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AN ENDPOINT ESTIMATE FOR THE CONE MULTIPLIER

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Abstract. In this paper we consider an endpoint estimate for high-dimensional cone multipliers.

1. Introduction and statements of the results

Let $\widehat{\psi} \in C_0^\infty(\mathbb{R})$ be real-valued and supported in $\{1 < \tau < 2\}$. Then for each $\delta > 0$, we consider the convolution operators T^δ associated with the smooth cone multipliers given by

$$\widehat{\mathrm{T}^\delta f}(\xi,\tau) = \widehat{f}(\xi,\tau)\left(1 - \frac{|\xi|^2}{|\tau|^2}\right)_+^\delta \widehat{\psi}(\tau), \quad (\xi,\tau) \in \mathbb{R}^d \times \mathbb{R}.$$

In the case $\delta > (d-1)/2$, the convolution kernel belongs to L^1; hence T^δ is an L^p-bounded operator for $1 \le p \le \infty$. In the case $0 < \delta < (d-1)/2$, it is conjectured that T^δ is an L^p-bounded operator for

$$\delta > \delta(p) := d|1/p - 1/2| - 1/2\ , \quad 1 < p < \infty.$$

This conjectured range is the same as for the d-dimensional Bochner-Riesz multiplier problem. Note that the cone multiplier problem implies the Bochner-Riesz multiplier problem. By now the Bochner-Riesz multiplier problem is understood in the range $p < (2d+4)/(d+4)$, $d \ge 2$ (see [8]). But compared to the Bochner-Riesz multiplier, little is known about the cone multiplier, and this conjecture still remains open for any $d \ge 2$. There are some partial results for this conjecture (see [3, 4, 5, 6, 7, 10]). In particular, in [10], the first author proved that T^δ is an L^p-bounded operator for $1 \le p \le 2(d-1)/(d+1)$, $\delta > \delta(p)$ and $d \ge 4$. This is the most recent result for high-dimensional cone multipliers. In this paper we consider an endpoint case $\delta = \delta(1) = (d-1)/2$.

Theorem 1.1. *If $d \ge 4$, then $\mathrm{T}^{(d-1)/2}$ maps $L^1(\mathbb{R}^{d+1})$ to $L^{1,\infty}(\mathbb{R}^{d+1})$.*

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In the case of the Bochner-Riesz means, this result is well known for any dimension $d \geq 2$ (see [1, 2]). But for the cone multiplier there are some additional difficulties; i.e., in the case of the Bochner-Riesz multiplier, the main contribution of the convolution kernel comes from

$$|x|^{-\delta-\frac{d+1}{2}} \mathrm{e}^{\pm 2\pi \mathrm{i}|x|}, \quad |x|>1, \quad x \in \mathbb{R}^d, \tag{1.1}$$

and in the case of the cone multiplier, the main contribution comes from

$$|x|^{-\delta-\frac{d+1}{2}} \psi(t \pm |x|), \quad |x|>1, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}. \tag{1.2}$$

As we can see in Lemma 2.3 and [1, 2], the term $\psi(t \pm |x|)$ in (1.2) plays a similar role as $\mathrm{e}^{\pm 2\pi \mathrm{i}|x|}$ in (1.1). But the convolution kernel (1.2) of the cone multiplier is defined on $\mathbb{R}^d \times \mathbb{R}$. This makes the cone multiplier problem more difficult than the Bochner-Riesz problem. Even though (1.2) is defined on $\mathbb{R}^d \times \mathbb{R}$, we can see that it is essentially supported in the cone $|t \pm |x|| \leq 1$, and we can use this advantage together with M. Christ's stopping time arguments (Lemma 2.2).

Remark 1.2. Recently, the second author, F. Nazarov and A. Seeger [12] obtained $L^p(\mathbb{R}^{d+1}) \to L^{p,\infty}(\mathbb{R}^{d+1})$ inequalities for the cone multiplier $(1-|\xi|^2/\tau^2)_+^{\delta(p)}$, $(\xi,\tau) \in \mathbb{R}^d \times \mathbb{R}$ with $1 < p < 2(d-1)/(d+1)$ and $d \geq 4$. See also [11] for the improvements upon the existing results in the so-called local smoothing problem for the wave equation in high dimensions.

2. Reductions and the proof of Theorem 1.1

Notation. If q is a dyadic cube in $\mathbb{R}^{d+1}$ with side-length 2^j, we write $\ell(q) = j$. For each $j \in \mathbb{Z}$, $\mathbf{D}_j$ denotes the collection of dyadic cubes $q \in \mathbb{R}^{d+1}$ with $\ell(q) = j$, and for each $q \in \mathbf{D}_j$, $2q$ denotes $q + [-2^j, 2^j]^{d+1}$. For two quantities A and B we shall write $A \lesssim B$ if $A \leq CB$ for some absolute positive constant C. Also we shall write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The Lebesgue measure on $\mathbb{R}^{d+1}$ of a subset E will be denoted by $|E|$.

We need to show that

$$\left|\left\{(x,t) : |\mathrm{T}^{(d-1)/2} f(x,t)| > \alpha\right\}\right| \leq C\alpha^{-1}\|f\|_1, \tag{2.1}$$

for each $\alpha > 0$. We may assume $f \geq 0$. Also by limiting arguments we may assume that $f \in L^1(\mathbb{R}^{d+1})$ have the form of a finite sum

$$f(x,t) = \sum_{q \in \mathbf{q}} \alpha_q \chi_q(x,t), \tag{2.2}$$

where $\alpha_q > 0$ and $\mathbf{q}$ is a finite, disjoint collection of dyadic cubes. Moreover if $\ell(q) \geq 0$, then by dividing q as smaller dyadic subcubes we may assume $\ell(q) = \ell \ll 0$ for all $q \in \mathbf{q}$. Let us explain the limiting arguments. Let $f \in L^1(\mathbb{R}^{d+1})$ with $f \geq 0$. Then there exists a sequence $\{f_m\}_{m=1}^\infty$ of functions such that each f_m has the form as in (2.2) and $\lim_{m\to\infty} \|f_m - f\|_1 = 0$. Also by choosing an appropriate subsequence we may assume

$$\|f_m\|_1 \leq 2\|f\|_1, \quad \|f_{m+1} - f_m\|_1 \leq 2^{-2m}\|f\|_1 \quad \text{for each } m.$$

Note that

$$\begin{aligned}\mathrm{T}^{\frac{d-1}{2}} f(x,t) &= \lim_{m\to\infty} \mathrm{T}^{\frac{d-1}{2}} f_m(x,t)\\ &= \mathrm{T}^{\frac{d-1}{2}} f_1(x,t) + \sum_{m=1}^{\infty} \mathrm{T}^{\frac{d-1}{2}}(f_{m+1}-f_m)(x,t)\end{aligned}$$

for almost every (x,t). Hence if we prove (2.1) under the condition (2.2), then

$$\begin{aligned}&\big|\big\{|\mathrm{T}^{\frac{d-1}{2}} f(x,t)| > 2\alpha\big\}\big|\\ &\le \big|\big\{|\mathrm{T}^{\frac{d-1}{2}} f_1(x,t)| > \alpha\big\}\big| + \sum_{m=1}^{\infty} \big|\big\{|\mathrm{T}^{\frac{d-1}{2}}(f_{m+1}-f_m)(x,t)| > 2^{-m}\alpha\big\}\big|\\ &\le C\alpha^{-1}\|f_1\|_1 + \sum_{m=1}^{\infty} C(2^{-m}\alpha)^{-1}\|f_{m+1}-f_m\|_1\\ &\le C\alpha^{-1}\|f\|_1.\end{aligned}$$

Therefore, from now on we assume that

$$f = \sum_{q\in\mathbf{q}} \alpha_q |q| \frac{\chi_q}{|q|} = \sum_{q\in\mathbf{q}} \lambda_q a_q, \tag{2.3}$$

where $\lambda_q = \alpha_q|q|$, $a_q = \chi_q/|q|$ and $\mathbf{q}$ is a finite, disjoint collection of dyadic cubes in $\mathbf{D}_\ell$ for some $\ell \ll 0$.

Next we compute the inverse Fourier transform $K^\delta(x,t)$ of $(1-|\xi/\tau|^2)_+^\delta \widehat{\psi}(\tau)$. First note that

$$K^\delta(x,t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}} \mathrm{e}^{2\pi\mathrm{i}\tau(x\cdot\xi+t)}(1-|\xi|^2)_+^\delta \tau^d \widehat{\psi}(\tau)\, d\tau\, d\xi,$$

and by integration by parts via

$$\frac{\partial}{\partial\tau}\mathrm{e}^{2\pi\mathrm{i}\tau(x\cdot\xi+t)} = 2\pi\mathrm{i}(x\cdot\xi+t)\mathrm{e}^{2\pi\mathrm{i}\tau(x\cdot\xi+t)},$$

we have

$$|K^\delta(x,t)| \le C_N \sup_{|\xi|\le 1}(1+|x\cdot\xi+t|)^{-N} \quad \text{for each } N>0.$$

From this, it is easy to see that $K^\delta(x,t)\chi_{(|x|\lesssim 1)} \in L^1(\mathbb{R}^{d+1})$. Therefore, from now on, we assume $|x| \gtrsim 1$. The inverse Fourier transform of the Bochner-Riesz multiplier $(1-|\xi|^2)_+^\delta$, $\xi\in\mathbb{R}^d$ is given by

$$\pi^{-\delta}\Gamma(1+\delta)|x|^{-\frac{d}{2}-\delta} J_{d/2+\delta}(2\pi|x|),$$

where $J_{d/2+\delta}$ is the Bessel function of order $d/2+\delta$. So the inverse Fourier transform $K^\delta(x,t)$ of $(1-|\xi/\tau|^2)_+^\delta\widehat{\psi}(\tau)$ is given by

$$K^\delta(x,t) = \int_{\mathbb{R}} \mathrm{e}^{2\pi\mathrm{i}\tau t}\Big(\pi^{-\delta}\Gamma(1+\delta)|\tau x|^{-\delta-d/2} J_{d/2+\delta}(2\pi\tau|x|)\Big)\tau^d\widehat{\psi}(\tau)\, d\tau.$$

It is well-known that, for all nonnegative integers N and n, as $r \to \infty$,

$$J_m(r) = \mathrm{e}^{\mathrm{i}r}\Big[r^{-1/2}\sum_{j=0}^{N} a_j r^{-j/2} + A_N(r)\Big] + \mathrm{e}^{-\mathrm{i}r}\Big[r^{-1/2}\sum_{j=0}^{N} b_j r^{-j/2} + B_N(r)\Big],$$

$$\frac{d^n}{dr^n}A_N(r) = O(r^{-n-\frac{N+1}{2}}), \quad \frac{d^n}{dr^n}B_N(r) = O(r^{-n-\frac{N+1}{2}}).$$

For reference, see pages $334-338$ in Stein's book [13]. Therefore from the asymptotic expansion of the Bessel function $J_{d/2+\delta}$, for any positive integers N and M we have

$$K^\delta(x,t) = \sum_{j=0}^{N} \alpha_j^{\pm}|x|^{-(\delta+\frac{d+1+j}{2})}\mathcal{F}^{-1}\big[\tau^{-\delta+\frac{d-1-j}{2}}\widehat{\psi}(\tau)\big](t\pm|x|) + F_N^\delta(x,t),$$

as $|x| \to \infty$; here $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and

$$|F_N^\delta(x,t)| \le C_{\delta,N,M}|x|^{-\delta-\frac{d+1+N}{2}}(1+||t|-|x||)^{-M}.$$

Note that

$$F_N^{(d-1)/2}\chi_{(|x|\ge 1)} \in L^1(\mathbb{R}^{d+1}) \quad \text{if } N \ge 1 \text{ and } M > 1.$$

Therefore it suffices to consider the terms

$$|x|^{-\delta-\frac{d+1+j}{2}}\mathcal{F}^{-1}\big[\tau^{-\delta+\frac{d-1-j}{2}}\widehat{\psi}(\tau)\big](t\pm|x|), \quad |x| \ge 1,$$

with $j = 0, 1$, and $\delta = (d-1)/2$. From now on we will concentrate on the term

$$|x|^{-\delta-\frac{d+1}{2}}\mathcal{F}^{-1}\big[\tau^{-\delta+\frac{d-1}{2}}\widehat{\psi}(\tau)\big](t-|x|), \quad |x| \ge 1,\ \delta = (d-1)/2,$$

and the other cases can be treated similarly. Now, we should treat the operation of convolution with

$$|x|^{-d}\psi(t-|x|), \quad |x| > 1.$$

For technical reasons, to obtain the convolution estimates in Lemma 2.3, fix a finite C^∞ partition of unity $\{\omega_i\}$ on the unit sphere $\mathbb{S}^{d-1}$, with each ω_i having very small support. Let ω be one of the C^∞ partition of unity $\{\omega_i\}$. Next choose $\eta \in C_0^\infty(\mathbb{R}^d)$, real-valued, radial and supported in $\{1/2 \le |x| \le 2\}$, so that $\sum_{j\in\mathbb{Z}}\eta(2^{-j}x) = 1$ on $\mathbb{R}^d \setminus \{0\}$. Also choose $\phi \in C_0^\infty(\mathbb{R})$, real-valued and supported in $\{|t| \le 2\}$, so that

$$\sum_{n\in\mathbb{Z}}\phi(t-n) := \sum_{n\in\mathbb{Z}}\phi_n(t) = 1.$$

Then it suffices to treat the operation of convolution with

$$\sum_{n\in\mathbb{Z}}\sum_{j\ge 0}|x|^{-d}\psi(t-|x|)\omega(x/|x|)\phi(t-|x|-n)\eta(2^{-j}x) := \sum_{n\in\mathbb{Z}}\sum_{j\ge 0}K_j^n(x,t).$$

Let f be as in (2.3). Then it suffices to show that

$$\Big|\Big\{(x,t) : \Big|\Big(\sum_{j\ge 0}K_j^n\Big) * f(x,t)\Big| > \alpha\Big\}\Big| \le C_N(1+|n|)^{-N}\alpha^{-1}\|f\|_1,$$

for each $n \in \mathbb{Z}$ and $\alpha > 0$. The following Lemma 2.1 is the standard Calderón-Zygmund decomposition, and we omit the proof (see Lemma 4.1 in [2]).

Lemma 2.1. *Suppose $\beta > 0$ is given. Then for any finite collection $\mathbf{q}$ of dyadic cubes q and associated positive scalars λ_q, there exists a collection of pairwise*

disjoint dyadic cubes $\{S : S \in \mathbf{S}\}$ *such that*

(1) $\sum_{q:\, q\subset S} \lambda_q \le 2^{d+1}\beta|S|$,

(2) $\sum_{S:\, S\in\mathbf{S}} |S| \le \beta^{-1}\sum_{q\in\mathbf{q}} \lambda_q$,

(3) $\left\|\sum_{\substack{q:\text{not contained}\\ \text{in any } S}} \frac{\lambda_q}{|q|}\chi_q\right\|_\infty \le \beta$.

Let $\mathbf{C}$ be the collection of dyadic cubes $q \in \mathbf{q}$ which are contained in some $S \in \mathbf{S}$. For each $q \in \mathbf{C}$ we define $S(q)$ as the unique $S \in \mathbf{S}$ containing q. The following is a refined Calderón-Zygmund decomposition whose proof relies on a stopping time argument.

Lemma 2.2 (cf. Lemma 5.1 in [2] or Lemma 5 in [9])**.** *Given* $\beta > 0$ *there exists a function* $\Gamma : \mathbf{C} \to \mathbb{Z}$ *and a measurable set* E *such that*

(1) $|E| \le C\left(\beta^{-1}\sum_{q\in\mathbf{C}} \lambda_q + \sum_{S\in\mathbf{S}} |S|\right)$,

(2) $\{q + \operatorname{supp}(K_j^n)\} \subset E$ *for all* $j < \Gamma(q)$ *and* $q \in \mathbf{C}$,

(3) $\ell(S(q)) < \Gamma(q)$ *for each* $q \in \mathbf{C}$,

(4) *for each* τ, $\ell \in \mathbb{Z}$ *with* $\ell \le \tau$, *and any* $Q \in \mathbf{D}_\ell$ *we have*

$$\sum_{q\subset Q,\, q\in\mathbf{C},\, \Gamma(q)\le\tau} \lambda_q \le \beta 2^{d(\tau+1)+\max(0,\ell)}.$$

For the proof we use the usual two-parameter stopping time arguments, and we construct an exceptional set E by combining stopping time arguments with the support condition of the kernel K_j^n; i.e., if $\ell(Q) \le \tau$ and $\tau \ge 0$, then

$$\Big|\bigcup_{j=0}^{\tau}\Big(Q + \operatorname{supp}(K_j^n)\Big)\Big| \le C2^{d\tau+\max(0,\ell(Q))}. \tag{2.4}$$

If $S \in \mathbf{S}$ has side-length 2^j, then (1) of Lemma 2.1 says that

$$\sum_{q:q\subset S} \lambda_q \le 2^{d+1}\beta|S|.$$

But if $Q \subset S$, then by (4) of Lemma 2.2 we have a more delicate estimate:

$$\sum_{q\subset Q, q\in\mathbf{C}, \Gamma(q)\le j} \lambda_q \le 2^d\beta 2^{jd+\max(0,\ell(Q))}.$$

This is why we referred to Lemma 2.2 as a refined Calderón-Zygmund decomposition. The proof will be given in Section 4.

Lemma 2.3. *Let* $\widetilde{K}_j^n$ *denote the conjugate of* K_j^n. *Then for each* $i < j$ *and every* $N > 0$ *we have*

(1) $|K_j^n * \widetilde{K}_j^n(x,t)| \le C_N(1+|n|)^{-2N}2^{-dj}\left(1+|(x,t)|\right)^{\frac{1-d}{2}}\chi_{(|(x,t)|\le 2^{j+4})}(x,t)$,

(2) $\left|K_j^n * \widetilde{K}_i^n(x,t)\right| \le C_N(1+|n|)^{-2N}2^{-dj}2^{\frac{1-d}{2}i}\chi_{(|x|\le 2^{j+4})}(x)\chi_{(|t-|x||\le 2^{i+4})}(x,t)$.

The proof of Lemma 2.3 will be given in Section 3. For the moment we assume Lemmas 2.2 and 2.3 and prove Theorem 1.1. Let f be as in (2.3). We need to show that

$$\Big|\Big\{(x,t):\Big|\Big(\sum_{j\ge 0}K_j^n\Big)*f(x,t)\Big|>\alpha\Big\}\Big|\le C_N(1+|n|)^{-N}\alpha^{-1}\sum\lambda_q, \tag{2.5}$$

for each $\alpha>0$. Apply Lemma 2.1 to the collection of dyadic cubes q and associated λ_q appearing in the definition of f with $\beta=(1+|n|)^N\alpha$. Let $\mathbf{S}$ be as in Lemma 2.1 and define

$$b=\sum_{S\in\mathbf{S}}\sum_{q\subset S}\lambda_q a_q,\quad g=f-b.$$

Then $\|g\|_\infty\le\beta$ and so by (1) of Lemma 2.3, for $d\ge 4$ we have

$$\begin{aligned}\Big\|\Big(\sum_j K_j^n\Big)*g\Big\|_2&\le\sum_j\|g\|_2\|K_j^n*\widetilde{K}_j^n\|_1^{1/2}\\&\le\|g\|_2\sum_j\Big(C_N(1+|n|)^{-N}2^{-j(d-3)/4}\Big)\\&\le C_N(1+|n|)^{-N}\beta^{1/2}\|f\|_1^{1/2}\\&\le C_N(1+|n|)^{-N/2}\alpha^{1/2}\|f\|_1^{1/2}.\end{aligned}$$

Therefore by Tchebychev's inequality we have

$$\begin{aligned}\Big|\Big\{(x,t):\Big|\Big(\sum_j K_j^n\Big)*g(x,t)\Big|>\alpha\Big\}\Big|&\le\alpha^{-2}\Big\|\Big(\sum_j K_j^n\Big)*g\Big\|_2^2\\&\le C_N(1+|n|)^{-N}\alpha^{-1}\|f\|_1.\end{aligned}\tag{2.6}$$

Let $\mathbf{S}$ be as above and $\mathbf{C}$ be the collection of q's appearing in the definition of b. Then apply Lemma 2.2 with $\mathbf{S}$, $\mathbf{C}$. By (1) of Lemma 2.2 we have an exceptional set E such that

$$|E|\le C\Big(\beta^{-1}\sum_{q\in\mathbf{C}}\lambda_q+\sum_{S\in\mathbf{S}}|S|\Big)\le C\beta^{-1}\|f\|_1\le C(1+|n|)^{-N}\alpha^{-1}\|f\|_1. \tag{2.7}$$

By (2.6) and (2.7), (2.5) will follow from

$$\Big|\Big\{(x,t)\in\mathbb{R}^{d+1}\setminus E:\Big|\Big(\sum_j K_j^n\Big)*b(x,t)\Big|>\alpha\Big\}\Big|\le C_N(1+|n|)^{-N}\alpha^{-1}\sum_q\lambda_q. \tag{2.8}$$

By Tchebychev's inequality, (2.8) will follow from

$$\Big\|\Big(\sum_j K_j^n\Big)*b\Big\|^2_{L^2(\mathbb{R}^{d+1}\setminus E)}\le C_N(1+|n|)^{-N}\alpha\sum_q\lambda_q. \tag{2.9}$$

By (2) of Lemma 2.2, for each $q\in\mathbf{C}$, $K_j^n*a_q$ is supported in E unless $j\ge\Gamma(q)$. Thus we have

$$\begin{aligned}&\Big\|\sum_j b*K_j^n\Big\|_{L^2(\mathbb{R}^{d+1}\setminus E)}\\&\le\Big\|\sum_j\Big(\sum_{\Gamma(q)\le j}\lambda_q a_q\Big)*K_j^n\Big\|_2\\&\le\Big\|\sum_j\Big(\sum_{\Gamma(q)\le 0}\lambda_q a_q\Big)*K_j^n\Big\|_2+\Big\|\sum_j\Big(\sum_{0<\Gamma(q)\le j}\lambda_q a_q\Big)*K_j^n\Big\|_2\end{aligned}\tag{2.10}$$

and

$$\Big\|\sum_j\Big(\sum_{0<\Gamma(q)\le j}\lambda_q a_q\Big)*K_j^n\Big\|_2=\Big\|\sum_j\sum_{0\le s<j}\Big(\sum_{\Gamma(q)=j-s}\lambda_q a_q\Big)*K_j^n\Big\|_2.$$

Note that

$$\sum_j\sum_{0\le s<j}=\sum_{s\ge 0}\sum_{j>s};$$

hence we have

$$\Big\|\sum_j\sum_{0\le s<j}\Big(\sum_{\Gamma(q)=j-s}\lambda_q a_q\Big)*K_j^n\Big\|_2\le\sum_{s\ge 0}\Big\|\sum_{j>s}\Big(\sum_{\Gamma(q)=j-s}\lambda_q a_q\Big)*K_j^n\Big\|_2. \tag{2.11}$$

Now by (2.10) and (2.11) we have

$$\Big\|\sum_j b*K_j^n\Big\|_{L^2(\mathbb{R}^{d+1}\setminus E)}\le \mathcal{F}_1+\mathcal{F}_2,$$

where

$$\mathcal{F}_1=\sum_j\Big\|\Big(\sum_{\Gamma(q)\le 0}\lambda_q a_q\Big)*K_j^n\Big\|_2,\quad \mathcal{F}_2=\sum_{s\ge 0}\Big\|\sum_{j>s}\Big(\sum_{\Gamma(q)=j-s}\lambda_q a_q\Big)*K_j^n\Big\|_2.$$

Estimation of part $\mathcal{F}_1$. Let $B_0=\sum_{\Gamma(q)\le 0}\lambda_q a_q$. For each $q\in\mathbf{C}$ there exists a unique $S(q)\in\mathbf{S}$ containing q; hence if $\Gamma(q)\le 0$, then by the condition $\ell(S(q))\le\Gamma(q)$ we have $\ell(S(q))\le 0$. Therefore by (1) of Lemma 2.1, we can see that

$$\|B_0\chi_Q\|_1=\sum_{q\subset Q,\Gamma(q)\le 0}\lambda_q\le C\beta\quad\text{for each } Q\in\mathbf{D}_0. \tag{2.12}$$

By (1) of Lemma 2.3, we have

$$\|K_j^n*\widetilde{K}_j^n*B_0\|_\infty\le C_N(1+|n|)^{-2N}2^{-jd}\sup_{x\in\mathbb{R}^{d+1}}\Big(\int_{|x-y|\lesssim 2^j}\frac{B_0(y)}{(1+|x-y|)^{\frac{d-1}{2}}}\,dy\Big).$$

By (2.12) it is easy to see that

$$\begin{aligned}\Big(\int_{|x-y|\lesssim 2^j}\frac{B_0(y)}{(1+|x-y|)^{\frac{d-1}{2}}}\,dy\Big)&\le\Big(\int_{|x-y|\lesssim 2^j}\frac{C\beta}{(1+|x-y|)^{\frac{d-1}{2}}}\,dy\Big)\\&\le C\beta 2^{\frac{d+3}{2}j}.\end{aligned}$$

Therefore we have

$$\|K_j^n*\widetilde{K}_j^n*B_0\|_\infty\le C_N(1+|n|)^{-2N}\beta 2^{\frac{3-d}{2}j},$$

and so for $d\ge 4$,

$$\begin{aligned}\mathcal{F}_1=\sum_j\big\|B_0*K_j^n\big\|_2&=\sum_j\big|\langle B_0,K_j^n*\widetilde{K}_j^n*B_0\rangle\big|^{1/2}\\&\le\|B_0\|_1^{1/2}\sum_j\|K_j^n*\widetilde{K}_j^n*B_0\|_\infty^{1/2}\\&\le C_N(1+|n|)^{-N}\beta^{1/2}\|f\|_1^{1/2}.\end{aligned}\tag{2.13}$$

Estimation of part F_2. For each $l > 0$, let $B_l = \sum_{\Gamma(q)=l} \lambda_q a_q$. Then

$$\begin{aligned}\Big\|\sum_{j>s} B_{j-s} * K_j^n\Big\|_2^2 &\le \sum_{j>s} \|B_{j-s} * K_j^n\|_2^2 \\ &\quad +2\sum_{j>s} \big|\langle B_{j-s} * K_j^n, B_{j-1-s} * K_{j-1}^n\rangle\big| \\ &\quad +2\sum_{j>s}\sum_{s<i<j-1} \big|\langle B_{i-s} * K_i^n * \widetilde{K}_j^n, B_{j-s}\rangle\big| \\ &:= A_1(s) + A_2(s) + A_3(s).\end{aligned}$$

For the part $A_1(s)$, we write

$$\begin{aligned}\|B_{j-s} * K_j^n\|_2^2 &= \sum_{\Gamma(q),\Gamma(q')=j-s} \lambda_q\lambda_{q'}\langle a_{q'} * K_j^n, a_q * K_j^n\rangle \\ &\le \sum_{m=0}^{j+4} \sum_{\substack{\Gamma(q),\Gamma(q')=j-s \\ 1+\mathrm{dist}(q,q')\sim 2^m}} \lambda_q\lambda_{q'}\big|\langle a_{q'} * K_j^n * \widetilde{K}_j^n, a_q\rangle\big| \\ &\le \sum_{m=0}^{j-s+2} \sum_{\substack{\Gamma(q),\Gamma(q')=j-s \\ 1+\mathrm{dist}(q,q')\sim 2^m}} + \sum_{m=j-s+3}^{j+4} \sum_{\substack{\Gamma(q),\Gamma(q')=j-s \\ 1+\mathrm{dist}(q,q')\sim 2^m}} \\ &:= \mathrm{I} + \mathrm{II}.\end{aligned}$$

Estimation of part I. For each fixed m and q' consider the contribution of all λ_q over all q so that $1 + \mathrm{dist}(q, q') \sim 2^m$. All such q are contained in the union of a fixed number of $\mathbf{D}_m$. Hence when $0 \le m \le j - s + 2$, by (1) of Lemma 2.3 we have

$$\mathrm{I} \le C_N(1 + |n|)^{-2N} 2^{-jd} \sum_{q'} \lambda_{q'} \sum_{m=0}^{j-s+2} \sum_{q:\, 1+\mathrm{dist}(q,q')\sim 2^m} 2^{\frac{1-d}{2}m}\lambda_q.$$

By (4) of Lemma 2.2 we have

$$\sum_{q:\, 1+\mathrm{dist}(q,q')\sim 2^m,\ \Gamma(q)=j-s} \lambda_q \le C\beta 2^{(j-s)d+m},$$

and so for $d \ge 4$,

$$\mathrm{I} \le C_N(1 + |n|)^{-2N} 2^{-ds}\beta \sum_{\Gamma(q')=j-s} \lambda_{q'}.$$

Estimation of part II. Next, consider all q with $\mathrm{dist}(q, q') \sim 2^m$ and $j - s + 3 \le m \le j + 4$. Recall that each $q \in \mathbf{C}$ is contained in some $S(q) \in \mathbf{S}$. Since $\ell(S(q)) < \Gamma(q) = j - s$, we have $\mathrm{dist}(S(q), q') \sim 2^m$ and so

$$\begin{aligned}\mathrm{II} &\le C_N(1 + |n|)^{-2N} 2^{-jd} \sum_{q'} \lambda_{q'} \sum_{m=j-s+3}^{j+4} \sum_{q:\, \mathrm{dist}(q,q')\sim 2^m} 2^{\frac{1-d}{2}m}\lambda_q \\ &\le C_N(1 + |n|)^{-2N} 2^{-jd} \sum_{q'} \lambda_{q'} \sum_{S\in\mathbf{S}\ 2^{j-s+2}\le \mathrm{dist}(S,q')\le 2^{j+5}} \sum_{q\subset S} \mathrm{dist}(S, q')^{\frac{1-d}{2}}\lambda_q.\end{aligned}$$

By (1) of Lemma 2.1, for each $S \in \mathbf{S}$ we have $\sum_{q\subset S} \lambda_q \le C\beta|S|$, and so

$$\begin{aligned}\sum_{\substack{S\in\mathbf{S}\\ 2^{j-s+2}\le \mathrm{dist}(S,q')\le 2^{j+5}}} \sum_{q\subset S} \mathrm{dist}(S,q')^{\frac{1-d}{2}}\lambda_q &\le C \sum_{\substack{S\in\mathbf{S}\\ 2^{j-s+2}\le \mathrm{dist}(S,q')\le 2^{j+5}}} \mathrm{dist}(S,q')^{\frac{1-d}{2}}(\beta|S|)\\ &\le C\beta \int_{y\in\mathbb{R}^{d+1},\, 2^{j-s+2}\le|y|\le 2^{j+5}} |y|^{\frac{1-d}{2}}dy\\ &\le C2^{\frac{3-d}{2}j}\beta.\end{aligned}$$

Now we have

$$\mathrm{II} \le C_N(1+|n|)^{-2N}2^{\frac{3-d}{2}j}\beta \sum_{\Gamma(q')=j-s}\lambda_{q'},$$

and so for $d \ge 4$,

$$\begin{aligned}A_1(s) &\le \sum_{j>s} C_N(1+|n|)^{-2N}\left(2^{-sd}+2^{\frac{3-d}{2}j}\right)\beta \sum_{\Gamma(q')=j-s}\lambda_{q'}\\ &\le C_N(1+|n|)^{-2N}2^{\frac{3-d}{2}s}\beta\sum_{q'}\lambda_{q'}.\end{aligned} \tag{2.14}$$

Similarly we have

$$A_2(s) \le C_N(1+|n|)^{-2N}2^{\frac{3-d}{2}s}\beta\sum_{q'}\lambda_{q'}. \tag{2.15}$$

For the part $A_3(s)$, if $\Gamma(q) = i-s$, then $\ell(q), S(q) \le i-s$. So by (1) of Lemma 2.1 and (2) of Lemma (2.3) together with the condition

$$\mathrm{supp}(K_i^n * \widetilde{K}_j^n) \subset \left\{(x,t) : |x| \le 2^{j+4},\, |t-|x|| \le 2^{i+4}\right\},$$

we have

$$\begin{aligned}\|B_{i-s} * K_i^n * \widetilde{K}_j^n\|_\infty &\le C_N(1+|n|)^{-2N}2^{-jd}2^{\frac{1-d}{2}i}\sum_{S\in\mathbf{S}}\sum_{\substack{q\subset S\\ \Gamma(q)=i-s}}\lambda_q\\ &\le C_N(1+|n|)^{-2N}2^{-jd}2^{\frac{1-d}{2}i}\sum_{S\in\mathbf{S}}\beta|S|,\end{aligned}$$

where the sums are taken over all $S \in \mathbf{S}$ such that

$$S \subset \left\{(x,t) : |x| \le 2^{j+4},\, |t-|x|| \le 2^{i+4}\right\}.$$

So we have $\sum_{S\in\mathbf{S}}|S| \le C2^{jd+i}$ and

$$\|B_{i-s} * K_i^n * \widetilde{K}_j^n\|_\infty \le C_N(1+|n|)^{-2N}2^{\frac{3-d}{2}i}\beta.$$

Therefore for $d \ge 4$,

$$\begin{aligned}A_3(s) &= \sum_{j>s}\sum_{s<i<j-1}\left|\langle B_{i-s} * K_i^n * \widetilde{K}_j^n, B_{j-s}\rangle\right|\\ &\le C_N(1+|n|)^{-2N}\sum_{j>s}\sum_{i>s}2^{\frac{3-d}{2}i}\beta\|B_{j-s}\|_1\\ &\le C_N(1+|n|)^{-2N}2^{\frac{3-d}{2}s}\beta\sum_q\lambda_q.\end{aligned} \tag{2.16}$$

Finally from (2.13), (2.14), (2.15) and (2.16), for $d \geq 4$ we have

$$\mathcal{F}_2 \leq \sum_{s\geq 0} (A_1(s) + A_2(s) + A_3(s))^{1/2} \leq C_N(1+|n|)^{-N}\beta^{1/2}\big(\sum_q \lambda_q\big)^{1/2},$$

and we are done.

3. Proof of Lemma 2.3

The proof is similar to Lemma 3.1 in [2]. For (1), let $\eta_1(x) := \eta(x)|x|^{-d}$. Then we have

$$K_j^n * \widetilde{K}_j^n(x,t) = 2^{-2dj}\int_{\mathbb{R}}\int_0^\infty F_j(x,t,s,r)\overline{\psi}(s-r)\eta_1(2^{-j}r)\phi_n(s-r)r^{d-1}drds,$$

where

$$F_j(x,t,s,r) = \int_{\mathbb{S}^{d-1}} \psi(t+s-|x+r\theta|)\omega\big[(x+r\theta)/|x+r\theta|\big]\omega(\theta) \\ \times\phi_n(t+s-|x+r\theta|)\eta_1\big[2^{-j}(x+r\theta)\big]\,d\theta.$$

Note that

$$\psi(t+s-|x+r\theta|) = \int_{\mathbb{R}} \mathrm{e}^{2\pi\mathrm{i}\tau(t+s-|x+r\theta|)}\widehat{\psi}(\tau)d\tau.$$

Hence we have

$$F_j(x,t,s,r) = \int_{\mathbb{R}} \mathrm{e}^{2\pi\mathrm{i}\tau(t+s)}\widehat{\psi}(\tau)G_j(x,t,s,r,\tau)\,d\tau, \tag{3.1}$$

where

$$G_j(x,t,s,r,\tau) = \int_{\mathbb{S}^{d-1}} \mathrm{e}^{-2\pi\mathrm{i}\tau|x+r\theta|}\,\omega\big[(x+r\theta)/|x+r\theta|\big]\omega(\theta) \\ \times\phi_n(t+s-|x+r\theta|)\eta_1\big[2^{-j}(x+r\theta)\big]d\theta. \tag{3.2}$$

It suffices to show that

$$|G_j(x,t,s,r,\tau)| \leq C\,(1+|(x,t)|)^{(1-d)/2},$$

uniformly for $\tau \sim 1$, s and $r \sim 2^j$. Suppose that the function ω from the partition of unity has sufficiently small support about $(0,\dots,0,1)$. Then we use the local coordinate chart

$$\theta = (\theta_1,\dots,\theta_d) = \Big(\theta_1,\dots,\theta_{d-1},\sqrt{1-\theta_1^2-\cdots-\theta_{d-1}^2}\Big).$$

Then by direct calculation, for $1 \leq l \leq d-1$,

$$\frac{\partial}{\partial\theta_l}(|x+r\theta|) = r\left(\frac{\theta_d x_l - \theta_l x_d}{|x+r\theta|\theta_d}\right), \tag{3.3}$$

$$\frac{\partial^2|x+r\theta|}{\partial\theta_l^2} = r\left(\frac{|x+r\theta|^2(\theta_d^2-\theta_l^2)(-x_d) - r\theta_d|\theta_d x_l - \theta_l x_d|^2}{|x+r\theta|^3\theta_d^3}\right), \tag{3.4}$$

and for $1 \leq l \neq m \leq d-1$,

$$\frac{\partial^2|x+r\theta|}{\partial\theta_l\partial\theta_m} = -r\left(\frac{\theta_l\theta_m x_d|x+r\theta|^2 + r\theta_d(\theta_d x_m - \theta_m x_d)(\theta_d x_l - \theta_l x_d)}{|x+r\theta|^3\theta_d^3}\right). \tag{3.5}$$

Note that

$$\frac{\partial^2|x+r\theta|}{\partial\theta_l^2}(0,\dots,0,1) = r\left(\frac{|x+r\theta|^2(-x_d) - x_l^2}{|x+r\theta|^3}\right) \tag{3.6}$$

and

$$\frac{\partial^2|x+r\theta|}{\partial\theta_l\partial\theta_m}(0,\dots,0,1)=\frac{-r^2x_lx_m}{|x+r\theta|^3},\quad 1\le l\ne m\le d-1. \tag{3.7}$$

In the case $|x_l|\gtrsim|x|$ for some $1\le l\le d-1$, by (3.3),

$$\Big|\frac{\partial|x+r\theta|}{\partial\theta_l}\Big|\gtrsim|x_l|$$

uniformly for $|x+r\theta|\sim 2^j$, $r\sim 2^j$. By integrating by parts via

$$\frac{\partial}{\partial\theta_l}\left(\mathrm{e}^{-2\pi\mathrm{i}\tau|x+r\theta|}\right)=-2\pi\mathrm{i}\tau\frac{\partial|x+r\theta|}{\partial\theta_l}\left(\mathrm{e}^{-2\pi\mathrm{i}\tau|x+r\theta|}\right),$$

we have

$$|G_j(x,t,s,r,\tau)|\le C_N\left(1+|x_l|\right)^{-N}\le C_N\left(1+|x|\right)^{-N}.$$

In the case $|x_d|\gtrsim|x|$, from (3.4), (3.5), (3.6), (3.7), together with the conditions $|x+r\theta|\sim 2^j$, $r\sim 2^j$, we can see that each absolute value of the eigenvalue of the Hessian matrix

$$\frac{\partial^2|x+r\theta|}{\partial^2\theta}$$

is bounded below by $C|x_d|$ uniformly for $2^{j-1}<r<2^{j+1}$ when θ and $(x+r\theta)/|x+r\theta|$ are both in the support of ω. By the methods of stationary phase, we have

$$|G_j(x,t,s,r,\tau)|\le C\left(1+|x_d|\right)^{(1-d)/2}\le\left(1+|x|\right)^{(1-d)/2}.$$

Hence we have

$$|K_j^n*\widetilde{K}_j^n(x,t)|\le C_N(1+|n|)^{-2N}2^{-jd}\left(1+|x|\right)^{(1-d)/2}. \tag{3.8}$$

Next recall that

$$\begin{aligned}K_j^n*\widetilde{K}_j^n(x,t)=2^{-2dj}\int_{\mathbb{R}}\int_{\mathbb{R}^d}&\psi(t+s-|x+y|)\overline{\psi}(s-|y|)\omega\big[(x+y)/|x+y|\big]\\&\times\omega(y/|y|)\phi_n(t+s-|x+y|)\phi_n(s-|y|)\eta_1\big[2^{-j}(x+y)\big]\eta_1(2^{-j}y)dyds.\end{aligned}$$

Hence $K_j^n*\widetilde{K}_j^n(x,t)$ is supported in

$$|t+s-|x+y|-n|\le 2,\quad |s-|y|-n|\le 2,$$

and by the triangle inequality, this implies that

$$|t|\le|t+s-|x+y|-n|+|-s+|y|+n|+|-|y|+|x-y||\le 2+2+|x|. \tag{3.9}$$

Therefore if $|x|\le 2$, then $|(x,t)|\le 6$, and (3.8) implies that

$$|K_j^n*\widetilde{K}_j^n(x,t)|\le C_N(1+|n|)^{-2N}2^{-jd}\left(1+|(x,t)|\right)^{-N}.$$

If $|x|\ge 2$, then by (3.9) we have $|t|\le 3|x|$, and (3.8) implies that

$$|K_j^n*\widetilde{K}_j^n(x,t)|\le C_N(1+|n|)^{-2N}2^{-jd}\left(1+|(x,t)|\right)^{(1-d)/2}.$$

For (2), let $F_j(x,t,s,r)$ and $G_j(x,t,s,r,\tau)$ be the same as in (3.1) and (3.2). Then we have

$$K_j^n*\widetilde{K}_i^n(x,t)=2^{-dj-di}\int_{\mathbb{R}}\int_0^\infty F_j(x,t,s,r)\overline{\psi}(s-r)\eta_1(2^{-i}r)\phi_n(s-r)r^{d-1}drds.$$

Since $|x+r\theta| \sim 2^j$, $r \sim 2^i$, we have $|x| \sim 2^j$. Hence in the case $|x_l| \gtrsim |x|$ for some $1 \le l \le d-1$, from (3.3) we can see that

$$\Big|\frac{\partial |x+r\theta|}{\partial \theta_l}\Big| \gtrsim 2^i,$$

uniformly for $|x+r\theta| \sim 2^j$ and $r \sim 2^i$, and by integrating by parts via

$$\frac{\partial}{\partial \theta_l}\left(\mathrm{e}^{-2\pi \mathrm{i}\tau|x+r\theta|}\right) = -2\pi \mathrm{i}\tau \frac{\partial |x+r\theta|}{\partial \theta_l}\left(\mathrm{e}^{-2\pi \mathrm{i}\tau|x+r\theta|}\right),$$

we have

$$|G_j(x,t,s,r,\tau)| \le C_N 2^{-Ni}.$$

In the case $|x_d| \gtrsim |x|$, each absolute value of the eigenvalue of the Hessian matrix

$$\frac{\partial^2 |x+r\theta|}{\partial^2 \theta}$$

is bounded below by $C2^i$ uniformly for $r \sim 2^i$ when θ and $(x+r\theta)/|x+r\theta|$ are both in support of ω. By the methods of stationary phase, we have

$$|G_j(x,t,s,r,\tau)| \le C 2^{\frac{1-d}{2}i}.$$

Hence in the case $|x_d| \gtrsim |x|$ we have

$$|K_j^n * \widetilde{K}_i^n(x,t)| \le C_N(1+|n|)^{-2N} 2^{-jd} 2^{\frac{1-d}{2}i}. \tag{3.10}$$

Note that $K_j^n * \widetilde{K}_i^n(x,t)$ is supported in

$$|t+s-|x+y|-n| \le 2,\ |s-|y|-n| \le 2,\ |x+y| \le 2^{j+1},\ |y| \le 2^{i+1}.$$

Hence $K_j^n * \widetilde{K}_i^n(x,t)$ is supported in

$$\begin{aligned} |t-|x|| &\le |t+s-|x+y|-n| + ||x+y|-|x|| + |-s+n| \\ &\le 2+|y|+(2+|y|) \le 2^{i+4}, \end{aligned} \tag{3.11}$$

and from (3.10) and (3.11) we have (2) of Lemma 2.3.

4. Proof of Lemma 2.2

The following are the usual two-parameter stopping-time arguments. These will be discussed in more detail below. We will combine these arguments with the support condition of the kernel K_j^n to construct an exceptional set E. Let $m = \min\{\ell(q) : q \in \mathbf{C}\}$. Select an integer τ_0 such that

$$\tau_0 > \max\{\ell(q) : q \in \mathbf{C}\}\ , \quad \sum_{q\in\mathbf{C}} \lambda_q < \beta 2^{d\tau_0 + \max(0,m)}.$$

For each fixed $\tau \in \mathbb{Z}$ with $\tau \le \tau_0$, we will define a sequence of functions $\Lambda_{\tau,\ell} : \mathbf{D}_\ell \to \mathbb{R}$ by a descending induction on $\ell \in \mathbb{Z}$ with $\ell \le \tau$, and proceed with the same construction by a descending induction on τ. At each step, we will define subsets $\mathbf{C}_1$, $\mathbf{C}_2$ of $\mathbf{C}$ which will increase as we proceed. Let $\mathbf{C}_1, \mathbf{C}_2 \subset \mathbf{C}$ and $\tau \in \mathbb{Z}$ be fixed for the moment, and define **Inner loop** as

Inner loop. Define $\Lambda_{\tau,\ell} : \mathbf{D}_\ell \to \mathbb{R}$ with $\ell \le \tau$. For each $Q \in \mathbf{D}_\ell$, define

$$\Lambda_{\tau,\ell}(Q) = \sum_{q\subset Q:\, q\notin \mathbf{C}_1\cup\mathbf{C}_2} \lambda_q.$$

Begin with $\ell = \tau$. If

$$\Lambda_{\tau,\ell}(Q) > \beta 2^{d\tau+\max(0,\ell)}, \tag{4.1}$$

then we say that "Q is selected at step (τ,ℓ)". Put into $\mathbf{C}_1$ every q such that $q \subset Q$ and define $\Gamma(q) = 1+\tau$. Repeat until $\ell < \min\{\ell(q) : q \in \mathbf{C}\}$. Actually this part of the process will be terminated once ℓ is smaller than m. Put into $\mathbf{C}_2$ every $q \in \mathbf{C}\setminus\mathbf{C}_1$ such that $\ell(q) \ge \tau$ and for such q define $\Gamma(q) = 1+\ell(S(q))$. Actually every $q \in \mathbf{C}\setminus\mathbf{C}_1\cup\mathbf{C}_2$ satisfies $\ell(q) \le \tau - 1$.

Perform **Inner loop** with $\mathbf{C}_1 = \emptyset = \mathbf{C}_2$ and $\tau = \tau_0$. Next replace τ by $\tau - 1$ and repeat **Inner loop**. Repeat until $\tau = m-1$. After this process we obtain $\mathbf{C} = \mathbf{C}_1 \cup \mathbf{C}_2$. Clearly all selected Q are disjoint and Γ is well defined. Note that there is the usual stopping-time condition

$$\Lambda_{\tau,\ell}(Q) \le \beta 2^{d(\tau+1)+\max(0,\ell)}, \tag{4.2}$$

which holds for all $Q \in \mathbf{D}_\ell$ when $\ell \le \tau \le \tau_0$. This is because if $\tau = \tau_0$, then the condition is clear from the choice of τ_0 and m. When $\ell \le \tau \le \tau_0$, suppose this fails. Then $\Lambda_{\tau+1,\ell}(Q) \ge \Lambda_{\tau,\ell}(Q) > \beta 2^{d(\tau+1)+\max(0,\ell)}$. This means that Q is selected at step $(\tau+1,\ell)$; hence $\Lambda_{\tau,\ell}(Q) = 0$, and we have the contradiction.

Next we show (4), which says, for each $Q \in \mathbf{D}_\ell$ with $\ell \le \tau$,

$$\sum_{q\subset Q:\, \Gamma(q)\le\tau} \lambda_q \le \beta 2^{d(\tau+1)+\max(0,\ell)}.$$

When $\tau \ge \tau_0$, this property is clear from the initial choice of τ_0. When $\tau < \tau_0$, if we show that

$$\Lambda_{\tau,\ell}(Q) = \sum_{q\subset Q:\, q\notin \mathbf{C}_1\cup\mathbf{C}_2} \lambda_q \ge \sum_{q\subset Q:\, \Gamma(q)\le\tau} \lambda_q \tag{4.3}$$

for each $Q \in \mathbf{D}_\ell$ with $\ell \le \tau < \tau_0$, then by combining (4.2) and (4.3), we have (4). Now (4.3) follows from the definition

$$\Lambda_{\tau,\ell}(Q) = \sum_{q\subset Q:\, q\notin \mathbf{C}_1\cup\mathbf{C}_2} \lambda_q$$

and the fact that

$$\Gamma(q) \le \tau \Rightarrow q \notin \mathbf{C}_1 \cup \mathbf{C}_2 \text{ at the beginning of step } (\tau,\ell).$$

This is because, if $q \in \mathbf{C}_1$, then $\Gamma(q) \ge 1+\tau > \tau$, and if $q \in \mathbf{C}_2$, then $\Gamma(q) = 1+\ell(S(q)) \ge 1+(1+\tau) > \tau$. Hence $\Gamma(q) \le \tau$ implies $q \notin \mathbf{C}_1 \cup \mathbf{C}_2$, and so we have (4.3).

Next, we construct an exceptional set E by using the above stopping-time arguments. If Q is selected at step (τ,ℓ), then we define $\tau(Q) = \tau$. If $\tau(Q) \ge 0$, then we define the tendril $T(Q)$ associated to Q by

$$T(Q) = \bigcup_{j<\tau(Q)+1} \Big(Q + \operatorname{supp}(K_j^n)\Big).$$

Also we define

$$E = E_1 \cup E_2 \ , \quad E_1 = \bigcup_{S\in\mathbf{S}} 2S \ , \quad E_2 = \bigcup_{Q\colon \text{selected}, \tau(Q)\geq 0} T(Q).$$

Thus we have

$$|E_1| \leq C \sum_{S\in\mathbf{S}} |S|.$$

By the measure condition (2.4) of $T(Q)$ and the condition (4.1) of selected Q,

$$\begin{aligned} |E_2| &\leq \sum_{Q\colon \text{selected}, \tau(Q)\geq 0} |T(Q)| \leq C \sum_{Q\colon \text{selected}} 2^{d\tau(Q)+\max(0,\ell(Q))} \\ &\leq C\frac{1}{\beta} \sum_{Q\colon \text{selected}} \Lambda_{\tau,\ell}(Q) \leq C\frac{1}{\beta}\sum \lambda_q. \end{aligned}$$

So we have (1). For (2), if $q \in \mathbf{C}_1$, then q belongs to some selected Q and $\Gamma(q) = \tau(Q) + 1$. Hence if $\Gamma(q) > j$, then

$$\bigcup_{j<\Gamma(q)} \Big(q + \operatorname{supp}(K_j^n)\Big) \subset T(Q) \subset E_2,$$

and if $q \in \mathbf{C}_2$, then q belongs to some $S(q) \in \mathbf{S}$ and $\Gamma(q) = 1 + \ell(S(q))$. Hence if $\Gamma(q) > j$, then

$$\bigcup_{j<\Gamma(q)} \{q + \operatorname{supp}(K_j^n)\} \subset 2S(q) \subset E_1.$$

For (3), let $\Gamma' := \Gamma$ and redefine

$$\Gamma(q) = \max\{\Gamma'(q), 1 + \ell(S(q))\}.$$

Then (1) and (3) are satisfied. We must check (2) and (4). For (2), if $\Gamma(q) = \Gamma'(q)$, then there is no problem. If $\Gamma(q) = 1 + \ell(S(q)) > j$, then the argument is the same as above. (4) follows from

$$\sum_{q\subset Q\colon \Gamma(q)\leq\tau} \lambda_q \leq \sum_{q\subset Q\colon \Gamma'(q)\leq\tau} \lambda_q,$$

and we have Lemma 2.2.

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