i(0) + \frac{\beta}{d} \varphi_i(-\tau_i(t)) e^{-\varphi_i(-\tau_i(t))}) > 0, \ t \in R^1. \end{aligned}$$

Theorem 4.1. *For system* (4.1)*, if* $\varphi \in C_+$ *with* $\varphi(0) > 0$*, then*

$$\lim_{t \to \infty} x(t; t_0, \varphi) = (\ln \frac{\beta}{d}, \ln \frac{\beta}{d}, \cdots, \ln \frac{\beta}{d}).$$

Proof. Since $A = (a_{ij})_{n \times n}$ is a cooperative and irreducible matrix, by Lemma 4.1 and Remark 4.1, the assumptions (A11), (A12) and (A3)–(A5) hold. Consequently, Theorem 4.1 follows from Theorem 3.2. □

Example 4.2. Consider the following non-autonomous delay differential systems:
(4.3)

$$x_i'(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + k_i x_i(t)[\alpha_i - b_{i0} x_i(t) - \sum_{j=1}^{m} b_{ij} x_i(t - \tau_j(t))], \text{ for } i = 1, 2, \cdots, n,$$

where $\tau_j(t) : R^1 \to (0, +\infty)$ is almost periodic, $k_i > 0$, $\alpha_i > 0$ and $b_{ij} \leq 0, j = 1, 2, \cdots, m$. $A = (a_{ij})_{n \times n}$ is a cooperative and irreducible matrix, $r_i = \sup_{t \in R^1} \tau_i(t)$ and $(r_1, r_2, \cdots, r_n) \in IntR_+^n$.

In this example, denote $b_i = \sum_{j=0}^{m} b_{ij} > 0$, and $\alpha_i^* = \frac{\alpha_i}{b_i}$, $x^* = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*)$. Also, denote $f(t, \varphi) = (f_1(t, \varphi), f_2(t, \varphi), \cdots, f_n(t, \varphi))$, where

$$f_i(t, \varphi) = \sum_{j=1}^{n} a_{ij} \varphi_j(0) + k_i \varphi_i(0)[\alpha_i - b_{i0} \varphi_i(0) - \sum_{j=1}^{m} b_{ij} \varphi_i(-\tau_j(t))] \text{ for } \varphi \in C_+.$$

Lemma 4.2. *If* $\alpha_i^* = \alpha_1^*$ *and* $\sum_{j\neq i} a_{ij} = -a_{ii}$ *for all* $i \in I$, *then* f *satisfies assumptions* (A3)–(A5) *of Section 3.*

Proof. Obviously, $x^* \in Int(R_+^n)$ and $f(t, x^*) = 0$; that is, (A3) follows. Let $\varphi_i(0) - x_i^* \geq ||\varphi - x^*||$ for some i and $\varphi \in C_+ \setminus \{x^*\}$. Then we have $\varphi_i(0) > x_i^*$ and

$$\begin{aligned} f_i(t,\varphi) &= \sum_{j=1}^{n} a_{ij}\varphi_j(0) + k_i\varphi_i(0)[\alpha_i - b_{i0}\varphi_i(0) - \sum_{j=1}^{m} b_{ij}\varphi_i(-\tau_j(t))] \\ &\leq \sum_{j=1}^{n} a_{ij}\varphi_i(0) + k_i\varphi_i(0)[\alpha_i - b_{i0}\varphi_i(0) - \sum_{j=1}^{m} b_{ij}\varphi_i(-\tau_j(t))] \\ &\leq \sum_{j=1}^{n} a_{ij}\varphi_i(0) + k_i\varphi_i(0)[\alpha_i - b_{i0}\varphi_i(0) - \sum_{j=1}^{m} b_{ij}\varphi_i(0)] \\ &\leq k_i\varphi_i(0)[\alpha_i - b_{i0}\varphi_i(0) - \sum_{j=1}^{m} b_{ij}\varphi_i(0)] \\ &\leq -k_i\varphi_i(0)(b_{i0} + \sum_{j=1}^{m} b_{ij})(\varphi_i(0) - \alpha_i^*) \\ &< 0, \end{aligned}$$

where $t \geq 0$. Thus, (A4) holds. By a similar argument as above, we can deduce that (A5) follows. This completes the proof. □

In what follows, we always assume that $\alpha_i^* = \alpha_1^*$ and $\sum_{j\neq i} a_{ij} = -a_{ii}$ for all $i \in I$.

Lemma 4.3. *If* $\varphi \in C_+$, *then* $x_t(t_0, \varphi)$ *exists and is unique on* $[t_0, +\infty)$.

Proof. Suppose $\varphi \in C_+$, Obviously, from Proposition 1.2 of [6], we obtain that $x_t(t_0, \varphi) \in C_+$ is unique for $t \in [t_0, \eta)$, where $[t_0, \eta)$ is the maximal right-interval of existence of $x(t; t_0, \varphi)$. We next show $\eta = \infty$. Denote $M_t = \sup\{|x_i(t + \theta; t_0, \varphi) - x_i^*| : i \in I \text{ and } \theta \in [t_0 - r_i, t_0]\}$ for all $t \in [t_0, \eta)$. We now claim M_t is non-increasing on $[t_0, \eta)$. Otherwise, without loss of generality, we may assume that there exists $i \in I$ and $t^* \geq t_0$ such that $x_i(t^*; t_0, \varphi) - \alpha_i^* > \sup\{|x_i(t; t_0, \varphi) - x_i^*| : i \in I \text{ and } t \in [t^* - r_i, t^*)\}$ and hence $x_i'(t^*; t_0, \varphi) \geq 0$. On the other hand, it follows from (4.3) that

$$\begin{aligned} x_i'(t^*; t_0, \varphi) \quad &= \quad \sum_{j=1}^{n} a_{ij}x_j(t^*; t_0, \varphi) + k_i x_i(t^*; t_0, \varphi)[\alpha_i - b_{i0}x_i(t^*; t_0, \varphi) \\ &\quad\quad - \sum_{j=1}^{m} b_{ij}x_i(t^* - \tau_j(t^*); t_0, \varphi)] \\ &\leq \quad k_i x_i(t^*; t_0, \varphi)[\alpha_i - b_{i0}x_i(t^*; t_0, \varphi) - \sum_{j=1}^{m} b_{ij}x_i(t^* - \tau_j(t^*); t_0, \varphi)] \\ &\leq \quad -k_i x_i(t^*; t_0, \varphi)(b_{i0} + \sum_{j=1}^{m} b_{ij})(x_i(t^*; t_0, \varphi) - \alpha_i^*) \\ &< \quad 0. \end{aligned}$$

This yields a contradiction, and hence the claim follows. Consequently, $M_t \leq M_0$ for all $t \in [t_0, \eta)$ and thus $x_i(t, t_0, \varphi) \leq M_0 + \alpha^*$ for all $t \in [t_0, \eta)$. Therefore, by Theorem 3.1 in [2, p. 45], $\eta = \infty$. This completes the proof. □

Theorem 4.2. *For system* (4.3), *if* $\varphi \in C_+ \setminus \{0\}$, *then* $\lim_{t\to\infty} x(t; t_0, \varphi) = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*)$.

Proof. Since $A = (a_{ij})_{n\times n}$ is a cooperative and irreducible matrix, by Lemmas 4.2 and 4.3, the assumptions (C1)–(C3) hold. Consequently, Theorem 4.2 follows from Theorem 3.3. □

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