BOCKSTEIN THEOREM FOR NILPOTENT GROUPS

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ABSTRACT. We extend the definition of Bockstein basis $\sigma(G)$ to nilpotent groups $G$. A metrizable space $X$ is called a Bockstein space if $\dim_G(X)$ is defined and is equal to $\sup\{\dim_H(X) \mid H \in \sigma(G)\}$ for all Abelian groups $G$. The Bockstein First Theorem says that all compact spaces are Bockstein spaces.

Here are the main results of the paper:

Theorem 0.1. Let $X$ be a Bockstein space. If $G$ is nilpotent, then $\dim_G(X)$ is defined and is equal to $\sup\{\dim_H(X) \mid H \in \sigma(G)\}$.

Theorem 0.2. $X$ is a Bockstein space if and only if $\dim_{Z(l)}(X) = \dim_{\hat{Z}(l)}(X)$ for all subsets $l$ of prime numbers.

1. INTRODUCTION

We use the Kuratowski notation $X \tau M$ in the case when every map from a closed subset of $X$ to $M$ can be extended over all $X$.

Recall that the cohomological dimension of a space $X$ with respect to an Abelian group $G$ is less than or equal to $n$, denoted by $\dim_G X \leq n$, if $H^{n+1}(X,A;G) = 0$ for all closed $A \subset X$.

Of basic importance in cohomological dimension theory is the Bockstein basis $\sigma(G)$ of an Abelian group $G$ (see Definition 3.1) and the following result of Bockstein (see [12] or [4]):

Theorem 1.1 (Bockstein First Theorem). If $X$ is a compact space, then $\dim_G(X) = \sup\{\dim_H(X) \mid H \in \sigma(G)\}$.

The aim of this paper is to generalize Theorem 1.1 to nilpotent groups. There are two issues to resolve first:

1. Define cohomological dimension with respect to non-Abelian groups.

2. Define the Bockstein basis of nilpotent groups.

The definition of $\dim_G(X)$ for $G$ non-Abelian was first introduced by A. Dranishnikov and D. Repovš [5] as follows: By [4], Theorem 1.1], $\dim_G(X) \leq n$ is equivalent (for Abelian groups $G$) to $X \tau K(G,n)$, where $K(G,n)$ is an Eilenberg-MacLane

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space. One ought to use the same equivalence in the definition of $\dim_G(X)$ for non-Abelian groups. For nilpotent groups $G$ this definition was used for the characterization of nilpotent CW complexes as absolute extensors of metric compacta (see [3]). There is no Eilenberg-MacLane space $K(G, n)$, $1 < n < \infty$, for non-Abelian groups $G$, so $\dim_G X \in \{0, 1, \infty\}$. Since $\dim_G X = 0$ means $\dim(X) = 0$ (see Lemma 1.2), the only interesting question is if $X \tau K(G, 1)$ holds.

Our definition of the Bockstein basis for nilpotent groups can be found in Definition 3.2.

The remainder of this section is devoted to elementary properties of cohomological dimension over non-Abelian groups.

**Lemma 1.2.** If $X$ is a metrizable space and $\dim_G(X) = 0$ for some group $G \neq 1$, then $\dim_H(X) = 0$ for any group $H$.

**Proof.** We will show that for a nontrivial group $G$ we have $\dim_G(X) = 0$ if and only if $\dim_{Z_2}(X) = 0$ and $\dim_{Z_2}(X) = 0$ means $\dim(X) = 0$.

Suppose $\dim_G(X) = 0$ for some $G \neq 1$. Let $i: Z_2 \to G$ be an injection and let $r: G \to Z_2$ be a map such that $r \circ i = id_{Z_2}$. Let $A$ be a closed subset of $X$ and $f: A \to K(Z_2, 0) = Z_2$ a map. Then there exists an extension $f': X \to G$ of $i \circ f: A \to G$. Then $F = r \circ f': X \to Z_2$ is an extension of $f$.

Suppose that $\dim_{Z_2}(X) = 0$. Let $A \subset X$ be a closed subset and let $f: A \to G$ be a continuous map. For every $g \in G$ we define $A_g = f^{-1}(g)$ and $B_g = \{ x \in X \mid d(x, A \setminus A_g) \leq d(x, A_g) \}$. Because $A \setminus A_g$ is a closed subset of $X$, $B_g$ is a closed subset of $X$. Because $B_g \cap A_g = \emptyset$ we can define a continuous map $f_g: A_g \cup B_g \to Z_2$ by $f_0(A_g) = 1$ and $f_0(B_g) = 0$. Let $F_g: X \to Z_2$ be an extension of $f_g$. For every $g \in G$ we define $X_g = F_g^{-1}(1)$. Sets $X_g$ are open and closed in $X$ and pairwise disjoint. So we can define a continuous map $F: X \to G$ as

$$F(x) = \begin{cases} g & ; x \in X_g, \\ e & ; x \not\in \bigcup_{g \in G} X_g. \end{cases}$$

The map $F$ is an extension of $f$. \hfill $\square$

**Lemma 1.3.** Let $X$ be a metrizable space. If $1 \to J \to G \to I \to 1$ is an exact sequence of groups and $\dim_J X \leq 1$, then $\dim_I(X) \leq 1$ if and only if $\dim_G(X) \leq 1$.

**Proof.** In view of Lemma 1.2 the only interesting case is that of $\dim_J X = 1$. Use the fibration $K(J, 1) \to K(G, 1) \to K(I, 1)$ and the fact that $X \tau K(J, 1)$ to conclude $X \tau K(I, 1)$ if and only if $X \tau K(G, 1)$. \hfill $\square$

If $G$ is a group, then $\text{Ab}(G)$ is its abelianization.

**Lemma 1.4.** If $X$ is a metrizable space, then $\dim_{\text{Ab}(G)}(X) \leq \dim_G X$ for any group $G$.

**Proof.** In view of Lemma 1.2 the only interesting case is that of $G$ non-Abelian and $\dim_G X = 1$. Since $X \tau K(G, 1)$ one gets $X \tau K(H_1(K(G, 1)), 1)$ by Theorem 3.4 of [5]. As $H_1(K(G, 1)) = \text{Ab}(G)$, we are done. \hfill $\square$

2. Nilpotent groups

If $A, B \subset G$ are subgroups, then the commutator subgroup $[A, B]$ is a group generated by all commutators $[a, b]$, $a \in A$ and $b \in B$. The lower central series $\{\Gamma_n(G)\}$ for a group $G$ is defined as follows: $\Gamma_1(G) = G$, and $\Gamma_{n+1}(G) = [\Gamma_n(G), G]$. 

If a group $G$ is nilpotent, then there exists an integer $c$ such that $\Gamma_c(G) \neq \{1\}$ but $\Gamma_{c+1}(G) = \{1\}$. The number $c$ (denoted by $h(G)$) is called the nilpotency class of the nilpotent group $G$ or its Hirsch length. Abelian groups are nilpotent of Hirsch length 1. By [14, Theorem 3.1], for every $n$ there exists an epimorphism

$$\otimes^n \text{Ab}(G) \rightarrow \Gamma_n(G)/\Gamma_{n+1}(G).$$

In particular, there is an epimorphism $\otimes^c \text{Ab}(G) \rightarrow \Gamma_c(G)$. It follows from the definition that $\Gamma_c(G)$ is in the center of $G$. Therefore $1 \rightarrow \Gamma_c(G) \rightarrow G \rightarrow G/\Gamma_c(G) \rightarrow 1$ is a central extension. A short calculation shows that the epimorphism $G \rightarrow G/\Gamma_c(G)$ induces the trivial homomorphism $\Gamma_c(G) \rightarrow \Gamma_c(G/\Gamma_c(G)) = 1$. Therefore $G/\Gamma_c(G)$ is nilpotent of nilpotency class strictly less than $c$. This motivates the following definition.

Definition 2.1. A central extension $K \rightarrow G \rightarrow I$ of groups where $G$ is nilpotent (or equivalently $I$ is nilpotent), for which there exists an epimorphism $\otimes^n \text{Ab}(G) \rightarrow K$ for some $n$, is called a nilpotent central extension.

Thus, for every (non-Abelian) nilpotent group $G$, there exists a nilpotent central extension $K \rightarrow G \rightarrow I$ such that the Hirsch length of $I$ is less than the Hirsch length of $G$.

Lemma 2.2. Let $1 \rightarrow K \rightarrow G \rightarrow I \rightarrow 1$ be a central extension of nilpotent groups.

(a) If $K$ and $I$ are $p$-divisible, then $G$ is $p$-divisible.

(b) If the extension is a nilpotent central extension and $G$ is $p$-divisible, then $K$ and $I$ are $p$-divisible.

Proof. Suppose $K$ and $I$ are $p$-divisible. Let $g \in G$. Then $\pi(g) = ip$ for some $i \in I$. Let $\tilde{g} \in G$ be such that $\pi(\tilde{g}) = i$. Then $\pi(\tilde{g}^p g^{-1}) = 1$, so $\tilde{g}^p g^{-1} = k^p$ for some $k \in K$. Because $k \in K \subset C(G)$, $g = (\tilde{g}g^{-1})^p$, so $G$ is $p$-divisible.

If $G$ is $p$-divisible, then any epimorphic image of $G$ is $p$-divisible. Thus both $I$ and $\text{Ab}(G)$ are $p$-divisible. As there is an epimorphism $\otimes^n \text{Ab}(G) \rightarrow K$, $K$ is also $p$-divisible. $$\square$$

Lemma 2.3. Suppose $\mathcal{P}_i$, $i = 1, 2$, are two classes of nilpotent groups such that for any nilpotent central extension $1 \rightarrow K \rightarrow G \rightarrow I \rightarrow 1$ where $h(I) < h(G)$ the following conditions hold:

(a) $K$ and $I$ belong to $\mathcal{P}_1$ if $G \in \mathcal{P}_1$,

(b) $G \in \mathcal{P}_2$ if $K, I \in \mathcal{P}_2$.

If $A \in \mathcal{P}_1 \Rightarrow A \in \mathcal{P}_2$ for all Abelian groups $A$, then $\mathcal{P}_1 \subset \mathcal{P}_2$.

Proof. We prove the implication $G \in \mathcal{P}_1 \Rightarrow G \in \mathcal{P}_2$ by induction on the Hirsch length of $G$. By assumption the implication holds for an Abelian group $G$. Therefore suppose $1 \rightarrow K \rightarrow G \rightarrow I \rightarrow 1$ is a nilpotent central extension of groups and $I$ is of lower Hirsch length than the Hirsch length of $G$. By (a) we have $K, I \in \mathcal{P}_1$. By the inductive hypothesis, $K, I \in \mathcal{P}_2$. The condition (b) yields $G \in \mathcal{P}_2$. $$\square$$

Corollary 2.4. Let $A$ be an Abelian group and $n \in \{1, 2\}$. Consider the following statements:

(1) $H_n(G; A) = 0$,

(2) $H_i(G; A) = 0$ for all $i \leq n$.

If (1) is equivalent to (2) for all Abelian groups $G$, then the two statements are equivalent for all nilpotent groups $G$. 
Proof. Let $\mathcal{P}_1 = \mathcal{P}_1^1$ (respectively, $\mathcal{P}_2 = \mathcal{P}_2^2$) be the class of all nilpotent groups $G$ such that $H_i(G; A) = 0$ for all $i \leq n$ (respectively, $\tilde{H}_i(G; A) = 0$ for all $i$) and $h(G) \leq r$. Our goal is to prove, by induction on $r$, that $\mathcal{P}_1^r = \mathcal{P}_2^r$. It is clearly so for $r = 1$. Assume $\mathcal{P}_1^m = \mathcal{P}_2^m$ for all $m < r$.

Suppose $1 \to K \to G \to I \to 1$ is a nilpotent central extension of groups such that $h(I) < h(G) = r$. If $G \in \mathcal{P}^r_j$, then $H_i(K; A) = 0$, which implies $H_i(G; A) \to H_i(I; A)$ is an epimorphism for $i \leq 2$, so $H_i(I; A) = 0$ for $1 \leq i \leq n$. By the inductive assumption, $H_i(I; A) = 0$ for all $i \geq 1$. If $H_i(K; A)$ is the first nontrivial reduced homology group of $K$, then the Leray-Serre spectral sequence implies $H_{i+1}(I; A) \neq 0$, a contradiction. Thus both $K$ and $I$ belong to $\mathcal{P}_j^r$. Conversely, if $K, I \in \mathcal{P}_j^{r-1}$, then (by the inductive assumption) they have trivial homology with coefficients in $A$ resulting in $G$ having trivial homology with coefficients in $A$ and $G \in \mathcal{P}_j^r$. Applying Lemma 2.3 one gets $\mathcal{P}_1^r = \mathcal{P}_2^r$. \hfill \Box

3. BOCKSTEIN BASIS

If $G$ is a group, then Tor$(G)$ is the subgroup generated by torsion elements of $G$, Tor$^p(G)$ is the subgroup generated by all elements of $G$ whose order is a power of $p$, $F_p(G) = G/Tor(G)$, and $F(G) = G/Tor(G)$.

The Bockstein groups are: rationals $\mathbb{Q}$, cyclic groups $\mathbb{Z}/p$ of $p$ elements, $p$-adic circles $\mathbb{Z}/p^\infty$, and $p$-localizations of integers $\mathbb{Z}_p = \{ \frac{m}{n} \in \mathbb{Q} \mid n \text{ is not divisible by } p \}$, where $p$ is a prime number. Here is a classical definition of the maximal Bockstein basis of an Abelian group $G$:

**Definition 3.1.** $\sigma(G)$ is a subset of all Bockstein groups satisfying the following conditions:

1. $\mathbb{Q} \in \sigma(G)$ if and only if $F(G) \neq 1$.
2. $\mathbb{Z}/p^\infty \in \sigma(G)$ if and only if Tor$^p(G) \neq 1$ or $F(G)$ is not divisible by $p$.
3. $\mathbb{Z}/p \in \sigma(G)$ if and only if Tor$^p(G)$ is not divisible by $p$ or $F(G)$ is not divisible by $p$.
4. $\mathbb{Z}_p \in \sigma(G)$ if and only if $F(G)$ is not divisible by $p$.

Notice that our Definition 3.1 differs from that in [14] in the sense that ours is maximal (if $\mathbb{Z}/p \in \sigma(G)$, then $\mathbb{Z}/p^\infty \in \sigma(G)$ and if $\mathbb{Z}_p \in \sigma(G)$, then $\mathbb{Z}/p \in \sigma(G)$) and the one in [14] is minimal (if $\mathbb{Z}/p \in \sigma(G)$ or $\mathbb{Z}_p \in \sigma(G)$, then $\mathbb{Z}/p^\infty \notin \sigma(G)$). From the point of view of the First Bockstein Theorem, both definitions are equivalent.

Here is a definition for nilpotent groups which is more convenient in this paper as it allows using localization of short exact sequences of nilpotent groups. Recall that a (multiplicative) group is $\bar{p}$-local iff the map $x \mapsto x^p$ is a bijection. We call a nilpotent group $p$-local iff it is $\bar{q}$-local for all primes $q \neq p$.

**Definition 3.2.** Let $G$ be a nilpotent group. Then the Bockstein basis $\sigma(G)$ is defined as follows:

1. $\mathbb{Q} \notin \sigma(G)$ if and only if $G = \text{Tor}(G)$.
2. $\mathbb{Z}/p^\infty \notin \sigma(G)$ if and only if $G$ is $p$-local.
3. $\mathbb{Z}/p \notin \sigma(G)$ if and only if $G$ is divisible by $p$.
4. $\mathbb{Z}_p \notin \sigma(G)$ if and only if $F(G)$ is $\bar{p}$-local.

**Remark 3.3.** Note that according to the above definition we have

$\mathbb{Z}_p \in \sigma(G) \Rightarrow \mathbb{Z}/p \in \sigma(G) \Rightarrow \mathbb{Z}/p^\infty \in \sigma(G)$. 

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Corollary 3.4. For a nilpotent group $G$ the following statements are equivalent:

1. $\mathbb{Q} \notin \sigma(G)$,
2. $H_s(G; \mathbb{Q}) = 0$, and
3. $H_1(G; \mathbb{Q}) = 0$.

Proof. For Abelian groups, (2) and (3) are equivalent [1, Theorem V.6.4(ii)]. With Corollary 3.5 we extend this equivalence to all nilpotent groups.

Let $\mathcal{P}_1$ be the class of torsion nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G; \mathbb{Q}) = 0$ (i.e., $\text{Ab}(G)$ is torsion). Use Lemma 2.3 to conclude that $\mathcal{P}_1 = \mathcal{P}_2$ and thus (1) and (3) are equivalent.

Corollary 3.5. For a nilpotent group $G$, $\mathbb{Z}/p \notin \sigma(G)$ if and only if $H_1(G; \mathbb{Z}/p) = 0$.

Proof. Let $\mathcal{P}_1$ be the class of $p$-divisible nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G; \mathbb{Z}/p) = 0$. Use Lemmas 2.4 and 2.2 to conclude that $\mathcal{P}_1 = \mathcal{P}_2$.

Corollary 3.6. For a nilpotent group $G$ the following statements are equivalent:

1. $\mathbb{Z}/p^\infty \notin \sigma(G)$,
2. $H_s(G; \mathbb{Z}/p^\infty) = 0$, and
3. $H_1(G; \mathbb{Z}/p^\infty) = H_2(G; \mathbb{Z}/p^\infty) = 0$.

Proof. If $\mathbb{Z}/p^\infty \notin \sigma(G)$, then $G$ is $\overline{p}$-local, so all its integral homology groups are $\overline{p}$-local and $H_i(G; \mathbb{Z}/p^\infty) = 0$ for all $i \geq 1$, which proves the implication (1) $\Rightarrow$ (2).

Notice (2) $\iff$ (3) by Corollary 2.1. Indeed, if $A$ is Abelian and $H_1(A; \mathbb{Z}/p^\infty) = H_2(A; \mathbb{Z}/p^\infty) = 0$, then $A$ must be $\overline{p}$-local.

Let $\mathcal{P}_1$ be the class of $\overline{p}$-local nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G; \mathbb{Z}/p^\infty) = H_2(G; \mathbb{Z}/p^\infty) = 0$. Use Lemma 2.3 to conclude that $\mathcal{P}_1 = \mathcal{P}_2$.

Corollary 3.7. For a nilpotent group $G$ the following statements are equivalent:

1. $\mathbb{Z}_{(p)} \notin \sigma(G)$,
2. $\mathbb{H}_s(F(G); \mathbb{Z}/p^\infty) = 0$, and
3. $H_1(F(G); \mathbb{Z}/p^\infty) = H_2(F(G); \mathbb{Z}/p^\infty) = 0$.

It is obvious that $\mathbb{Z}/p^\infty \notin \sigma(G)$ if and only if $G \to G_{(p)}$ is an isomorphism. The following lemma characterizes $\mathbb{Z}_{(p)} \notin \sigma(G)$ via localizations.

Lemma 3.8. For a nilpotent group $G$ the following statements are equivalent:

1. $\mathbb{Z}_{(p)} \notin \sigma(G)$,
2. $G \to G_{(p)}$ is an epimorphism.

Proof. If $G \to G_{(p)}$ is an epimorphism, then its kernel has trivial $\overline{p}$-localization (by exactness of the localization functor) and must be a torsion group. Therefore $F(G) = F(G_{(p)})$ is $\overline{p}$-local. If $F(G)$ is $\overline{p}$-local, then apply exactness of the localization functor to the short exact sequence $1 \to \text{Tor}(G) \to G \to F(G) \to 1$ and derive that $G \to G_{(p)}$ is an epimorphism.

Definition 3.9. The torsion-divisible Bockstein basis $\sigma_{TD}(G)$ of $G$ consists of all $\mathbb{Z}/p^\infty$ belonging to $\sigma(G)$. We set $\sigma_{NTD}(G) = \sigma(G) \setminus \sigma_{TD}(G)$.

Lemma 3.10. If $G \to I$ is an epimorphism of nilpotent groups. Then $\sigma_{NTD}(I) \subset \sigma_{NTD}(G)$.
Proof. Suppose \( Q \notin \sigma(G) \). Then \( G \) is a torsion group. So \( I \) is a torsion group; hence \( Q \notin \sigma(I) \).

Let \( Z/p \notin \sigma(G) \). Then \( G \) is \( p \)-divisible and then also \( I \) is \( p \)-divisible.

Let \( Z_{(p)} \notin \sigma(G) \). Then \( G \to G_{(p)} \) is an epimorphism. Because \( p \)-localization is an exact functor, the map \( G_{(p)} \to I_{(p)} \) is an epimorphism and hence \( I \to I_{(p)} \) is an epimorphism.

\[ \square \]

Lemma 3.11. Let \( 1 \to K \to G \to I \to 1 \) be a central extension of nilpotent groups. Then \( \sigma(G) \subset \sigma(K) \cup \sigma(I) \).

Proof. Let \( Q \notin \sigma(K) \cup \sigma(I) \). Then \( K \) and \( I \) are torsion groups. Hence \( G \) is torsion, so \( Q \notin \sigma(G) \).

Let \( Z/p \notin \sigma(K) \cup \sigma(I) \). Then \( K \) and \( I \) are \( p \)-divisible. By Lemma 2.2, also \( G \) is \( p \)-divisible.

Let \( Z/p^\infty \notin \sigma(K) \cup \sigma(I) \). Then \( K \to K_{(p)} \) and \( I \to I_{(p)} \) are isomorphisms. Using the Five Lemma and the fact that \( p \)-localization is an exact functor, we conclude that also \( G \to G_{(p)} \) is an isomorphism.

Let \( Z_{(p)} \notin \sigma(K) \cup \sigma(I) \). Then \( K \to K_{(p)} \) and \( I \to I_{(p)} \) are epimorphisms. By the Three Lemma, also \( G \to G_{(p)} \) is an epimorphism.

\[ \square \]

Lemma 3.12. Let \( 1 \to K \to G \to I \to 1 \) be a nilpotent central extension of groups. If \( Z_{(p)} \notin \sigma(\text{Ab}(G)) \) for some prime \( p \), then \( Z_{(p)} \notin \sigma(K) \) and \( Z_{(p)} \notin \sigma(\text{Ab}(I)) \).

Proof. Assume \( Z_{(p)} \notin \sigma(\text{Ab}(G)) \). That means the map \( F(\text{Ab}(G)) \to F(\text{Ab}(G))_{(p)} \) is an isomorphism. Because \( \text{Ab}(G) \to \text{Ab}(I) \) is an epimorphism and \( F \) is a right exact functor, the map \( F(\text{Ab}(G)) \to F(\text{Ab}(I)) \) is an epimorphism. Hence the map \( F(\text{Ab}(I)) \to F(\text{Ab}(I))_{(p)} \) is an epimorphism. Its kernel is a \( p \)-torsion group, so the kernel is trivial and the map \( F(\text{Ab}(I)) \to F(\text{Ab}(I))_{(p)} \) is an isomorphism. That means \( Z_{(p)} \notin \sigma(\text{Ab}(I)) \).

There exists an epimorphism \( \otimes^n \text{Ab}(G) \to K \) for some integer \( n \). Because \( F(\text{Ab}(\otimes^n G)) \to F(\text{Ab}(\otimes^n G))_{(p)} \) is an isomorphism in the same way as in the previous paragraph, we can prove that \( F(K) \to F(K)_{(p)} \) is an isomorphism. Hence \( Z_{(p)} \notin \sigma(K) \).

\[ \square \]

Lemma 3.13. Let \( G \) be a nilpotent group. Then \( \sigma_{\text{NTD}}(G) = \sigma_{\text{NTD}}(\text{Ab}(G)) \) and \( \sigma(\text{Ab}(G)) \subset \sigma(G) \).

Proof. The inclusion \( \sigma_{\text{NTD}}(\text{Ab}(G)) \subset \sigma_{\text{NTD}}(G) \) follows from Lemma 3.10

Let us prove \( \sigma_{\text{NTD}}(G) \subset \sigma_{\text{NTD}}(\text{Ab}(G)) \). Suppose \( Q \notin \sigma_{\text{NTD}}(\text{Ab}(G)) \). Because \( G \) is a torsion group if and only if \( \text{Ab}(G) \) is a torsion group, \( Q \notin \sigma_{\text{NTD}}(G) \).

Suppose \( Z/p \notin \sigma_{\text{NTD}}(\text{Ab}(G)) \). Because \( G \) is \( p \)-divisible if and only if \( \text{Ab}(G) \) is \( p \)-divisible, \( Z/p \notin \sigma_{\text{NTD}}(G) \).

Consider the class \( \mathcal{P} \) of all nilpotent groups such that \( Z_{(p)} \notin \sigma_{\text{NTD}}(\text{Ab}(G)) \) implies \( Z_{(p)} \notin \sigma_{\text{NTD}}(G) \). \( \mathcal{P} \) clearly contains all Abelian groups. To show \( \mathcal{P} \) equals the class \( \mathcal{N} \) of all nilpotent groups it suffices to show (see Lemma 2.3) that for any nilpotent central extension \( A \to G \to G' \) such that the Hirsch length of \( G' \) is less than \( h(G) \), \( A, G' \in \mathcal{P} \) implies \( G \in \mathcal{P} \). Assume \( Z_{(p)} \notin \sigma_{\text{NTD}}(\text{Ab}(G)) \). By Lemma 3.12 we conclude \( Z_{(p)} \notin \sigma_{\text{NTD}}(G') \) as \( G' \in \mathcal{P} \) and \( Z_{(p)} \notin \sigma_{\text{NTD}}(A) \). By Lemma 3.11 \( Z_{(p)} \notin \sigma_{\text{NTD}}(G) \).

Let us now prove that \( \sigma(\text{Ab}(G)) \subset \sigma(G) \). By Lemma 3.10 \( \sigma_{\text{NTD}}(\text{Ab}(G)) \subset \sigma_{\text{NTD}}(G) \). Suppose \( Z/p^\infty \notin \sigma(G) \). Then \( G \) is uniquely \( p \)-divisible, so \( H_*(G; Z) \) is
uniquely $p$-divisible. In particular $\text{Ab}(G) = H_1(G; \mathbb{Z})$ is uniquely $p$-divisible; hence $\mathbb{Z}/p\mathbb{Z} \not\in \sigma(\text{Ab}(G)). \hfill \square$

**Theorem 3.14.** If $1 \to K \to G \to I \to 1$ is a nilpotent central extension, then $\sigma(G) = \sigma(K) \cup \sigma(I)$.

**Proof.** By Lemma 3.11, $\sigma(G) \subset \sigma(K) \cup \sigma(I)$.

Let us prove that $\sigma(K) \cup \sigma(I) \subset \sigma(G)$. Suppose $Q \not\in \sigma(G)$. Then $G$ is a torsion group. Therefore $K$ and $I$ are also torsion groups, so $Q \not\in \sigma(K) \cup \sigma(I)$.

Suppose $\mathbb{Z}/p \not\in \sigma(G)$. Then $G$ is $p$-divisible. By Lemma 2.2, $K$ and $I$ are $p$-divisible, so $\mathbb{Z}/p \not\in \sigma(K) \cup \sigma(I)$.

Suppose $\mathbb{Z}/p^\infty \not\in \sigma(G)$. Then $G \to G_{(p)}$ is an isomorphism. Because $p$-localization is an exact functor, the map $K \to K_{(p)}$ is a monomorphism and the map $\text{Ab}(G) \to (\text{Ab}(G))_{(p)}$ is an epimorphism. Because there exists an epimorphism $\otimes^n \text{Ab}G \to K$, the map $K \to K_{(p)}$ is also an epimorphism; hence it is an isomorphism. By the Five Lemma, also the map $I \to I_{(p)}$ is an isomorphism, so $\mathbb{Z}/p^\infty \not\in \sigma(K) \cup \sigma(I)$.

If $\mathbb{Z}_{(p)} \not\in \sigma(G)$, then $\mathbb{Z}_{(p)} \not\in \sigma(K)$ and $\mathbb{Z}_{(p)} \not\in \sigma(I)$ by Lemmas 3.12 and 3.13 \hfill \square

## 4. Bockstein Spaces

**Definition 4.1.** A metrizable space $X$ is called a Bockstein space if $\dim_G X = \sup\{\dim_H X \mid H \in \sigma(G)\}$ for all Abelian groups $G$.

**Remark 4.2.** In the above definition observe that $\dim_G X$ is an element of $\mathbb{N} \cup \{0, \infty\}$ and not only in $\{0, 1, \infty\}$ as in the case of non-Abelian groups $G$.

Dranishnikov-Repovš-Shchepin \[\text{[6]}\] showed the existence of a separable metric space $X$ of dimension 2 such that $\dim_{\mathbb{Z}_{(p)}} X = 1$ for all primes $p$. Thus, $X$ is not a Bockstein space as $\dim_{\mathbb{Z}} X = 2 > 1 = \sup\{\dim_H X \mid H \in \sigma(\mathbb{Z})\}$.

**Problem 4.3.** Is every metric ANR a Bockstein space?

**Proposition 4.4.** Suppose $X = \bigcup_{n=1}^{\infty} X_n$ is metrizable and each $X_n$ is closed in $X$. If all $X_n$ are Bockstein spaces, then so is $X$.

**Proof.** Suppose $G$ is an Abelian group and $H \in \sigma(G)$. If $\dim_G X \leq m$, then $\dim_G(X_n) \leq m$ for all $n$ and $\dim_H(X_n) \leq m$ for all $n$ resulting in $\dim_H X \leq m$.

If $\dim_H X \leq m$ for all $H \in \sigma(G)$, then $\dim_H(X_n) \leq m$ for all $n$ and $\dim_G(X_n) \leq m$ for all $n$ resulting in $\dim_G X \leq m$. \hfill \square

For a subset $l \subset \mathbb{P}$ of prime integers let $\mathbb{Z}_l = \{\frac{m}{n} \in \mathbb{Q} \mid n \text{ is not divisible by any } p \in l\}$ and let $\mathbb{Z}_l$ be the group of $l$-adic integers.

**Theorem 4.5.** A metrizable space $X$ is a Bockstein space if and only if $\dim_{\mathbb{Z}_l} X = \dim_{\mathbb{Z}_l} X$ for all subsets $l \subset \mathbb{P}$ of the set of prime numbers.

**Proof.** Since $\sigma(\mathbb{Z}_l) = \sigma(\mathbb{Z}_l)$ for all $l \subset \mathbb{P}$, $\dim_{\mathbb{Z}_l} X = \dim_{\mathbb{Z}_l} X$ holds for any Bockstein space $X$. 

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Assume \( \dim_{\mathbb{Z}} X = \dim_{\mathbb{Z}} X \) for all subsets \( l \subset \mathbb{P} \) of the set of prime numbers. Suppose \( G \) is a torsion-free Abelian group \( G \). If \( \mathbb{Z}(p) \in \sigma(G) \), then Theorem B(d) of [7] says that \( \dim_{\mathbb{Z}(p)} X \leq \dim_G X \). Therefore \( \dim_G X \geq \sup \{ \dim_H X \mid H \in \sigma(G) \} \). Suppose \( \sup \{ \dim_H X \mid H \in \sigma(G) \} = n \) and consider \( l = \{ p \mid p \cdot G \neq G \} \). Theorem B(f) of [7] says that \( \dim_G X \leq \dim_{\mathbb{Z}(p)} X \). Since \( \sigma(G) = \sigma(\mathbb{Z}(p)) \), \( \dim_G X \leq \dim_{\mathbb{Z}(p)} X = \sup \{ \dim_H X \mid H \in \sigma(\mathbb{Z}(p)) \} = \sup \{ \dim_H X \mid H \in \sigma(G) \} = n \). That proves \( \dim_G X = \sup \{ \dim_H X \mid H \in \sigma(G) \} \) for all torsion-free Abelian groups. The same equality holds for all torsion Abelian groups by Theorem B(a) of [7]. In the case of arbitrary Abelian groups \( G \), as \( \sigma(G) = \sigma(F(G)) \cup \sigma(\text{Tor}(G)) \) and \( \dim_G = \max(\dim_{F(G)} X, \dim_{\text{Tor}(G)} X) \) (see Theorem B(b) of [7]) one gets \( \dim_G X = \sup \{ \dim_H X \mid H \in \sigma(G) \} \) as well.

\( \square \)

**Remark 4.6.** Notice that it is not sufficient to assume \( \dim_{\mathbb{Z}(p)} X = \dim_{\mathbb{Z}(p)} X \) for all primes \( p \) in Theorem 4.5. Indeed, the space \( X \) in [6] has that property as \( 1 = \dim_{\mathbb{Z}(p)} X \geq \dim_{\mathbb{Z}(p)} X \geq 1 \) for all primes \( p \).

**Theorem 4.7.** Let \( X \) be a Bockstein space. If \( G \) is nilpotent, then \( \dim_G(X) \leq 1 \) if and only if \( \sup \{ \dim_H(X) \mid H \in \sigma(G) \} \leq 1 \).

**Proof.** Let \( \mathcal{P}_1 \) be the class of all nilpotent groups and let \( \mathcal{P}_2 \) be the class of nilpotent groups \( G \) such that \( \dim_G(X) \leq 1 \) if and only if \( \sup \{ \dim_H(X) \mid H \in \sigma(G) \} \leq 1 \). Since \( \mathcal{P}_2 \) contains all Abelian groups, in view of Lemma 3.1 it suffices to show that for any nilpotent central extension \( K \rightarrow G \rightarrow I \) the conditions \( K, I \in \mathcal{P}_2 \) imply \( G \in \mathcal{P}_2 \). It is so if \( G \) is Abelian, so assume \( G \) is not Abelian. Moreover, as \( \sigma(\text{Ab}(G)) \subset \sigma(G) \) by Lemma 3.13 and \( \dim_G X \leq 1 \) implies \( \dim_{\text{Ab}(G)} X \leq 1 \) (see Lemma 1.4), either \( \dim_G(X) \leq 1 \) or \( \sup \{ \dim_H(X) \mid H \in \sigma(G) \} \leq 1 \) implies \( \dim_{\text{Ab}(G)} X \leq 1 \), so we may as well assume \( \dim_{\text{Ab}(G)} X \leq 1 \).

In view of Lemma 1.2, and the fact that \( \text{Ab}(G) = 1 \) implies \( 1 \), the equivalence of conditions \( \dim_G(X) \leq 1 \) and \( \sup \{ \dim_H(X) \mid H \in \sigma(G) \} \leq 1 \) may fail only if \( \dim_{\text{Ab}(G)} X = 1 \), so assume \( \dim_{\text{Ab}(G)} X = 1 \).

Suppose \( \dim_H(X) = n > 1 \) for some \( H \in \sigma(K) \). If \( H \neq \mathbb{Z}/p^{\infty} \), then \( H \in \sigma_{\text{NTD}}(G) = \sigma_{\text{NTD}}(\text{Ab}(G)) \) (Theorem 3.11 and Lemma 3.13), a contradiction. So \( H = \mathbb{Z}/p^{\infty} \) for some prime \( p \) and \( \mathbb{Z}/p^{\infty} \notin \sigma(\text{Ab}(G)) \). Hence \( \text{Ab}(G) \) is \( p \)-divisible and this is equivalent to \( G \) being \( p \)-divisible [2, Lemma 5.1]. Because \( \mathbb{Z}/p^{\infty} \notin \sigma(\text{Ab}(G)) \), by Lemma 3.4 \( \hat{H}_1(\text{Ab}(G); \mathbb{Z}/p^{\infty}) = 0 \), so also \( \hat{H}_1(G; \mathbb{Z}/p^{\infty}) = 0 \). By Lemma 3.6 \( H_2(G; \mathbb{Z}/p^{\infty}) \neq 0 \) as \( \mathbb{Z}/p^{\infty} \notin \sigma(G) \). This implies that \( F(H_2(G; \mathbb{Z})) \) is not \( p \)-divisible, so \( \mathbb{Z}(p) \in \sigma(H_2(G; \mathbb{Z})) \). If \( G \) is not a torsion group, then \( \text{Ab}(G) \) is not a torsion group; hence by definition \( Q \in \sigma(\text{Ab}(G)) \). Therefore \( \dim_Q(X) \leq 1 \). Using that fact and the Bockstein Inequalities (B15, B16 [12]), we get \( \dim_{\mathbb{Z}/p^{\infty}}(X) = \dim_{\mathbb{Z}(p)}(X) - 1 \). Because \( \mathbb{Z}(p) \in \sigma(H_2(G; \mathbb{Z})) \), the dimension \( \dim_{\mathbb{Z}(p)}(X) \leq \dim_{H_2(G; \mathbb{Z})}(X) \leq 2 \) as \( X \) is a Bockstein space and then \( \dim_{\mathbb{Z}/p^{\infty}}(X) = \dim_{\mathbb{Z}(p)}(X) - 1 \leq 1 \), a contradiction.

Thus \( G \) is a torsion group and is a product of \( q \)-groups \( G = \prod_{q \in \mathbb{P}} G_q \). Hence \( \text{Ab}(G) = \prod_{q \in \mathbb{P}} \text{Ab}(G_q) \). Because \( G \) is not \( p \)-local, \( G_p \neq 1 \), but \( \text{Ab}(G) \) is uniquely \( p \)-divisible, so \( \text{Ab}(G_p) = 1 \). Therefore \( G_p \) is a perfect nilpotent group, but such a group is trivial, a contradiction.

Thus \( \dim_H X \leq 1 \) for all \( H \in \sigma(K) \) and \( \dim_K X = \sup \{ \dim_H(X) \mid H \in \sigma(K) \} \leq 1 \) as \( K \) is Abelian and \( X \) is a Bockstein space.
Suppose \( \sup(\dim_H(X) | H \in \sigma(G)) \leq 1 \). By Theorem 3.13 \( \sigma(G) = \sigma(K) \cup \sigma(I) \).
Therefore \( \sup(\dim_H(X) | H \in \sigma(I)) \leq 1 \) and \( \dim_I X \leq 1 \) as \( I \in \mathcal{P}_2 \). Consequently, \( \dim_G X \leq 1 \) by Lemma 1.3.

Suppose \( \dim_L X \leq 1 \). By Lemma 1.3 one gets \( \dim_L X \leq 1 \) and \( \sup(\dim_H(X) | H \in \sigma(I)) \leq 1 \) as \( I \in \mathcal{P}_2 \). By Theorem 3.14 \( \sigma(G) = \sigma(K) \cup \sigma(I) \); hence
\[
\sup(\dim_H(X) | H \in \sigma(G)) = \sup(\dim_H(X) | H \in \sigma(K) \cup \sigma(I)) \leq 1.
\]

\textbf{Corollary 4.8.} Let \( L \) be a connected nilpotent CW complex. If \( X \) is a Bockstein space and \( \dim_{H_n(L)} X \leq n \) for all \( n \geq 1 \), then \( \dim_{\pi_n(L)} X \leq n \) for all \( n \geq 1 \).

\textit{Proof.} It is shown in [2] that \( \dim_{\pi_n(L)} X \leq n \) for all \( n \geq 2 \), so it suffices to prove \( \dim_{\pi_1(L)} X \leq 1 \). If that inequality is false, then there is \( H \in \sigma(\pi_1(L)) \) such that \( \dim_H(X) > 1 \) (see Theorem 4.7). In view of Lemma 3.13 as \( \dim_{H_1(L)}(X) \leq 1 \), \( H = \mathbb{Z}/p^\infty \) for some prime \( p \). Also, \( H_1(L) \) is not a torsion group, so \( \dim_H(X) \leq 1 \). Using Bockstein Inequalities one gets \( \dim_{\mathbb{Z}/p}(X) \geq 1 \). Therefore the \( i \)-th homology groups of both \( L \) and \( \tilde{L} \) with coefficients in \( \mathbb{Z}/p^\infty \) vanish for \( i \leq 2 \). From the fibration \( \tilde{L} \to L \to K(\pi_1(L), 1) \) one gets that the \( i \)-th homology groups of \( K(\pi_1(L), 1) \) with coefficients in \( \mathbb{Z}/p^\infty \) vanish for \( i \leq 2 \). However, in view of Corollary 4.7 that means \( \mathbb{Z}/p^\infty \notin \sigma(\pi_1(L)) \), a contradiction.

\textbf{Corollary 4.9.} Let \( L \) be a connected nilpotent CW complex and let \( X \) be a Bockstein space such that \( \dim_{H_n(L)} X \leq n \) for all \( n \geq 1 \). If \( X \) is finite dimensional or \( X \in \text{ANR} \), then \( X \tau L \).

\textit{Proof.} By Corollary 4.8 one gets \( \dim_{\pi_n(L)} X \leq n \) for all \( n \geq 1 \) and by Theorem G in [7] one has \( X \tau L \).


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