CHARACTERIZATIONS OF THE SOLVABLE RADICAL

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Abstract. We prove that there exists a constant \( k \) with the property: if \( C \) is a conjugacy class of a finite group \( G \) such that every \( k \) elements of \( C \) generate a solvable subgroup, then \( C \) generates a solvable subgroup. In particular, using the Classification of Finite Simple Groups, we show that we can take \( k = 4 \). We also present proofs that do not use the Classification Theorem. The most direct proof gives a value of \( k = 10 \). By lengthening one of our arguments slightly, we obtain a value of \( k = 7 \).

1. Introduction

The aim of this paper is to prove the following theorem:

**Theorem A.** There exists a constant \( k \) with the property: if \( C \) is a conjugacy class of the finite group \( G \) such that every \( k \) elements of \( C \) generate a solvable subgroup, then \( C \) generates a solvable subgroup.

The most direct proof of Theorem A uses a value of \( k = 10 \). The ideas involved are similar to those used in the proofs of Hall's extended Sylow Theorems together with a little representation theory. After a preprint containing a proof of Theorem A was circulated, Gordeev et al. \[6\] used the Classification of Finite Simple Groups to prove Theorem A with a value of \( k = 8 \). By lengthening one of our arguments we are able to obtain a classification free proof with a value of \( k = 7 \). Using deeper representation theory, better results are possible; see \[2\] and \[5\]. As conjectured by Gordeev et al. \[6\], which has since been announced by them independently (see \[7\] \[8\]) we prove that, in fact, we can take \( k = 4 \). Our result was announced in \[12\]. Both proofs for \( k = 4 \) rely on the Classification Theorem. Note also that 4 is best possible (consider the conjugacy class of transpositions in \( S_n, n > 4 \)).

The second author, in a forthcoming paper, has classified all conjugacy classes of involutions in almost simple groups in which three conjugates always generate a solvable subgroup.

For many conjugacy classes, it is worth noting that Theorem 1.1 below implies that it is enough to consider pairs of elements in \( C \).

**Theorem 1.1** ([10]). Let \( C \) be a conjugacy class of the finite group \( G \) consisting of elements of prime order \( p \geq 5 \). Then \( C \) generates a solvable subgroup if and only if every pair of elements of \( C \) generates a solvable subgroup.
The corresponding result for nilpotency is true without any restriction on \( p \). This is the Baer–Suzuki theorem, which is well known and reasonably elementary. These results do not hold for all groups, but the finite case does yield the following results.

**Corollary 1.2.** Let \( k \) be a field and \( G \) a subgroup of \( GL(n, k) \).

1. If \( g \in G \), then the normal closure of \( g \) in \( G \) is solvable if and only if every 4 conjugates of \( g \) generate a solvable subgroup.
2. If \( k \) has characteristic 0 or \( p > 3 \) and \( g \in G \) is a unipotent element, then the normal closure of \( g \) in \( G \) is solvable if and only if every 2 conjugates of \( g \) generate a solvable subgroup.

Finally, we remark that we prove two results of independent interest. One is Lemma 3.4, where we show that if \( G \) is solvable and generated by a conjugacy class \( C \) and \( V \) is an irreducible \( G \)-module, then \( \dim C V(a) \leq (3/4) \dim V \) for \( a \in C \) (and this is best possible). This gives Corollary 3.5: if \( V \) is a minimal normal self-centralizing subgroup of a finite solvable group and \( G/V \) is generated by 5 elements in the conjugacy class \( C \), then so is \( G \). We also prove Theorem 2.4, which asserts that any involution in the automorphism group of a Chevalley group in odd characteristic inverts an element of odd prime order other than a long root element.

We would like to thank the referee for a careful reading of the initial version of this paper and for helpful suggestions. These included reorganizing the proof, and in particular stating Theorem 2.4 and Corollary 3.5 as separate results.

**2. Proof of Theorem A using the classification theorem**

We use standard terminology and notation. See [1] for the basic definitions.

Let \((C, G)\) be a minimal counterexample. Then every four elements of \( C \) generate a solvable subgroup, yet the subgroup generated by \( C \) is not solvable. Since \( |G| \) is minimal, it is clear that the solvable radical of \( G \) is trivial. The following lemma [10, Lemma 1] will be used to show that \( G \) must be almost simple.

**Lemma 2.1.** Suppose that \( G \) is a finite group such that the Fitting subgroup \( F(G) \) is trivial. Let \( L \) be a component of \( G \).

(a) If \( x \) is an element of \( G \) such that \( x \notin N_G(L) \) and \( x^2 \notin C_G(L) \), then there exists an element \( g \) in \( G \) such that \( \langle x, x^g \rangle \) is not solvable.

(b) If \( x \) is an element of \( G \) such that \( x \notin N_G(L) \) and \( x^2 \in C_G(L) \), then there exist elements \( g_1 \) and \( g_2 \) in \( G \) such that \( \langle x, x^{g_1}, x^{g_2} \rangle \) is not solvable.

Part (a) of Lemma 2.1 relies on the so-called \( \frac{3}{2} \)-generation result of Guralnick and Kantor [11]. Part (b) of Lemma 2.1 only relies on the fact that every finite simple group can be generated by two elements (see [3]). We can now show that \( G \) is almost simple.

**Lemma 2.2.** \( G \) is almost simple.

**Proof.** Let \( x \in C \) such that every four conjugates of \( x \) generate a solvable subgroup of \( G \) but that \( M := \langle x^C \rangle \) is not solvable.

Let \( N \) be a minimal normal subgroup of \( G \). Since \( G \) has no solvable normal subgroups, \( N = L \times \cdots \times L \) with \( L \) a nonabelian simple group.

By minimality, \( MN/N \) is solvable. If \( [x, N] = 1 \), then \( [M, N] = 1 \) and so \( M \) embeds in \( G/N \), whence \( M \) is solvable.
Table 1. List of exceptions to Theorem 2.3

<table>
<thead>
<tr>
<th>$G_0$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{PSL}(n, 3)$, $n &gt; 2$</td>
<td>transvection</td>
</tr>
<tr>
<td>$\text{PSp}(2n, 3)$, $n &gt; 1$</td>
<td>transvection</td>
</tr>
<tr>
<td>$\text{PSU}(n, 3)$, $n &gt; 2$</td>
<td>transvection</td>
</tr>
<tr>
<td>$\text{PSU}(n, 2)$, $n &gt; 3$</td>
<td>pseudoreflection of order 3</td>
</tr>
<tr>
<td>$\text{PΩ}^+(n, 3)$, $n &gt; 6$</td>
<td>long root element</td>
</tr>
<tr>
<td>$E_6(3), F_4(3), 2E_6(3), 3D_4(3)$</td>
<td>long root element</td>
</tr>
<tr>
<td>$G_2(3)$</td>
<td>long or short root element</td>
</tr>
<tr>
<td>$G_2(2)' \cong \text{PSU}(3, 3)$</td>
<td>transvection</td>
</tr>
</tbody>
</table>

Set $H = \langle x, N \rangle$. The normal closure of $x$ in $H$ contains $[x, N]$, which is a nontrivial normal subgroup of $N$, whence is not solvable. Thus, by minimality, $G = H$, whence $x$ acts transitively on the direct factors of $N$. By Lemma 2.1 this implies that $N = L$ is simple. Since $C_{G_0}(N)$ is central in $G$, this is trivial, whence $N$ is the unique minimal normal subgroup of $G$. Thus, $G$ is almost simple. □

Let $G_0$ be the socle of the almost simple group $G$. The Classification of Finite Simple Groups implies that $G_0$ is an alternating group, a simple group of Lie type, or a sporadic group. Since the solvable radical of $G$ is trivial and $(C, G)$ is a counterexample to the theorem, every four elements of $C$ generate a solvable group. Observe that it suffices to assume that the elements of $C$ have prime order. Indeed, the following theorem implies that we may assume that $C$ is a conjugacy class of involutions. It also precludes the vast majority of possibilities for $G_0$.

**Theorem 2.3** ([10]). Let $G$ be a finite almost simple group with socle $G_0$. Suppose that $x$ is an element of odd prime order in $G$. Then one of the following holds:

(i) There exists $g \in G$ such that $\langle x, x^g \rangle$ is not solvable.

(ii) $x^3 = 1$ and $(x, G_0)$ belongs to a short list of exceptions given in Table 1. Moreover, there exist $g_1, g_2 \in G$ such that $\langle x, x^{g_1}, x^{g_2} \rangle$ is not solvable, unless $G_0 \cong \text{PSU}(n, 2)$ or $\text{PSp}(2n, 3)$. In any case, there exist $g_1, g_2, g_3 \in G$ such that $\langle x, x^{g_1}, x^{g_2}, x^{g_3} \rangle$ is not solvable.

Now observe that if $\langle x, x^g \rangle$ is a 2-group for all $g \in G$, then $\langle x^G \rangle$ is nilpotent by the Baer–Suzuki Theorem. So if $x$ is an involution in an almost simple group there must exist a conjugate $x^{g_0}$ such that $\langle x, x^{g_0} \rangle$ is not a 2-group. Thus, $x$ inverts an element $y$ of odd prime order. So $(C, G)$ cannot be a minimal counterexample unless:

1. $G_0$ is one of the groups in Table 1
2. $C$ is a conjugacy class of involutions; and
3. if $x \in C$, then $x$ inverts no elements of odd prime order other than those in the listed conjugacy classes of Table 1

Thus, the result will follow from the next two results.

We first rule out the case that $G_0$ is a Chevalley group over the field of 3 elements by proving the following result.

**Theorem 2.4.** Let $G$ be a simple group of Lie type of rank at least 2 defined over a field of odd characteristic $p$. Let $R$ be the conjugacy class of long root elements
in $G$. If $G = G_2(3^k)$, let $\mathcal{R}'$ denote the conjugacy of short root elements of $G$. If $x \in \text{Aut}(G)$ is an involution, then $x$ inverts some element $h$ of odd prime order not in $\mathcal{R}$ (or $\mathcal{R} \cup \mathcal{R}'$ if $G = G_2(3^k)$).

Proof. Suppose the result is false and that $G$ is a minimal counterexample. Let $A = \langle x, G \rangle$.

By the Baer–Suzuki theorem, $x$ inverts some $z \in \mathcal{R}$. Let $P$ be the parabolic subgroup that normalizes $C_G(z)$. Let $L$ be a Levi subgroup of $P$ and $U$ the unipotent radical of $P$. Now observe that $N_A(L)$ has odd index in $N_A(U)$, and so a Sylow 2-subgroup of $N_A(U)$ is contained in $N_A(L)$. Thus we may assume that $x$ normalizes $L$. If $x$ centralizes $L$, then $x$ is in the central torus of $L$ (in particular, it is in a split torus). By conjugating by an element of the Weyl group, we may assume that $x$ does not centralize $[L, L]$. By minimality, it follows that either $[L, L]$ has rank 1 or $x$ inverts some $p$-element outside $\mathcal{R}$. If $[L, L]$ has rank 1, then either $G$ has rank 2 or $G = \text{SL}(4, q)$.

If $G = \text{SL}(4, q)$ and $x$ induces a graph automorphism, field or field-graph automorphism, then $x$ normalizes but does not centralize an $\text{SL}(3, q)$. If $x$ is an inner diagonal automorphism (of order 2), then either $x \in \text{Sp}(4, q)$ or $x$ normalizes but does not centralize an $\text{SL}(3, q)$. Thus, this case is eliminated by minimality.

If $G = \text{Sp}(4, q)$, $\text{SU}(4, q)$ or $\text{SU}(5, q)$, then $x$ acts on $Q := C_G(C_G(z))$, the largest normal $p$-subgroup of $C_G(z)$. This is a special group with commutator and center both equal to the root subgroup containing $z$. Moreover, all root elements in $Q$ are contained in $Z(Q)$. Since $x$ does not centralize $Z(Q)$, it cannot centralize $Q/Z(Q)$ and so it inverts some element of $Q/Z(Q)$, and this contradicts the choice of $x$.

Suppose that $G = \text{SL}(3, q)$. If $x$ is a field, graph or field-graph automorphism, then we can choose $x$ to normalize a maximal torus of order $(q^3 - 1)/(q - 1)$ and so it will invert some semisimple element of odd order. The one remaining conjugacy class of involutions is the unique conjugacy class of inner-diagonal involutory automorphisms. It is straightforward to see that $x$ inverts a regular unipotent element of order $p$.

Suppose that $G = G_2(q)$. Note that if $3|q$, then $x$ cannot involve a graph automorphism (for then $x$ interchanges the conjugacy classes of long and short roots). So in all cases, we may assume that $x$ is in the group generated by inner and field automorphisms. Now $G$ contains subgroups $H$ of the form $\text{SU}(3, q)$ and $\text{SL}(3, q)$. Moreover, the normalizer of one of these will contain a Sylow 2-subgroup of $A$. So we may assume that our involution normalizes $H$ (and does not centralize $H$ since the centralizer of $H$ has odd order). By minimality, $x$ inverts some element of odd prime order in $H$ (which is not a long root element in $H$ and so not a root element in $G_2$).

The only remaining case to deal with is $G = ^3D_4(q)$. We can reduce to one of $\text{PGL}(3, q)$, $\text{PGU}(3, q)$ whose normalizer contains a Sylow 2-subgroup of $\text{Aut}(G)$. □

Next we rule out the case that $G = \text{PSU}(n, q)$ with $n > 3$ and $q = 2$.

Lemma 2.5. Let $G = \text{PSU}(n, 2), n > 3$. If $x \in \text{Aut}(G)$ is an involution, then $x$ inverts an element of odd order other than a pseudoreflection.

Proof. We can view any involution as acting semilinearly on the natural module for $\text{GU}(n, q)$. If the result fails, then by Baer–Suzuki, $x$ inverts some pseudoreflection and therefore normalizes $H := \text{SU}(n - 1, q)$. Since the centralizer of $H$ has order dividing 3, we may assume that $x$ normalizes but does not centralize $H$.
So the result follows by induction once we have handled the case $G = \text{SU}(4,2)$. Since $\text{PSU}(4,2) \cong \text{PSp}(4,3)$ and pseudoreflections correspond to root elements, the previous result applies. \hfill \Box

3. A classification free approach

The purpose of this section is to explore what can be proved using only elementary means.

For $G$ a nontrivial solvable group, we let $f(G)$ denote the Fitting height of $G$. This is the smallest integer $n$ such that $G$ possesses a series

$$1 = F_0 \leq F_1 \leq \cdots \leq F_n = G$$

with $F_{i+1}/F_i$ nilpotent for all $i$. The trivial group has Fitting height 0; a nontrivial nilpotent group has Fitting height 1; and if $G \neq 1$, then $f(G/F(G)) = f(G) - 1$.

If $G \neq 1$ is solvable we define

$$\psi(G) = \bigcap \{K \leq G \mid f(G/K) < f(G)\}.$$

Now $G/\psi(G)$ is isomorphic to a subgroup of a direct product of groups each with Fitting height less than $f(G)$. Thus $f(G/\psi(G)) < f(G)$. It follows that

$$1 \neq \psi(G) \leq F(G)$$

and that $\psi(G)$ is the unique smallest normal subgroup of $G$ such that the corresponding quotient group has Fitting height less than the Fitting height of $G$.

**Lemma 3.1.** Let $H$ be a subgroup of the solvable group $G \neq 1$. If $f(H) = f(G)$, then

$$\psi(H) \leq \psi(G) \leq F(G).$$

**Proof.** Set $\overline{G} = G/\psi(G)$, so that $f(\overline{G}) < f(G)$. Then $f(\overline{H}) \leq f(\overline{G}) < f(G) = f(H)$, so the definition of $\psi(H)$ implies that $\psi(H) \leq \psi(G)$. We have already seen that $\psi(G) \leq F(G)$. \hfill \Box

**Lemma 3.2.** Let $G$ be a solvable group, let $N \trianglelefteq G$, set $\overline{G} = G/N$ and suppose that $\overline{G} \neq 1$. Then the following are equivalent:

(i) $\psi(G) \neq 1$.
(ii) $f(\overline{G}) = f(G)$.
(iii) $\psi(\overline{G}) = \psi(G)$.

**Proof.** Suppose that $\psi(\overline{G}) \neq 1$. Then $\psi(G) \not\leq N$, so the definition of $\psi(G)$ implies that $f(\overline{G}) = f(G)$. Thus (i) implies (ii).

Suppose that $f(\overline{G}) = f(G)$. Now $G/\psi(G)$ is a homomorphic image of $G/\psi(G)$, so $f(G/\psi(G)) \leq f(G/\psi(G)) < f(G) = f(\overline{G})$, whence $\psi(\overline{G}) \leq \psi(\overline{G})$. Let $K$ be the full inverse image of $\psi(\overline{G})$ in $G$. Then $G/K \cong G/\psi(G)$, so $f(G/K) < f(\overline{G}) = f(G)$, whence $\psi(G) \leq K$ and then $\psi(G) \leq K = \psi(\overline{G})$. We deduce that $\psi(\overline{G}) = \psi(G)$. Thus (ii) implies (iii).

Since $\overline{G} \neq 1$ we have $\psi(\overline{G}) \neq 1$, so (iii) implies (i). \hfill \Box

The next lemma is well known.

**Lemma 3.3.** Suppose that the solvable group $G$ possesses a unique minimal normal subgroup $V$. If $V$ has a complement in $G$, then $V$ acts transitively by conjugation on the set of complements to $V$ in $G$. 

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Proof. Suppose that $A$ is a complement to $V$ and let $Q$ be a minimal normal subgroup of $A$, so that $Q$ is a $q$-group for some prime $q$. Set $K = QV$. Now $C_V(Q) = 1$ since otherwise $Q$ would be another minimal normal subgroup of $G$. It follows that $V$ is an $r$-group for some prime $r \neq q$. Then $Q$ is a Sylow $q$-subgroup of $K$, and any complement to $V$ in $G$ is the normalizer of a Sylow $q$-subgroup of $K$. The result now follows from Sylow’s Theorem. □

The following extends a result that appears in [17, page 82]. One can construct examples to see that the result is best possible (take $G$ to be a subgroup of the monomial subgroup of $GL(4, k)$ with $a$ a transposition in $S_4 \leq G$).

Lemma 3.4. Let $G$ be a solvable group that possesses an element $a$ such that $G = \langle a^G \rangle$. Let $k$ be a field. Let $V$ be a nontrivial irreducible $kG$-module. Then

$$\dim C_V(a) \leq \frac{3}{4} \dim V.$$  

Proof. First note that by replacing $k$ by $\text{End}_{kG}(V)$, we may assume that $V$ is absolutely irreducible. Then we can extend scalars and assume that $k$ is algebraically closed. Clearly, we may assume that $G$ acts faithfully on $V$.

Assume false, so that $\dim C_V(a) > \frac{3}{4} \dim V$. We will construct a normal subgroup of $G$ that has more than one homogeneous component on $V$. The proof then proceeds by analyzing the permutation action of $G$ on those components.

Let $g, h \in G$. The subspaces $C_V(a)$ and $C_V(a^g)$ both have dimension greater than $\frac{3}{4} \dim V$, so their intersection has dimension greater than $\frac{1}{2} \dim V$. Since $[g, a]$ acts trivially on this intersection it follows that

$$\dim C_V([g, a]) > \frac{1}{2} \dim V.$$  

Repeating this argument, we deduce that

$$C_V([g, a]) \neq 0 \quad \text{and} \quad C_V([g, a, h]) \neq 0$$  

for all $g, h \in G$.

Since $G$ acts irreducibly and faithfully on $V$ we have

$$C_V(z) = 0$$  

for all $z \in Z(G)^\#$. In particular, $a \not\in Z(G)$. Let $N$ be a normal subgroup of $G$ chosen minimal subject to the fact that $N$ is not central in $G$. Now $N$ is solvable, so $N' < N$, whence $N' \leq Z(G)$. We claim that $N$ is abelian. If not, then $Z(N) < N$, whence $Z(N) \leq Z(G)$.

Since $G = \langle a^G \rangle$ and $N$ is not central, we see that $[a, N] \neq 1$. Choose $g \in N$ such that $[g, a] \neq 1$. Since $[g, a]$ fixes a nonzero vector in $V$, $[g, a]$ is a noncentral element of $N$. Now choose $h \in N$ with $1 \neq [g, a, h] \in N' \leq Z(G)$. Thus, $[g, a, h]$ is a nontrivial scalar on $V$, but by Equations 1, 2 this is not the case. Thus, $N$ is abelian.

Since $N$ is abelian and not central in $G$, $V = V_1 \oplus \ldots \oplus V_r$ is a direct sum of the $N$ eigenspaces $V_i$ with $r > 1$. Set $\Omega = \{V_1, \ldots, V_r\}$. Since $V$ is irreducible, $G$ acts transitively on $\Omega$. Since $G$ is generated by the conjugates of $a$, $a$ acts nontrivially on $\Omega$. Set $c = \dim V_i$.

We claim that $a$ fixes no more than $d/2$ points in any transitive permutation action of $G$ of degree $d > 1$. It suffices to prove this for a primitive action (if $a$ fixes no more than $1/2$ the blocks, it fixes no more than $1/2$ the points). In any
Let $\Delta$ be an $a$-orbit on $\Omega$ and $V_\Delta = \sum_{i \in \Delta} V_i$, then dim $C_V(a) \leq e$. Thus, dim $C_V(a) \leq ef$, where $f$ is the number of orbits of $a$ on $\Omega$. Since $a$ fixes at most $r/2$ points, it has at most $3r/4$ orbits on $\Omega$, whence dim $C_V(a) \leq (3/4)$ dim $V$, a contradiction. \hfill\square

We point out the following corollary.

**Corollary 3.5.** Let $G$ be a finite solvable group with $V$ a minimal normal self-centralizing subgroup of $G$. If $G = \langle a_1, \ldots, a_5, V \rangle$ with the $a_i$ all conjugate to $a$, then $G = \langle b_1, \ldots, b_5 \rangle$, where each $b_j$ is conjugate to $a$.

**Proof.** Clearly, $G = \langle a^G \rangle$ (since $a^G$) is a normal subgroup and acts irreducibly and nontrivially on $V$, whence it contains $V$). Now apply Lemma 3.3 to conclude that dim $C_V(a) \leq (3/4)$ dim $V$. Choose $v_i \in \langle a_i, V \rangle$. Note that $a_i v_i$ is conjugate to $a_i$ and so to $a$. Let $H(v_1, \ldots, v_5) = \langle a_1 v_1, \ldots, a_5 v_5 \rangle$. Note that any such subgroup covers $G/V$, and so either the result follows or each $H(v_1, \ldots, v_5)$ is a complement to $V$ in $G$.

Assume that this is the case. Note that the map $(v_1, \ldots, v_5) \rightarrow H(v_1, \ldots, v_5)$ is an injection (for if $a_i v_i$ and $a_i v_i'$ are both in $H(v_1, \ldots, v_5)$, then $V \cap H(v_1, \ldots, v_5) \neq 1$ and so $V \leq H(v_1, \ldots, v_5)$), whence $H(v_1, \ldots, v_5) = G$.

Thus, the number of complements to $V$ is at least $|a, V|^5 \geq |V|^5/4$. On the other hand, the number of complements to $V$ is at most $|V|$ by Lemma 3.3 This contradiction completes the proof. \hfill\square

**Lemma 3.6.** Let $G$ be a solvable group and let $a$ be an element of $G$. Suppose that $A$ is a subgroup of $G$ with the following properties:

(i) $A = \langle a_1, \ldots, a_5 \rangle$, where $a_1, \ldots, a_5$ are conjugate to $a$ in $G$ and conjugate to one another in $A$.

(ii) $A$ has maximal Fitting height subject to (i).

Then

$$\psi(A) \leq F(G).$$

**Proof.** Assume false and let $G$ be a minimal counterexample, so that $\psi(A) \not\leq F(G)$. Let $V$ be a minimal normal subgroup of $G$ and set

$$\overline{G} = G/V.$$

Now $V$ is abelian, so $V \leq F(G)$. In particular, $\psi(A) \not\leq V$, and then the definition of $\psi(A)$ implies that

$$f(\overline{A}) = f(A).$$

We claim that $\overline{A}$ satisfies (i) and (ii) when $G$ is replaced by $\overline{G}$ and $a$ by $\overline{a}$. Certainly (i) is satisfied. As for (ii), let $\overline{B}$ be a subgroup of $\overline{G}$ such that $\overline{B} = \langle \overline{b}_1, \ldots, \overline{b}_5 \rangle$ with $\overline{b}_1, \ldots, \overline{b}_5$ conjugate to $\overline{a}$ in $\overline{G}$ and conjugate to one another in $\overline{B}$. Let $b_1$ be a conjugate of $a$ that maps onto $\overline{b}_1$ and let $B$ be an inverse image of $\overline{B}$ that is minimal subject to $b_j \in B$. Choose $g_2, \ldots, g_5 \in B$ such that $\overline{b}_1 = \overline{g}_1$. Then $\langle b_1, b_1^{g_2}, \ldots, b_1^{g_5} \rangle$ is a subgroup of $B$ that maps onto $\overline{B}$. The minimality of $B$ forces

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\[ B = \langle b_1, b_2, \ldots, b_{1168} \rangle \] and as \( g_2, \ldots, g_5 \in B \) we see that \( B \) is a subgroup of \( G \) that satisfies (i). Consequently
\[ f(A) \geq f(B). \]
Now \( B \) is a homomorphic image of \( B \), so \( f(B) \geq f(B) \), and then using (3) we have
\[ f(A) \geq f(B). \]
This proves the claim.

The minimality of \( G \) and the previous paragraph imply that \( \psi(A) \leq F(G) \). Lemma 3.2 and 3 imply that \( \psi(A) = \overline{\psi(A)} \), so we deduce that
\[ \overline{\psi(A)} \leq F(G). \]

It follows readily that \( V \) is the unique minimal normal subgroup of \( G \). Indeed, if \( U \) were another such subgroup, then \( \langle \psi(A) \rangle \) would embed into the nilpotent group \( F(G/U) \times F(G/V) \), contrary to the fact that \( \psi(A) \not\leq F(G) \).

Since \( \psi(A) \) and \( F(G) \) are nilpotent and since \( \psi(A) \not\leq F(G) \), there exists a prime \( q \) such that \( O_q(\psi(A)) \not\leq O_q(G) \). Set \( Q = O_q(\psi(A)) \). By (1) we have \( Q \leq \overline{O_q(G)} \).

Let \( K \) be the full inverse image of \( O_q(G) \) in \( G \), so that \( Q \leq K \leq G \) and \( K/V \) is a \( q \)-group. Now \( G \) is solvable, so \( V \) is an elementary abelian \( r \)-group for some prime \( r \). Moreover, \( Q \not\leq O_q(G) \), so \( K \) is not a \( q \)-group and hence \( r \neq q \). Since \( V \) is the unique minimal normal subgroup of \( G \), we deduce that \( O_q(G) = 1 \).

We claim that
\[ C_K(V) = V. \]

Indeed, choose \( S \in \text{Syl}_q(K) \). Since \( K/V \) is a \( q \)-group we have \( K = SV \), whence \( C_K(V) = C_S(V) \times V \). Then \( C_S(V) = O_q(C_K(V)) \leq O_q(K) \leq O_q(G) = 1 \), proving the claim.

Suppose that \( AV \neq G \). Then the minimality of \( G \) implies that \( Q \leq O_q(AV) \), so as \( V \leq O_r(AV) \) we see that \( [Q, V] = 1 \), contrary to (5). We deduce that
\[ G = AV. \]

If \( f(A) = f(G) \), then Lemma 3 implies that \( Q \leq O_q(G) \), contrary to the choice of \( q \). Thus \( f(A) < f(G) \), and then the definition of \( A \) implies that \( G \) cannot be generated by 5 conjugates of \( a \). Moreover, we have \( A \neq G \), so using (3) and the fact that \( V \) is a minimal normal subgroup of \( G \) we deduce that \( A \cap V = 1 \), so \( A \) is a complement to \( V \) in \( G \).

Since \( A = \langle a_1, \ldots, a_5 \rangle \), Corollary 3.5 implies that \( G \) can be generated by 5 conjugates of \( a \), a contradiction. \( \square \)

The following lemma proves Theorem A.

**Lemma 3.7.** Let \( \mathcal{C} \) be a conjugacy class of the group \( G \). If every 10 members of \( \mathcal{C} \) generate a solvable subgroup, then \( \mathcal{C} \) generates a solvable subgroup.

**Proof.** Assume false and let \( G \) be a minimal counterexample. Then \( G \) possesses no nontrivial normal solvable subgroups, and we may suppose that the elements of \( \mathcal{C} \) have prime order. Let \( a \in \mathcal{C} \) and let \( A \) be a subgroup of \( G \) that satisfies

(1) \( A = \langle a_1, \ldots, a_5 \rangle \), where \( a_1, \ldots, a_5 \) are conjugate to \( a \) in \( G \) and conjugate to one another in \( A \), and

(2) \( A \) has maximal Fitting height subject to (1).
Replacing $A$ by a suitable conjugate, we may suppose that $a_1 = a$. Let $Q = \psi(A)$, so that $Q \neq 1$. Let $g \in G$ and set $H = \langle A, A^g \rangle$. By hypothesis, $H$ is solvable, so Lemma 3.6 with $H$ in place of $G$ yields $Q \leq F(H)$. Similarly, $Q^g \leq F(H)$. We deduce that $\langle Q, Q^g \rangle$ is nilpotent for all $g \in G$. The Baer–Suzuki Theorem implies that

$$Q \leq F(G).$$

This contradicts the fact that $G$ has no nontrivial normal solvable subgroups and completes the proof. □

Using a slightly longer argument we are able to replace 10 by 7. First we need:

**Lemma 3.8.** Suppose $A \neq 1$ is solvable and that $\psi(A) \leq Z(A)$. Then $A$ is abelian.

**Proof.** We have

$$f(A/\psi(A)) < f(A).$$

On the other hand, for any $n \geq 1$, the class of solvable groups of Fitting height $n$ is closed under central extensions. This forces $f(A/\psi(A)) = 0$, whence $A = \psi(A)$. As $\psi(A) \leq Z(A)$, the conclusion follows. □

**Theorem 3.9.** Let $C$ be a conjugacy class of the group $G$. If every 7 members of $C$ generate a solvable subgroup, then $C$ generates a solvable subgroup.

**Proof.** Proceed as in the proof of the previous lemma and construct the subgroup $A$.

We claim there is a prime $p$, a conjugate $b$ of $a$ and a $p$-subgroup $P$ with $1 \neq P \leq \psi(A) \cap \langle a, b \rangle$. If $[\psi(A), a] \neq 1$ there exists a prime $p$ and $x \in O_p(\psi(A))$ with $[a, x] \neq 1$. Put $b = a^x$ and $P = \langle [a, x] \rangle$. Suppose that $[\psi(A), a] = 1$. As $A = \langle a^A \rangle$, it follows that $\psi(A) \leq Z(A)$. The previous lemma implies that $A$ is abelian. Then $A = \langle a \rangle$. Put $b = a$ and $P = A$.

Let $g \in G$ and set $H = \langle A, a^g, b^g \rangle$. By hypothesis, $H$ is solvable. Lemma 3.6 implies $P \leq O_p(H)$. As $P^g \leq \langle a, b \rangle^g \leq H$ it follows that $\langle P, P^g \rangle$ is a $p$-group. A contradiction follows from the Baer–Suzuki Theorem. □

**4. Proof of the corollary**

The proof of Corollary 1.2 is standard. We first prove (1). We first note the well-known fact that if $H$ is a solvable subgroup of $GL(n, k)$, then the derived length of $H$ is bounded by a function $f = f(n)$.

So suppose that the normal closure $N$ of $g$ in $H$ is not solvable. Then there is some nontrivial element $x$ in the $f$th term in the derived series of $N$. We may pass to a subgroup of $G$ and assume that $G$ is finitely generated, and so $G \leq GL(n, R)$, where $R$ is a finitely generated ring over the prime field of $k$. We can choose a maximal ideal $M$ of $R$ such that $x$ is not in the congruence kernel of the map $\phi : GL(n, R) \to GL(n, R/M)$. Thus, $\phi(N)$ is not solvable and $\phi(G)$ is finite, whence some four conjugates of $\phi(g)$ generate a nonsolvable subgroup. Thus, the same is true for $G$.

The proof of (2) is essentially the same. First, as above, reduce to the case that $G$ is finitely generated and contained in $GL(n, R)$, where $R$ is a finitely generated ring over $\mathbb{Z}$. Now argue exactly as above (except that if the characteristic is 0, take $M$ to be a maximal ideal containing some prime $p > 3$), and so our unipotent element in the image has order divisible by the characteristic, a prime at least 5.
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