AN ENDPOINT ESTIMATE FOR THE CONE MULTIPLIER

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ABSTRACT. In this paper we consider an endpoint estimate for high-dimensional cone multipliers.

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

Let \( \hat{\psi} \in C_0^\infty(\mathbb{R}) \) be real-valued and supported in \( \{1 < \tau < 2\} \). Then for each \( \delta > 0 \), we consider the convolution operators \( T^\delta \) associated with the smooth cone multipliers given by

\[
\hat{T^\delta f}(\xi, \tau) = \hat{f}(\xi, \tau) \left( 1 - \frac{|\xi|^2}{|\tau|^2} \right)^{\delta} \hat{\psi}(\tau), \quad (\xi, \tau) \in \mathbb{R}^d \times \mathbb{R}.
\]

In the case \( \delta > (d - 1)/2 \), the convolution kernel belongs to \( L^1 \); hence \( T^\delta \) is an \( L^p \)-bounded operator for \( 1 \leq p \leq \infty \). In the case \( 0 < \delta < (d - 1)/2 \), it is conjectured that \( T^\delta \) is an \( L^p \)-bounded operator for

\[
\delta > \delta(p) := d|1/p - 1/2| - 1/2, \quad 1 < p < \infty.
\]

This conjectured range is the same as for the \( d \)-dimensional Bochner-Riesz multiplier problem. Note that the cone multiplier problem implies the Bochner-Riesz multiplier problem. By now the Bochner-Riesz multiplier problem is understood in the range \( p < (2d + 4)/(d + 4) \), \( d \geq 2 \) (see [5]). But compared to the Bochner-Riesz multiplier, little is known about the cone multiplier, and this conjecture still remains open for any \( d \geq 2 \). There are some partial results for this conjecture (see [3, 4, 5, 6, 7, 10]). In particular, in [10], the first author proved that \( T^\delta \) is an \( L^p \)-bounded operator for \( 1 \leq p \leq 2(d - 1)/(d + 1), \delta > \delta(p) \) and \( d \geq 4 \). This is the most recent result for high-dimensional cone multipliers. In this paper we consider an endpoint case \( \delta = \delta(1) = (d - 1)/2 \).

**Theorem 1.1.** If \( d \geq 4 \), then \( T^{(d-1)/2} \) maps \( L^1(\mathbb{R}^{d+1}) \) to \( L^{1,\infty}(\mathbb{R}^{d+1}) \).

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In the case of the Bochner-Riesz means, this result is well known for any dimension $d \geq 2$ (see [12]). But for the cone multiplier there are some additional difficulties; i.e., in the case of the Bochner-Riesz multiplier, the main contribution of the convolution kernel comes from

$$|x|^{-\delta - \frac{d+1}{2}} e^{\pm 2\pi i |x|}, \quad |x| > 1, \quad x \in \mathbb{R}^d,$$

and in the case of the cone multiplier, the main contribution comes from

$$|x|^{-\delta - \frac{d+1}{2}} \psi(t \pm |x|), \quad |x| > 1, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}.$$

As we can see in Lemma 2.2 and [1, 2], the term $\psi(t \pm |x|)$ in (1.2) plays a similar role as $e^{\pm 2\pi i |x|}$ in (1.1). But the convolution kernel (1.2) of the cone multiplier is defined on $\mathbb{R}^d \times \mathbb{R}$. This makes the cone multiplier problem more difficult than the Bochner-Riesz problem. Even though (1.2) is defined on $\mathbb{R}^d \times \mathbb{R}$, we can see that it is essentially supported in the cone $|t \pm |x|| \leq 1$, and we can use this advantage together with M. Christ’s stopping time arguments (Lemma 2.2).

**Remark.** Recently, the second author, F. Nazarov and A. Seeger [12] obtained $L^p(\mathbb{R}^{d+1}) \to L^{p, \infty}(\mathbb{R}^{d+1})$ inequalities for the cone multiplier $(1 - |\xi|^2/\tau^2)\chi_\pm^{(1)}$, $(\xi, \tau) \in \mathbb{R}^d \times \mathbb{R}$ with $1 < p < 2(d-1)/(d+1)$ and $d \geq 4$. See also [11] for the improvements upon the existing results in the so-called local smoothing problem for the wave equation in high dimensions.

2. **Reductions and the Proof of Theorem 1.1**

**Notation.** If $q$ is a dyadic cube in $\mathbb{R}^{d+1}$ with side-length $2^j$, we write $\ell(q) = j$. For each $j \in \mathbb{Z}$, $D_j$ denotes the collection of dyadic cubes $q \in \mathbb{R}^{d+1}$ with $\ell(q) = j$, and for each $q \in D_j$, $2q$ denotes $q + [-2^j, 2^j]^{d+1}$. For two quantities $A$ and $B$ we shall write $A \lesssim B$ if $A \leq CB$ for some absolute positive constant $C$. Also we shall write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The Lebesgue measure on $\mathbb{R}^{d+1}$ of a subset $E$ will be denoted by $|E|$.

We need to show that

$$\left| \left\{ (x, t) : |T^{(d-1)/2} f(x, t)| > \alpha \right\} \right| \leq C \alpha^{-1} \|f\|_1,$$

for each $\alpha > 0$. We may assume $f \geq 0$. Also by limiting arguments we may assume that $f \in L^1(\mathbb{R}^{d+1})$ have the form of a finite sum

$$f(x, t) = \sum_{q \in Q} \alpha_q \chi_q(x, t),$$

where $\alpha_q > 0$ and $Q$ is a finite, disjoint collection of dyadic cubes. Moreover if $\ell(q) \geq 0$, then by dividing $q$ as smaller dyadic subcubes we may assume $\ell(q) = \ell \ll 0$ for all $q \in Q$. Let us explain the limiting arguments. Let $f \in L^1(\mathbb{R}^{d+1})$ with $f \geq 0$. Then there exists a sequence $\{f_m\}_{m=1}^\infty$ of functions such that each $f_m$ has the form as in (2.2) and $\lim_{m \to \infty} \|f_m - f\|_1 = 0$. Also by choosing an appropriate subsequence we may assume

$$\|f_m\|_1 \leq 2\|f\|_1, \quad \|f_{m+1} - f_m\|_1 \leq 2^{-2m}\|f\|_1$$

for each $m$. 

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Note that

\[ T^{\frac{d}{2}} f(x, t) = \lim_{m \to \infty} T^{\frac{d}{2}} f_m(x, t) \]

\[ = T^{\frac{d}{2}} f_1(x, t) + \sum_{m=1}^{\infty} T^{\frac{d}{2}} (f_{m+1} - f_m)(x, t) \]

for almost every \((x, t)\). Hence if we prove (2.1) under the condition (2.2), then

\[ |\{ |T^{\frac{d}{2}} f(x, t)| > 2^\alpha \}| \]

\[ \leq \bigg| \{ |T^{\frac{d}{2}} f_1(x, t)| > \alpha \} \bigg| + \sum_{m=1}^{\infty} \bigg| \{ |T^{\frac{d}{2}} (f_{m+1} - f_m)(x, t)| > 2^{-m} \alpha \} \bigg| \]

\[ \leq C\alpha^{-1} \|f_1\|_1 + \sum_{m=1}^{\infty} C(2^{-m}\alpha)^{-1} \|f_{m+1} - f_m\|_1 \]

\[ \leq C\alpha^{-1} \|f\|_1. \]

Therefore, from now on we assume that

\[ f = \sum_{q \in \mathbf{Q}} \alpha_q |q| \frac{\chi_q}{|q|} = \sum_{q \in \mathbf{Q}} \lambda_q a_q, \]

where \(\lambda_q = \alpha_q |q|, a_q = \chi_q/|q|\) and \(\mathbf{Q}\) is a finite, disjoint collection of dyadic cubes in \(D_\varepsilon\) for some \(\varepsilon \ll 0\).

Next we compute the inverse Fourier transform \(K^\delta(x, t)\) of \((1 - |\xi/\tau|^2)^\frac{\delta}{d} \hat{\psi}(\tau)\). First note that

\[ K^\delta(x, t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i r(x, \xi + t)(1 - \frac{|\xi|^2}{\tau^2})^\frac{\delta}{2}} e^{2\pi i x \tau} \hat{\psi}(\tau) d\tau d\xi, \]

and by integration by parts via

\[ \frac{\partial}{\partial \tau} e^{2\pi i r(x, \xi + t)} = 2\pi i (x \cdot \xi + t) e^{2\pi i r(x, \xi + t)}, \]

we have

\[ |K^\delta(x, t)| \leq C_N \sup_{|\xi| \leq 1} (1 + |x \cdot \xi + t|)^{-N} \quad \text{for each } N > 0. \]

From this, it is easy to see that \(K^\delta(x, t)\chi_{\{|x| \leq 1\}} \in L^1(\mathbb{R}^{d+1})\). Therefore, from now on, we assume \(|x| \geq 1\). The inverse Fourier transform of the Bochner-Riesz multiplier \((1 - |\xi|^2)^\frac{\delta}{2} \), \(\xi \in \mathbb{R}^d\) is given by

\[ \pi^{-\delta} \Gamma(1 + \delta) |x|^{-\frac{d}{2} - \delta} J_{d/2 + \delta}(2\pi |x|), \]

where \(J_{d/2 + \delta}\) is the Bessel function of order \(d/2 + \delta\). So the inverse Fourier transform \(K^\delta(x, t)\) of \((1 - |\xi/\tau|^2)^\frac{\delta}{d} \hat{\psi}(\tau)\) is given by

\[ K^\delta(x, t) = \int_{\mathbb{R}} e^{2\pi i r(x, \xi + t)(1 + \delta)\tau x |x|^{-\delta - d/2} J_{d/2 + \delta}(2\pi|\tau| |x|)} \tau^d \hat{\psi}(\tau) d\tau. \]
It is well-known that, for all nonnegative integers $N$ and $n$, as $r \to \infty$,
\[ J_m(r) = e^{ir} \left[ r^{-1/2} \sum_{j=0}^{N} a_j r^{-j/2} + A_N(r) \right] + e^{-ir} \left[ r^{-1/2} \sum_{j=0}^{N} b_j r^{-j/2} + B_N(r) \right], \]
\[ \frac{d^n}{dr^n} A_N(r) = O\left(r^{-n-\frac{N+1}{2}}\right), \quad \frac{d^n}{dr^n} B_N(r) = O\left(r^{-n-\frac{N+1}{2}}\right). \]

For reference, see pages 334–338 in Stein’s book [13]. Therefore from the asymptotic expansion of the Bessel function $J_{d/2+\delta}$, for any positive integers $N$ and $M$ we have
\[ K^\delta(x,t) = \sum_{j=0}^{N} a_j^\pm \left| x \right|^{-\left(\delta + \frac{d+1}{2} \right)} \mathcal{F}^{-1} \left[ \hat{\psi}(\tau) \right] (t \pm |x|) + F^\delta_N(x,t), \]
as $|x| \to \infty$; here $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and
\[ |F^\delta_N(x,t)| \leq C_{\delta,N,M} \left( |x|^{-\delta - \frac{d+1}{2}} \right) (1 + |t| - |x|)^{-M}. \]

Note that
\[ F^{(d-1)/2} N \chi_{\{|x| \geq 1\}} \in L^1(\mathbb{R}^{d+1}) \quad \text{if} \quad N \geq 1 \quad \text{and} \quad M > 1. \]
Therefore it suffices to consider the terms
\[ |x|^{-\delta - \frac{d+1}{2}} \mathcal{F}^{-1} \left[ \hat{\psi}(\tau) \right] (t \pm |x|), \quad |x| \geq 1, \]
with $j = 0, 1$, and $\delta = (d - 1)/2$. From now on we will concentrate on the term
\[ |x|^{-\delta - \frac{d+1}{2}} \mathcal{F}^{-1} \left[ \hat{\psi}(\tau) \right] (t - |x|), \quad |x| \geq 1, \quad \delta = (d - 1)/2, \]
and the other cases can be treated similarly. Now, we should treat the operation of convolution with
\[ |x|^{-d} \psi(t - |x|), \quad |x| > 1. \]

For technical reasons, to obtain the convolution estimates in Lemma [2.3], fix a finite $C^\infty$ partition of unity $\{\omega_i\}$ on the unit sphere $\mathbb{S}^{d-1}$, with each $\omega_i$ having very small support. Let $\omega$ be one of the $C^\infty$ partition of unity $\{\omega_i\}$. Next choose $\eta \in C_0(\mathbb{R}^d)$, real-valued, radial and supported in $\{1/2 \leq |x| \leq 2\}$, so that $\sum_{j \in \mathbb{Z}} \eta(2^{-j} x) = 1$ on $\mathbb{R}^d \setminus \{0\}$. Also choose $\phi \in C_0(\mathbb{R})$, real-valued and supported in $\{|t| \leq 2\}$, so that
\[ \sum_{n \in \mathbb{Z}} \phi(t - n) = \sum_{n \in \mathbb{Z}} \phi_n(t) = 1. \]
Then it suffices to treat the operation of convolution with
\[ \sum_{n \in \mathbb{Z}} \sum_{j \geq 0} |x|^{-d} \psi(t - |x|) \omega(x/|x|) \phi(t - |x| - n) \eta(2^{-j} x) := \sum_{n \in \mathbb{Z}} \sum_{j \geq 0} K^\eta_j(x,t). \]

Let $f$ be as in (2.3). Then it suffices to show that
\[ \left| \left\{ (x,t) : \left( \sum_{j \geq 0} K^\eta_j \right) * f(x,t) > \alpha \right\} \right| \leq C_N (1 + |n|)^{-N} \alpha^{-1} \|f\|_1, \]
for each $n \in \mathbb{Z}$ and $\alpha > 0$. The following Lemma 2.1 is the standard Calderón-Zygmund decomposition, and we omit the proof (see Lemma 4.1 in [2]).

**Lemma 2.1.** Suppose $\beta > 0$ is given. Then for any finite collection $q$ of dyadic cubes $q$ and associated positive scalars $\lambda_q$, there exists a collection of pairwise...
disjoint dyadic cubes \( \{ S : S \in \mathbf{S} \} \) such that
\[
\begin{align*}
(1) \quad & \sum_{q \in S} \lambda_q \leq 2^{d+1} |S|, \\
(2) \quad & \sum_{S \in \mathbf{S}} |S| \leq \beta^{-1} \sum_{q \in \mathbf{q}} \lambda_q, \\
(3) \quad & \left\| \sum_{q \not\in \text{not contained in any } S} \frac{\lambda_q}{|q|} \right\|_\infty \leq \beta.
\end{align*}
\]

Let \( \mathbf{C} \) be the collection of dyadic cubes \( q \in \mathbf{q} \) which are contained in some \( S \in \mathbf{S} \). For each \( q \in \mathbf{C} \) we define \( S(q) \) as the unique \( S \in \mathbf{S} \) containing \( q \). The following is a refined Calderón-Zygmund decomposition whose proof relies on a stopping time argument.

**Lemma 2.2** (cf. Lemma 5.1 in \([2]\) or Lemma 5 in \([9]\)). Given \( \beta > 0 \) there exists a function \( \Gamma : \mathbf{C} \to \mathbb{Z} \) and a measurable set \( E \) such that
\[
\begin{align*}
(1) \quad & |E| \leq C \left( \beta^{-1} \sum_{q \in \mathbf{C}} \lambda_q + \sum_{S \in \mathbf{S}} |S| \right), \\
(2) \quad & \{ q + \text{supp}(K^n_j) \} \subset E \quad \text{for all } j < \Gamma(q) \text{ and } q \in \mathbf{C}, \\
(3) \quad & \ell(S(q)) < \Gamma(q) \quad \text{for each } q \in \mathbf{C}, \\
(4) \quad & \text{for each } \tau, \ell \in \mathbb{Z} \text{ with } \ell \leq \tau, \text{ and any } Q \in D_\ell \text{ we have}
\sum_{q \subset Q, q \in \mathbf{C}, \Gamma(q) \leq \tau} \lambda_q \leq \beta 2^{d(\tau+1)+\max(0,\ell)}.
\end{align*}
\]

For the proof we use the usual two-parameter stopping time arguments, and we construct an exceptional set \( E \) by combining stopping time arguments with the support condition of the kernel \( K^n_j \); i.e., if \( \ell(Q) \leq \tau \) and \( \tau \geq 0 \), then
\[
(2.4) \quad \left| \bigcup_{j=0}^\tau (Q + \text{supp}(K^n_j)) \right| \leq C 2^{d\tau+\max(0,\ell(Q))}.
\]

If \( S \in \mathbf{S} \) has side-length \( 2^j \), then (1) of Lemma 2.1 says that
\[
\sum_{q \subset S} \lambda_q \leq 2^{d+1} |S|.
\]

But if \( Q \subset S \), then by (4) of Lemma 2.2 we have a more delicate estimate:
\[
\sum_{q \subset Q, q \in \mathbf{C}, \Gamma(q) \leq j} \lambda_q \leq 2^d 2^{jd+\max(0,\ell(Q))}.
\]

This is why we referred to Lemma 2.2 as a refined Calderón-Zygmund decomposition. The proof will be given in Section 4.

**Lemma 2.3.** Let \( \overline{K}^n_j \) denote the conjugate of \( K^n_j \). Then for each \( i < j \) and every \( N > 0 \) we have
\[
\begin{align*}
(1) \quad & |K^N_i * \overline{K}^n_j(x,t)| \leq C_N (1 + |n|)^{-2N} 2^{-d_j} (1 + |(x,t)|)^{-2} \chi(|(x,t)| \leq 2^{j+1})(x,t), \\
(2) \quad & |K^N_i * \overline{K}^n_j(x,t)| \leq C_N (1 + |n|)^{-2N} 2^{-d_j} 2^{\frac{1-d}{2}} \chi(|x| \leq 2^{j+1})(x) \chi(|t-x| \leq 2^{j+1})(x,t).
\end{align*}
\]
The proof of Lemma 2.3 will be given in Section 3. For the moment we assume Lemmas 2.2 and 2.3 and prove Theorem 1.1. Let $f$ be as in (2.3). We need to show that
\begin{equation}
\left\{(x, t) : \left(\sum_{j \geq 0} K^n_j \ast f(x, t)\right) > \alpha\right\} \leq C \left(1 + |n|\right)^{-N} \alpha^{-1} \sum_{q} \lambda_q,
\end{equation}
for each $\alpha > 0$. Apply Lemma 2.1 to the collection of dyadic cubes $q$ and associated $\lambda_q$ appearing in the definition of $f$ with $\beta = \left(1 + |n|\right)^{N} \alpha$. Let $S$ be as in Lemma 2.1 and define
\[ b = \sum_{S \in q \subset S} \lambda_q a_q, \quad g = f - b. \]
Then $\|g\|_{\infty} \leq \beta$ and so by (1) of Lemma 2.3 for $d \geq 4$ we have
\[ \left\| \left( \sum_{j} K^n_j \right) \ast g \right\|_2 \leq \sum_{j} ||g||_2 \|K^n_j \ast K^n_j\|_1^{1/2} \]
\[ \leq \sum_{j} (C \left(1 + |n|\right)^{-N} 2^{-j(d-3)/4}) \]
\[ \leq C \left(1 + |n|\right)^{-N} \beta^{1/2} ||f||_1^{1/2} \]
\[ \leq C \left(1 + |n|\right)^{-N} \alpha^{1/2} ||f||_1^{1/2}. \]
Therefore by Tchebychev’s inequality we have
\begin{equation}
\left\{(x, t) : \left(\sum_{j} K^n_j \ast g(x, t)\right) > \alpha\right\} \leq \alpha^{-2} \left\| \left(\sum_{j} K^n_j \right) \ast g \right\|_2^2 \leq C \left(1 + |n|\right)^{-N} \alpha^{-1} ||f||_1.
\end{equation}
Let $S$ be as above and $C$ be the collection of $q$’s appearing in the definition of $b$. Then apply Lemma 2.2 with $S, C$. By (1) of Lemma 2.2 we have an exceptional set $E$ such that
\begin{equation}
|E| \leq C \left(\sum_{q \in C} \lambda_q + \sum_{S \in S} |S|\right) \leq C \beta^{-1} ||f||_1 \leq C \left(1 + |n|\right)^{-N} \alpha^{-1} ||f||_1.
\end{equation}
By (2.6) and (2.7), (2.3) will follow from
\begin{equation}
\left\{(x, t) \in \mathbb{R}^{d+1} \setminus E : \left(\sum_{j} K^n_j \ast b(x, t)\right) > \alpha\right\} \leq C \left(1 + |n|\right)^{-N} \alpha^{-1} \sum_{q} \lambda_q.
\end{equation}
By Tchebychev’s inequality, (2.8) will follow from
\begin{equation}
\left\| \left( \sum_{j} K^n_j \right) \ast b \right\|_{L^2(\mathbb{R}^{d+1} \setminus E)} \leq C \left(1 + |n|\right)^{-N} \alpha \sum_{q} \lambda_q.
\end{equation}
By (2) of Lemma 2.2 for each $q \in C$, $K^n_j \ast a_q$ is supported in $E$ unless $j \geq \Gamma(q)$. Thus we have
\begin{equation}
\left\| \sum_{j} b \ast K^n_j \right\|_{L^2(\mathbb{R}^{d+1} \setminus E)} \leq \left\| \sum_{j \leq \Gamma(q)} \left( \sum_{\Gamma(q) \leq j} \lambda_q a_q \right) \ast K^n_j \right\|_2 \leq \left\| \sum_{j \leq \Gamma(q)} \lambda_q a_q \ast K^n_j \right\|_2 + \left\| \sum_{0 < \Gamma(q) \leq j} \lambda_q a_q \ast K^n_j \right\|_2.
\end{equation}
By (2.12) it is easy to see that

$$(2.13) \quad \sum_s \sum_{s<j} |\langle B_0, K^n_j \rangle| \leq \sum_s \sum_{s=j}^\infty |\langle B_0, K^n_j \rangle|.$$ 

Note that

$$\sum_s \sum_{s<j} = \sum_s \sum_{s=j}^\infty,$$

hence we have

$$(2.11) \quad \left\| \sum_s \sum_{s<j} \left( \sum_{\Gamma(q)=j-s} \lambda_q a_q \right) K^n_j \right\|_2 \leq \sum_s \sum_{s=j}^\infty \left\| \sum_{\Gamma(q)=j-s} \lambda_q a_q \right\|_2 \left\| K^n_j \right\|_2.$$

Now by (2.10) and (2.11) we have

$$\left\| \sum_j b_j K^n_j \right\|_{L^2(\mathbb{R}^{d+1}\setminus E)} \leq F_1 + F_2,$$

where

$$F_1 = \sum_j \left\| \left( \sum_{\Gamma(q)=j} \lambda_q a_q \right) K^n_j \right\|_2, \quad F_2 = \sum_j \sum_{s=j} \left( \sum_{\Gamma(q)=j-s} \lambda_q a_q \right) \left\| K^n_j \right\|_2.$$

**Estimation of part $F_1$.** Let $B_0 = \sum_{\Gamma(q)=0} \lambda_q a_q$. For each $q \in \mathcal{C}$ there exists a unique $S(q) \in \mathcal{S}$ containing $q$; hence if $\Gamma(q) \leq 0$, then by the condition $\ell(S(q)) \leq \Gamma(q)$ we have $\ell(S(q)) \leq 0$. Therefore by (1) of Lemma 2.1, we can see that

$$\|B_0\chi_q\|_1 = \sum_{q \leq Q, \Gamma(q)=0} \lambda_q \leq C \beta \quad \text{for each } Q \in \mathcal{D}_0.$$

By (1) of Lemma 2.3 we have

$$\|K^n_j * \tilde{K}_n^j * B_0\|_\infty \leq C_N (1 + |n|)^{-2N} 2^{-jd} \sup_{x \in \mathbb{R}^{d+1}} \left( \int_{|x-y| \leq 2^j} \frac{B_0(y)}{(1 + |x-y|)^{2^j}} \, dy \right).$$

By (2.12) it is easy to see that

$$\left( \int_{|x-y| \leq 2^j} \frac{B_0(y)}{(1 + |x-y|)^{2^j}} \, dy \right) \leq \left( \int_{|x-y| \leq 2^j} \frac{C \beta}{(1 + |x-y|)^{2^j}} \, dy \right) \leq C \beta 2^{\frac{jd}{2^j}}.$$

Therefore we have

$$\|K^n_j * \tilde{K}_n^j * B_0\|_\infty \leq C_N (1 + |n|)^{-2N} \beta 2^{\frac{jd}{2^j}},$$

and so for $d \geq 4$,

$$(2.13) \quad F_1 = \sum_j \|B_0 * K^n_j\|_2 \leq \sum_j \|\langle B_0, K^n_j \rangle\|^{1/2} \leq \|B_0\|^{1/2} \sum_j \|K^n_j * \tilde{K}_n^j + B_0\|^{1/2} \leq C_N (1 + |n|)^{-N} \beta^{1/2} \|f\|^{1/2}.$$
Estimation of part $F_2$. For each $l > 0$, let $B_l = \sum_{\Gamma(q) = l} \lambda_q a_q$. Then
\[
\left\| \sum_{j > s} B_{j-s} * K^n_j \right\|_2^2 \leq \sum_{j > s} \left\| B_{j-s} * K^n_j \right\|_2^2 + 2 \sum_{j > s} \left| \langle B_{j-s} * K^n_j, B_{j-1-s} * K^n_{j-1} \rangle \right|
\]
\[
+ 2 \sum_{j > s} \sum_{s < t < j-1} \left| \langle B_{t-s} * K^n_t, B_{j-s} \rangle \right|
\]
\[= A_1(s) + A_2(s) + A_3(s).\]

For the part $A_1(s)$, we write
\[
\left\| B_{j-s} * K^n_j \right\|_2^2 = \sum_{\Gamma(q), \Gamma(q') = j-s} \lambda_q \lambda_{q'} \langle a_{q'} * K^n_j, a_q * K^n_j \rangle
\]
\[
\leq \sum_{m=0}^{j+4} \sum_{1 + \text{dist}(q, q') \sim 2^m} \lambda_q \lambda_{q'} \langle a_{q'} * K^n_j, a_q \rangle
\]
\[
\leq \sum_{m=0}^{j-s+2} \sum_{q: 1 + \text{dist}(q, q') \sim 2^m} + \sum_{m=j-s+3}^{j+4} \sum_{q: 1 + \text{dist}(q, q') \sim 2^m} 2^{d-4} m \lambda_q.
\]

Estimation of part $I$. For each fixed $m$ and $q'$ consider the contribution of all $\lambda_q$ over all $q$ so that $1 + \text{dist}(q, q') \sim 2^m$. All such $q$ are contained in the union of a fixed number of $D_m$. Hence when $0 \leq m \leq j-s+2$, by (1) of Lemma 2.3 we have
\[
I \leq C_N (1 + |n|)^{-2} 2^{-j \beta} \sum_{q: 1 + \text{dist}(q, q') \sim 2^m} \lambda_q \leq C_N (1 + |n|)^{-2} 2^{-j \beta} \sum_{\Gamma(q) = j-s} \lambda_q
\]
By (1) of Lemma 2.2 we have
\[
\sum_{q: 1 + \text{dist}(q, q') \sim 2^m, \Gamma(q') = j-s} \lambda_q \leq C \beta 2^{(j-s) \beta + m},
\]
and so for $d \geq 4$,
\[
I \leq C_N (1 + |n|)^{-2} 2^{-j \beta} \sum_{\Gamma(q') = j-s} \lambda_{q'}.
\]

Estimation of part $II$. Next, consider all $q$ with $\text{dist}(q, q') \sim 2^m$ and $j-s+3 \leq m \leq j+4$. Recall that each $q \in C$ is contained in some $S(q) \subset S$. Since $\ell(S(q)) < \Gamma(q) = j-s$, we have $\text{dist}(S(q), q') \sim 2^m$ and so
\[
II \leq C_N (1 + |n|)^{-2} 2^{-j \beta} \sum_{q'} \lambda_{q'} \sum_{m=j-s+3}^{j+4} \sum_{q: \text{dist}(q, q') \sim 2^m} 2^{j-d} m \lambda_q
\]
\[
\leq C_N (1 + |n|)^{-2} 2^{-j \beta} \sum_{q'} \lambda_{q'} \sum_{m=j-s+2}^{j+5} \sum_{q \in C} \text{dist}(S(q), q')^{j-d} \lambda_q.
\]
By (1) of Lemma 2.1 for each $S \in \mathbf{S}$ we have $\sum_{q \subset S} \lambda_q \leq C|\beta||S|$, and so

$\sum_{S \in \mathbf{S}} \sum_{q \subset S} \text{dist}(S, q')^{\frac{1+\beta}{d}} \lambda_q \leq C \sum_{S \in \mathbf{S}} \sum_{q \subset S} \text{dist}(S, q')^{\frac{1+\beta}{d}} (\beta|S|) \leq C\beta \int_{y \in \mathbb{R}^{d+1}, |y| \leq 2^{j+5}} |y|^{\frac{1+\beta}{d}} dy \leq C2^{\frac{2+\beta}{d}j}\beta.$

Now we have

$$\Pi \leq C_N(1 + |n|)^{-2N} 2^{\frac{3+\beta}{d}j} \beta \sum_{\Gamma(q') = j-s} \lambda_{q'},$$

and so for $d \geq 4$,

$$A_1(s) \leq \sum_{j > s} C_N(1 + |n|)^{-2N} \left(2^{-sd} + 2^{\frac{3+\beta}{d}j}\right) \beta \sum_{\Gamma(q') = j-s} \lambda_{q'} \tag{2.14}$$

$$\leq C_N(1 + |n|)^{-2N} 2^{\frac{3+\beta}{d}s} \beta \sum_{q'} \lambda_{q'}.$$  

Similarly we have

$$A_2(s) \leq C_N(1 + |n|)^{-2N} 2^{\frac{3+\beta}{d}s} \beta \sum_{q'} \lambda_{q'}.$$

For the part $A_3(s)$, if $\Gamma(q) = i - s$, then $\ell(q), S(q) \leq i - s$. So by (1) of Lemma 2.1 and (2) of Lemma 2.3 together with the condition

$$\text{supp}(K^n_i + \tilde{K}^n_j) \subset \{(x, t) : |x| \leq 2^{j+4}, |t - |x|| \leq 2^{j+4}\},$$

we have

$$\|B_{i-s} * K^n_i * \tilde{K}^n_j\|_{\infty} \leq C_N(1 + |n|)^{-2N} 2^{-jd} 2^{\frac{1-\beta}{d}} \sum_{S \in \mathbf{S}} \sum_{\Gamma(q) = i - s} \beta_{q} \lambda_{q} \leq C_N(1 + |n|)^{-2N} 2^{-jd} 2^{\frac{1-\beta}{d}} \sum_{S \in \mathbf{S}} \beta_{|S|},$$

where the sums are taken over all $S \in \mathbf{S}$ such that

$$S \subset \{(x, t) : |x| \leq 2^{j+4}, |t - |x|| \leq 2^{j+4}\}.$$  

So we have $\sum_{S \in \mathbf{S}} |S| \leq C2^{d+1}$ and

$$\|B_{i-s} * K^n_i * \tilde{K}^n_j\|_{\infty} \leq C_N(1 + |n|)^{-2N} 2^{\frac{2+\beta}{d}i}\beta.$$

Therefore for $d \geq 4$,

$$A_3(s) = \sum_{j > s} \sum_{0 < i < j - 1} \|B_{i-s} * K^n_i * \tilde{K}^n_j, B_{j-s}\|_1 \leq C_N(1 + |n|)^{-2N} 2^{\frac{2+\beta}{d}s} \beta \sum_{q} \lambda_{q},$$  

$$\tag{2.16} \leq C_N(1 + |n|)^{-2\beta} 2^{\frac{2+\beta}{d}s} \beta \sum_{q} \lambda_{q}.$$
Finally from (2.13), (2.14), (2.15) and (2.16), for $d \geq 4$ we have
\[ F_2 \leq \sum_{s \geq 0} (A_1(s) + A_2(s) + A_3(s))^{1/2} \leq C_N (1 + |n|)^{-N} \beta^{1/2} (\sum_q \lambda_q)^{1/2}, \]

and we are done.

3. Proof of Lemma 2.3

The proof is similar to Lemma 3.1 in [2]. For (11), let $\eta_1(x) := \eta(x)|x|^{-d}$. Then we have
\[ K^n \ast K^n_j(x, t) = 2^{-2d} \int_{\mathbb{R}} \int_0^\infty F_j(x, t, s, r) \psi(s-r) \eta_1(2^{-j}r) \phi_n(s-r) r^{d-1} dr ds, \]

where
\[ F_j(x, t, s, r) = \int_{\mathbb{R}^{d-1}} \psi(t+s-|x+r\theta|) \omega((x+r\theta)/|x+r\theta|) \omega(\theta) \times \phi_n(t+s-|x+r\theta|) \eta_1(2^{-j}(x+r\theta)) d\theta. \]

Note that
\[ \psi(t+s-|x+r\theta|) = \int_{\mathbb{R}} e^{2\pi i r(t+s-|x+r\theta|)} \tilde{\psi}(\tau) d\tau. \]

Hence we have
\[ F_j(x, t, s, r) = \int_{\mathbb{R}} e^{2\pi i r(t+s)} \tilde{\psi}(\tau) G_j(x, t, s, r, \tau) d\tau, \]

where
\[ G_j(x, t, s, r, \tau) = \int_{\mathbb{R}^{d-1}} e^{-2\pi i r|x+r\theta|} \omega((x+r\theta)/|x+r\theta|) \omega(\theta) \times \phi_n(t+s-|x+r\theta|) \eta_1(2^{-j}(x+r\theta)) d\theta. \]

It suffices to show that
\[ |G_j(x, t, s, r, \tau)| \leq C (1 + |(x, t)|)^{(1-d)/2}, \]

uniformly for $\tau \sim 1$, $s$ and $r \sim 2^j$. Suppose that the function $\omega$ from the partition of unity has sufficiently small support about $(0, \ldots, 0, 1)$. Then we use the local coordinate chart
\[ \theta = (\theta_1, \ldots, \theta_d) = (\theta_1, \ldots, \theta_{d-1}, \sqrt{1 - \theta_1^2 - \cdots - \theta_{d-1}^2}). \]

Then by direct calculation, for $1 \leq l \leq d - 1$,
\[ \frac{\partial}{\partial \theta_l} (|x + r\theta|) = r \left( \frac{\theta_d x_l - \theta_l x_d}{|x + r\theta|^2} \right), \]
\[ \frac{\partial^2 |x + r\theta|}{\partial \theta_l^2} = r \left( \frac{|x + r\theta|^2 (\theta_l^2 - \theta_d^2) (-x_d) - r \theta_d (\theta_d x_l - \theta_l x_d)^2}{|x + r\theta|^3 \theta_d^3} \right), \]

and for $1 \leq l \neq m \leq d - 1$,
\[ \frac{\partial^2 |x + r\theta|}{\partial \theta_l \partial \theta_m} = -r \left( \frac{\theta_l \theta_m x_d |x + r\theta|^2 + r \theta_d (\theta_d x_m - \theta_m x_d) (\theta_d x_l - \theta_l x_d)}{|x + r\theta|^3 \theta_d^3} \right). \]

Note that
\[ \frac{\partial^2 |x + r\theta|}{\partial \theta_l^2}(0, \ldots, 0, 1) = r \left( \frac{|x + r\theta|^2 (-x_d) - x_l^2}{|x + r\theta|^3} \right). \]
Therefore if \( r \theta \) is bounded below by \( r \theta \) for some \( 1 \leq l \leq d - 1 \), by (3.3),

\[
\left| \frac{\partial (r \theta)}{\partial l} \right| \geq |x|
\]

uniformly for \( |x + r \theta| \sim 2^i \), \( r \sim 2^i \). By integrating by parts via

\[
\frac{\partial}{\partial t_i} \left( e^{-2\pi i r |x + r \theta|} \right) = -2\pi i r \frac{\partial |x + r \theta|}{\partial t_i} \left( e^{-2\pi i r |x + r \theta|} \right),
\]

we have

\[
|G_j(x, t, s, r, \tau)| \leq C_N (1 + |x_i|)^{-N} \leq C_N (1 + |x|)^{-N}.
\]

In the case \( |x_d| \gtrsim |x| \), from (3.4), (3.5), (3.6), (3.7), together with the conditions \( |x + r \theta| \sim 2^i \), \( r \sim 2^i \), we can see that each absolute value of the eigenvalue of the Hessian matrix

\[
\frac{\partial^2 |x + r \theta|}{\partial s \partial \theta}
\]

is bounded below by \( C|x_d| \) uniformly for \( 2^{i-1} < r < 2^{i+1} \) when \( \theta \) and \( (x + r \theta)/|x + r \theta| \) are both in the support of \( \omega \). By the methods of stationary phase, we have

\[
|G_j(x, t, s, r, \tau)| \leq C (1 + |x_d|)^{(1-d)/2} \leq (1 + |x|)^{(1-d)/2}.
\]

Hence we have

\[
(3.8) \quad |K^n_j(x, t)| \leq C_N (1 + |n|)^{-2N} 2^{-jd} (1 + |x|)^{(1-d)/2}.
\]

Next recall that

\[
K_j^n(x, t) = 2^{-2d_j} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(t + s - |x + y|) \overline{\psi}(s - |y|) \varphi[(x + y)/|x + y|] \times \omega(y/|y|) \phi_n(t + s - |x + y|) \phi_n(s - |y|) \eta_1(2^{-j}(x + y)) \eta_1(2^{-j} y)dy ds.
\]

Hence \( K_j^n(x, t) \) is supported in

\[
|t + s - |x + y| - n| \leq 2, \quad |s - |y| - n| \leq 2,
\]

and by the triangle inequality, this implies that

\[
|t| \leq |t + s - |x + y| - n| + |s - |y| + n| + |- |y| + |x - y|| \leq 2 + 2 + |x|.
\]

Therefore if \( |x| \leq 2 \), then \(|(x, t)| \leq 6 \), and (3.8) implies that

\[
|K_j^n(x, t)| \leq C_N (1 + |n|)^{-2N} 2^{-jd} (1 + |(x, t)|)^{-N}.
\]

If \( |x| \geq 2 \), then by (3.9) we have \(|t| \leq 3|x| \), and (3.8) implies that

\[
|K_j^n(x, t)| \leq C_N (1 + |n|)^{-2N} 2^{-jd} (1 + |(x, t)|)^{(1-d)/2}.
\]

For (2), let \( F_j(x, t, s, r) \) and \( G_j(x, t, s, r, \tau) \) be the same as in (3.1) and (3.2). Then we have

\[
K_j^n(x, t) = 2^{-d_j-d_i} \int_{\mathbb{R}^d} \int_0^\infty F_j(x, t, s, r) \overline{\psi}(s - r) \eta_1(2^{-i} r) \phi_n(s - r) r^{d-1} dr ds.
\]
Since $|x + r\theta| \sim 2^j$, $r \sim 2^j$, we have $|x| \sim 2^j$. Hence in the case $|x| \gtrsim |x|$ for some $1 \leq l \leq d - 1$, from (3.3) we can see that
\[
\left| \frac{\partial |x + r\theta|}{\partial \theta_l} \right| \gtrsim 2^j,
\]
uniformly for $|x + r\theta| \sim 2^j$ and $r \sim 2^j$, and by integrating by parts via
\[
\frac{\partial}{\partial \theta_l} \left( e^{-2\pi i \tau |x + r\theta|} \right) = -2\pi i \frac{\partial |x + r\theta|}{\partial \theta_l} \left( e^{-2\pi i \tau |x + r\theta|} \right),
\]
we have
\[
|G_j(x, t, s, r, \tau)| \leq C N 2^{-Nj}.
\]
In the case $|x_d| \gtrsim |x|$, each absolute value of the eigenvalue of the Hessian matrix
\[
\frac{\partial^2 |x + r\theta|}{\partial^2 \theta}
\]
is bounded below by $C 2^j$ uniformly for $r \sim 2^j$ when $\theta$ and $(x + r\theta)/|x + r\theta|$ are both in support of $\omega$. By the methods of stationary phase, we have
\[
|G_j(x, t, s, r, \tau)| \leq C 2^{\frac{j-2}{2}}.
\]
Hence in the case $|x_d| \gtrsim |x|$ we have
\[
(3.10) \quad |K_j^n \ast \widetilde{K}_k^n(x, t)| \leq C N (1 + |n|)^{-2N} 2^{-j} d^{\frac{j-2}{2}}.
\]
Note that $K_j^n \ast \widetilde{K}_k^n(x, t)$ is supported in
\[
|t + s - |x + y| - n| \leq 2, |s - |y| - n| \leq 2, \quad |x + y| \leq 2^{i+1}, |y| \leq 2^{i+1}.
\]
Hence $K_j^n \ast \widetilde{K}_k^n(x, t)$ is supported in
\[
|t - x| \leq |t + s - |x + y| - n| + ||x + y| - |x|| + | - s + n| \leq 2 + |y| + (2 + |y|) \leq 2^{i+4},
\]
and from (3.10) and (3.11) we have (2) of Lemma 2.3.

4. Proof of Lemma 2.2

The following are the usual two-parameter stopping-time arguments. These will be discussed in more detail below. We will combine these arguments with the support condition of the kernel $K_j^n$ to construct an exceptional set $E$. Let $m = \min \{\ell(q) : q \in \mathbb{C} \}$. Select an integer $\tau_0$ such that
\[
\tau_0 > \max \{\ell(q) : q \in \mathbb{C} \}, \quad \sum_{q \in \mathbb{C}} \lambda_q < \beta 2^{4\tau_0 + \max(0, m)}.
\]
For each fixed $\tau \in \mathbb{Z}$ with $\tau \leq \tau_0$, we will define a sequence of functions $\Lambda_{\tau, \ell} : \mathbb{D}_k \to \mathbb{R}$ by a descending induction on $\ell \in \mathbb{Z}$ with $\ell \leq \tau$, and proceed with the same construction by a descending induction on $\tau$. At each step, we will define subsets $C_1, C_2$ of $\mathbb{C}$ which will increase as we proceed. Let $C_1, C_2 \subset \mathbb{C}$ and $\tau \in \mathbb{Z}$ be fixed for the moment, and define $\textbf{Inner loop}$ as

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Inner loop. Define $\Lambda_{\tau,\ell} : D_\ell \to \mathbb{R}$ with $\ell \leq \tau$. For each $Q \in D_\ell$, define
\[
\Lambda_{\tau,\ell}(Q) = \sum_{q \in Q: \gamma \in C_1 \cup C_2} \lambda_q.
\]

Begin with $\ell = \tau$. If
\[
(4.1) \quad \Lambda_{\tau,\ell}(Q) > \beta 2^{d\tau + \max(0,\ell)},
\]
then we say that “$Q$ is selected at step $(\tau, \ell)$”. Put into $C_1$ every $q$ such that $q \subseteq Q$ and define $\Gamma(q) = 1 + \tau$. Repeat until $\ell < \min\{\ell(q) : q \in C\}$. Actually this part of the process will be terminated once $\ell$ is smaller than $m$. Put into $C_2$ every $q \in C \setminus C_1$ such that $\ell(q) \geq \tau$ and for such $q$ define $\Gamma(q) = 1 + \ell(S(q))$. Actually every $q \in C \setminus C_1 \cup C_2$ satisfies $\ell(q) \leq \tau - 1$.

Perform Inner loop with $C_1 = \emptyset = C_2$ and $\tau = \tau_0$. Next replace $\tau$ by $\tau - 1$ and repeat Inner loop. Repeat until $\tau = m - 1$. After this process we obtain $C = C_1 \cup C_2$. Clearly all selected $Q$ are disjoint and $\Gamma$ is well defined. Note that there is the usual stopping-time condition
\[
(4.2) \quad \Lambda_{\tau,\ell}(Q) \leq \beta 2^{d(\tau + 1) + \max(0,\ell)},
\]
which holds for all $Q \in D_\ell$ when $\ell \leq \tau \leq \tau_0$. This is because if $\tau = \tau_0$, then the condition is clear from the choice of $\tau_0$ and $m$. When $\ell \leq \tau \leq \tau_0$, suppose this fails. Then $\Lambda_{\tau+1,\ell}(Q) \geq \Lambda_{\tau,\ell}(Q) > \beta 2^{d(\tau + 1) + \max(0,\ell)}$. This means that $Q$ is selected at step $(\tau + 1, \ell)$; hence $\Lambda_{\tau,\ell}(Q) = 0$, and we have the contradiction.

Next we show (4), which says, for each $Q \in D_\ell$ with $\ell \leq \tau$,
\[
\sum_{q \in Q: \Gamma(q) \leq \tau} \lambda_q \leq \beta 2^{d(\tau + 1) + \max(0,\ell)}.
\]

When $\tau \geq \tau_0$, this property is clear from the initial choice of $\tau_0$. When $\tau < \tau_0$, if we show that
\[
(4.3) \quad \Lambda_{\tau,\ell}(Q) = \sum_{q \in Q: q \notin C_1 \cup C_2} \lambda_q \geq \sum_{q \in Q: \Gamma(q) \leq \tau} \lambda_q
\]
for each $Q \in D_\ell$ with $\ell \leq \tau < \tau_0$, then by combining (4.2) and (4.3), we have (4).

Now (4.3) follows from the definition
\[
\Lambda_{\tau,\ell}(Q) = \sum_{q \in Q: q \notin C_1 \cup C_2} \lambda_q
\]
and the fact that
\[
\Gamma(q) \leq \tau \Rightarrow q \notin C_1 \cup C_2 \text{ at the beginning of step } (\tau, \ell).
\]
This is because, if $q \in C_1$, then $\Gamma(q) \geq 1 + \tau > \tau$, and if $q \in C_2$, then $\Gamma(q) = 1 + \ell(S(q)) \geq 1 + (1 + \tau) > \tau$. Hence $\Gamma(q) \leq \tau$ implies $q \notin C_1 \cup C_2$, and so we have (4.3).

Next, we construct an exceptional set $E$ by using the above stopping-time arguments. If $Q$ is selected at step $(\tau, \ell)$, then we define $\tau(Q) = \tau$. If $\tau(Q) \geq 0$, then we define the tendril $T(Q)$ associated to $Q$ by
\[
T(Q) = \bigcup_{j < \tau(Q) + 1} \left( Q + \text{supp}(K_j^n) \right).
\]
Also we define
\[ E = E_1 \cup E_2, \quad E_1 = \bigcup_{S \in S} 2S, \quad E_2 = \bigcup_{Q: \text{selected}, \tau(Q) \geq 0} T(Q). \]
Thus we have
\[ |E_1| \leq C \sum_{S \in S} |S|. \]
By the measure condition (2.4) of \( T(Q) \) and the condition (4.1) of selected \( Q \),
\[ |E_2| \leq \sum_{Q: \text{selected}, \tau(Q) \geq 0} |T(Q)| \leq C \sum_{Q: \text{selected}} 2^{d\tau(Q) + \max(0, \ell(Q))} \leq C_1 \beta \sum_{Q: \text{selected}} \Lambda_{r, \ell}(Q) \leq C_1 \beta \sum \lambda_q. \]
So we have \( \mathbf{1} \). For \( \mathbf{2} \), if \( q \in C_1 \), then \( q \) belongs to some selected \( Q \) and \( \Gamma(q) = \tau(Q) + 1 \). Hence if \( \Gamma(q) > j \), then
\[ \bigcup_{j < \Gamma(q)} \left( q + \text{supp}(K^n_j) \right) \subset T(Q) \subset E_2, \]
and if \( q \in C_2 \), then \( q \) belongs to some \( S(q) \in S \) and \( \Gamma(q) = 1 + \ell(S(q)) \). Hence if \( \Gamma(q) > j \), then
\[ \bigcup_{j < \Gamma(q)} \{ q + \text{supp}(K^n_j) \} \subset 2S(q) \subset E_1. \]
For \( \mathbf{3} \), let \( \Gamma' := \Gamma \) and redefine
\[ \Gamma(q) = \max \{ \Gamma'(q), 1 + \ell(S(q)) \}. \]
Then \( \mathbf{1} \) and \( \mathbf{3} \) are satisfied. We must check \( \mathbf{2} \) and \( \mathbf{4} \). For \( \mathbf{2} \), if \( \Gamma(q) = \Gamma'(q) \), then there is no problem. If \( \Gamma(q) = 1 + \ell(S(q)) > j \), then the argument is the same as above. \( \mathbf{4} \) follows from
\[ \sum_{q \in Q: \Gamma(q) \leq \tau} \lambda_q \leq \sum_{q \in Q: \Gamma'(q) \leq \tau} \lambda_q, \]
and we have Lemma 2.2.

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