APPLICATIONS OF MAÑÉ’S \textit{C}^2 CONNECTING LEMMA

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Abstract. We consider a few applications of Mañé’s \textit{C}^2 Connecting Lemma. These are the \textit{C}^2 creation of homoclinic points associated to a basic set (i.e., isolated transitive hyperbolic set), a \textit{C}^2 locally generic criterion to know whether a given point belongs to the stable set of hyperbolic homoclinic classes, and that measurably hyperbolic diffeomorphisms (i.e., having the closure of supports of all invariant measures as a countable union of disjoint basic sets) are \textit{C}^2 generically uniformly hyperbolic diffeomorphisms.

Introduction

Contrary to developed structure theories of individual \textit{C}^2 diffeomorphisms, we have only a few \textit{C}^2 perturbation results, which is a big obstacle for the understanding of global dynamics in the space of \textit{C}^2 diffeomorphisms on a compact manifold (see [Pa] for instance). So, \textit{C}^2 generic properties are quite valuable even for very special cases. In [M1], Mañé developed \textit{C}^2 perturbation techniques to connect the stable and unstable sets of isolated hyperbolic sets. In his \textit{C}^2 Connecting Lemma, Mañé assumed a condition that some type of invariant measures have positive value on an isolated hyperbolic set, but surprisingly it works for general isolated hyperbolic sets and compact manifolds. (For other results on the \textit{C}^2 creation of homoclinic points on specific manifolds, see [P1] and [H] for the two-sphere, and [O] for the two-torus.)

In particular, when an isolated hyperbolic set can be decomposed by basic sets, some problems (the \textit{C}^1 Stability Conjecture for instance [M2]) are reduced to creating homoclinic points associated to a basic component of the isolated hyperbolic set. In the \textit{C}^1 case, we have the \textit{C}^1 Connecting Lemma ([H1]) to solve such problems, and its \textit{C}^2 version is expected to play a similar role in the \textit{C}^2 case even if it requires some assumptions. The first theorem, just a modification of Mañé’s result, deals with the \textit{C}^2 creation of homoclinic points associated to a basic component of an isolated hyperbolic set. The second theorem is a \textit{C}^2 locally generic criterion to know whether a given point belongs to the stable set of hyperbolic homoclinic classes. To prove the theorem, we need a key lemma concerning the existence of a so-called pseudo-cycle. Using these theorems, we prove the third theorem: \textit{C}^2 generically, measurably hyperbolic diffeomorphisms (i.e., having the closure of...
supports of all invariant measures as a countable union of disjoint basic sets) are uniformly hyperbolic ones (Axiom A with no cycles). More precisely, in a $C^2$ residual subset at which every homoclinic class varies continuously with respect to $C^2$ perturbations, we apply the first theorem to show that the closure of supports of all invariant measures for every measurably hyperbolic diffeomorphism is a finite union of disjoint basic sets that are also homoclinic classes. Then, we use the second theorem in order to extend the hyperbolicity to the whole nonwandering set for every measurably hyperbolic diffeomorphism in some other $C^2$ residual subset. In the proofs of the last two theorems, another ingredient is Palis’ perturbation ([P]) decreasing the lengths of cycles of basic sets. Applying his perturbation, we finally create a homoclinic point associated to a basic set (1-cycle) under the assumption that the theorems are not true, which contradicts the choices of the residual subsets above. As a consequence, we see that, in the $C^2$ generic viewpoint, it is sufficient to find a measurably hyperbolic diffeomorphism in a residual subset to obtain a uniformly hyperbolic diffeomorphism.

Let $M$ be a smooth compact manifold without boundary, and let $\text{Diff}^2(M)$ be the space of $C^2$ diffeomorphisms with the $C^2$ topology. Denote by $\mathcal{M}(M)$ the set of probabilities on the Borel $\sigma$-algebra $\mathcal{B}$ of $M$ endowed with its usual topology ($\mu_n \to \mu$ if and only if $\int \varphi d\mu_n \to \int \varphi d\mu$ for every continuous function $\varphi: M \to \mathbb{R}$). For $f \in \text{Diff}^2(M)$, let $\mathcal{M}_f(M)$ be the set of $f$-invariant elements of $\mathcal{M}(M)$. For $x \in M$ and $n \in \mathbb{Z}^+$, define a probability $\mu(x, n) \in \mathcal{M}(M)$ by

$$\mu(x, n) = \frac{1}{n} \sum_{j=1}^{n} \delta_{f^j(x)}.$$  

Denote by $\mathcal{M}(x, f)$ the set of $\mu \in \mathcal{M}(M)$ such that there exists a sequence of positive integers $n_1 < n_2 < \ldots$ satisfying

$$\mu = \lim_{i \to +\infty} \mu(x, n_i).$$

Note that $\mathcal{M}(x, f) \subset \mathcal{M}_f(M)$.

We say that $\Lambda \subset M$ is a hyperbolic set of $f$ if it is compact $f$-invariant and there exist a $Df$-invariant continuous splitting $TM|\Lambda = E^s \oplus E^u$ and constants $C > 0$, $0 < \lambda < 1$ such that

$$\|Df^n|E^s(x)\| \leq C\lambda^n$$

and

$$\|Df^{-n}|E^u(x)\| \leq C\lambda^n$$

for all $x \in \Lambda$, $n \geq 1$. In particular, a hyperbolic set $\Lambda$ is isolated if there exists an isolating block $U$ of $\Lambda$, that is, a compact neighborhood $U$ of $\Lambda$ such that $\bigcap_{n \in \mathbb{Z}} f^n(U) = \Lambda$. For an isolated hyperbolic set $\Lambda$, we define the local stable and unstable sets of $\Lambda$ by

$$W^s_\varepsilon(\Lambda, f) = \bigcup_{x \in \Lambda} W^s_\varepsilon(x, f),$$

$$W^u_\varepsilon(\Lambda, f) = \bigcup_{x \in \Lambda} W^u_\varepsilon(x, f).$$
where $W^s(x, f)$ and $W^u(x, f)$ are the local stable and unstable manifolds at $x \in \Lambda$ with size $\varepsilon$ for some small $\varepsilon > 0$. Then,
\[
W^s(\Lambda, f) = \bigcup_{n \geq 0} f^{-n}(W^s_\varepsilon(\Lambda, f)),
\]
\[
W^u(\Lambda, f) = \bigcup_{n \geq 0} f^n(W^u_\varepsilon(\Lambda, f)),
\]
where $W^s(\Lambda, f)$ and $W^u(\Lambda, f)$ are the stable and unstable sets of $\Lambda$ defined by
\[
W^s(\Lambda, f) = \{ y \in M : \lim_{n \to +\infty} d(f^n(y), \Lambda) = 0 \},
\]
\[
W^u(\Lambda, f) = \{ y \in M : \lim_{n \to -\infty} d(f^n(y), \Lambda) = 0 \}.
\]
It is well known that if $\Lambda$ is an isolated hyperbolic set of $f$ with the nonwandering set $\Omega(f|\Lambda) = \Lambda$, then $\Lambda$ can be uniquely decomposed by disjoint finite basic sets (i.e., isolated transitive hyperbolic sets) as:
\[
\Lambda = \bigcup_{t=1}^l \Lambda_t,
\]
which is called the basic decomposition of $\Lambda$, and each $\Lambda_t$ is called a basic component of $\Lambda$. When
\[
W^s(\Lambda_t, f) \cap W^u(\Lambda_t, f) \setminus \Lambda_t \neq \emptyset,
\]
we say that $f$ exhibits a homoclinic point associated to a basic component $\Lambda_t$ of $\Lambda$.
We call a sequence $\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_\ell$ of basic sets a reordered subsequence of $\Lambda_1, \ldots, \Lambda_l$ if $\tilde{\Lambda}_i = \Lambda_{j_i}$ (1 ≤ $i$ ≤ $\ell$) for some $\{j_1, \ldots, j_\ell\} \subset \{1, \ldots, l\}$ with $j_i \neq j_{i'}$ when $1 \leq i \neq i' \leq \ell$. An $\ell$-cycle on $\Lambda$ is a reordered subsequence $\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_\ell$ of $\Lambda_1, \ldots, \Lambda_l$ such that
\[
W^u(\tilde{\Lambda}_i, f) \cap W^s(\tilde{\Lambda}_{i+1}, f) \setminus \Lambda \neq \emptyset
\]
for all 1 ≤ $i$ ≤ $\ell$ with $\tilde{\Lambda}_{\ell+1} = \tilde{\Lambda}_1$. Here, $\ell$ is called the length of the $\ell$-cycle. Note that the existence of a 1-cycle implies that of a homoclinic point associated to a basic component of $\Lambda$. Now let us recall a $C^2$ Connecting Lemma ([12 Theorem I.1]) by Mañé:

**Mañé’s $C^2$ Connecting Lemma.** Let $f \in \text{Diff}^2(M)$ and let $\Lambda$ be an isolated hyperbolic set of $f$. Let $\bar{x} \notin \Lambda$ be a point such that there exist a sequence $\{x_k\} \subset M$ converging to $\bar{x}$ and a sequence of integers $0 < m_1 < m_2 < \ldots$ such that the sequence of probabilities $\mu(x_k, m_k), k \geq 1,$ converges to a probability $\mu$ that satisfies $\mu(\Lambda) > 0$. Then, for given neighborhoods $U$ of $f$ in $\text{Diff}^2(M)$ and $U$ of $\Lambda$ in $M$, there exist $g \in U$ and a neighborhood $V \subset U$ of $\Lambda$ such that
\[
f| (V \cup U^c) = g| (V \cup U^c),
\]
\[
f^{-1}| (V \cup U^c) = g^{-1}| (V \cup U^c)
\]
and moreover they satisfy one of the following properties:

(I) There exist $k \geq 1$ and $0 < s < m_k$ such that
\[
g^j(x_k) = f^j(x_k)
\]
for all $0 \leq j < s$ and
\[
g^{s+1}(x_k) \in \bigcap_{n \geq 0} g^{-n}(V) = \bigcap_{n \geq 0} f^{-n}(V).
\]
There exist \( k \geq 1 \) and \( 0 < s_0 < s_1 < m_k \) such that
\[
\begin{align*}
(\text{a}) & \quad g^j(f^{s_0}(x_k)) = f^j(f^{s_0}(x_k)) \text{ for } 0 \leq j < s_1 - s_0, \\
(\text{b}) & \quad g^{-2}(f^{s_0}(x_k)) \in \bigcap_{n \geq 0} g^n(V) = \bigcap_{n \geq 0} f^n(V), \\
(\text{c}) & \quad g^{(s_1-s_0)+1}(f^{s_0}(x_k)) \in \bigcap_{n \geq 0} g^{-n}(V) = \bigcap_{n \geq 0} f^{-n}(V).
\end{align*}
\]

This statement includes connecting an orbit with the stable set \( W^s(\Lambda, f) \) or the unstable set \( W^u(\Lambda, f) \) of \( \Lambda \) by an arbitrarily small \( C^2 \) perturbation without changing \( f \) in a neighborhood of \( \Lambda \). In fact, property (I) implies \( x_k \in W^s(\Lambda, g) \) and property (II) implies \( W^s(\Lambda, g) \cap W^u(\Lambda, g) \setminus \Lambda \neq \emptyset \). Note that, when \( \Omega(f|\Lambda) = \Lambda \), unless \( \Lambda \) is a single basic set, property (II) does not always imply the creation of a homoclinic point associated to a basic component of \( \Lambda \). In order to always get such a homoclinic point from the hypothesis of Mañe’s \( C^2 \) Connecting Lemma, we apply this to a special case where \( \{x_k\} \) is a sequence of periodic points of \( f \) not contained in \( \Lambda \) and \( m_k \) is the period of \( x_k \). Note that the convergence \( \lim_{k \to +\infty} x_k = \hat{x} \notin \Lambda \) can be assumed by replacing \( x_k \) by some periodic point on the same periodic orbit with \( x_k \) and taking a subsequence of \( k = 1, 2, \ldots \) if necessary. Consequently, we obtain the following theorem:

**Theorem A.** Let \( f \in \text{Diff}^2(M) \) and let \( \Lambda \) be an isolated hyperbolic set of \( f \) with \( \Omega(f|\Lambda) = \Lambda \). Suppose that there exists a sequence \( \{p_k\} \) of periodic points of \( f \) with \( p_k \notin \Lambda \) such that the probabilities \( \mu(p_k, \ell_k) \) with \( \ell_k \), the period of \( p_k \), \( k \geq 1 \), converge to a probability \( \mu \) that satisfies \( \mu(\Lambda) > 0 \). Then, for every neighborhood \( U \) of \( f \) in \( \text{Diff}^2(M) \), there exists \( g \in U \) coinciding with \( f \) in a neighborhood of \( \Lambda \) and exhibiting a homoclinic point associated to a basic component of \( \Lambda \).

A homoclinic class is the closure of the transversal intersection between the stable and unstable manifolds of a hyperbolic periodic orbit. It is well-known that every hyperbolic homoclinic class is a basic set. The following theorem is obtained as applications of Mañe’s \( C^2 \) Connecting Lemma and Palis’ perturbation decreasing the lengths of cycles of basic sets:

**Theorem B.** Let \( f \in \text{Diff}^2(M) \) have a hyperbolic set \( \Lambda \) and let \( U \) be a neighborhood of \( f \) such that the continuation \( \Lambda(g) \) of \( \Lambda \) is defined for all \( g \in U \). Then, when \( \Lambda(g) \) is a finite union of homoclinic classes for all \( g \) in a residual subset of \( U \), there exists an open and dense subset \( O(U) \) of \( U \) such that given \( x \in M \), if \( \mu(\Lambda(g)) > 0 \) for all \( \mu \in \mathcal{M}(x, g) \) with \( g \in O(U) \), then \( x \in W^s(\Lambda(g), g) \).

**Remark.** Mañe used the conclusion of this theorem (for the \( C^1 \) case) without the generic condition in his proof of the \( C^1 \) Stability Conjecture [M2, Lemma V.5]. In this author’s opinion, his proof without the generic condition contains a gap coming from that of the proof of [M1, Theorem D]. But the statement of [M1, Theorem D] for the \( C^1 \) case itself is true by the \( C^1 \) Connecting Lemma [H2].

We say that the diffeomorphism \( f \) is measurably hyperbolic if the closure of supports of all \( f \)-invariant measures
\[
S(f) = \{x \in \text{supp}(\mu) : \mu \in \mathcal{M}_f(M)\}
\]
is a countable union of disjoint basic sets. The following theorem shows that: \( C^2 \) generically, every measurably hyperbolic diffeomorphism is a uniformly hyperbolic one.

**Theorem C.** There exists a residual subset \( \mathcal{R} \) of \( \text{Diff}^2(M) \) such that if \( f \in \mathcal{R} \) is measurably hyperbolic, then \( f \) is an Axiom A diffeomorphism with no cycles.
In Section I, Theorem A is proved following Mañé’s perturbation framework. In Section II, Theorem B is proved after the proof of the key lemma. Finally, in Section III, Theorem C is proved using Theorems A and B. All the perturbations we made in this paper are those with respect to the $C^2$ topology.

I. PROOF OF THEOREM A

Let us recall Mañé’s perturbation framework. For $f$, $\Lambda$ and $U$ given in Theorem A, define

$$V^+ = W^s_\varepsilon(\Lambda, f) \quad \text{and} \quad V^- = W^u_\varepsilon(\Lambda, f),$$

where $\varepsilon > 0$ is so small that

$$W^u_\varepsilon(\Lambda, f) \cup W^s_\varepsilon(\Lambda, f) \subset U_0$$

for an isolating block $U_0$ of $\Lambda$ contained in $U$. Then, $f(V^+) \subset V^+$, $f^{-1}(V^-) \subset V^-$ and

$$\Lambda = \bigcup_{t=1}^{l} \Lambda_t = V^+ \cap V^-$$

for some $l \in \mathbb{Z}^+$. To obtain a perturbation exhibiting a homoclinic point associated to a basic component of $\Lambda$, we need to take a component-wise perturbation framework as considered in [H1]. Restricting this to each basic component $\Lambda_t$, we have

$$\Lambda_t = V^{t,+} \cap V^{t,-},$$

where

$$V^{t,+} = W^s_\varepsilon(\Lambda_t, f) \quad \text{and} \quad V^{t,-} = W^u_\varepsilon(\Lambda_t, f).$$

Let

$$V^n_t = \{x \in M : d(x, V^{t,+}) \leq r_n, \ d(x, V^{t,-}) \leq r_n\}$$

with $r_n = r_{n-1}^{1+\delta}$ $(n = 1, 2, \ldots)$ for some $0 < \delta < 1$, where $r_0 > 0$ is a sufficiently small positive number. We can suppose $r_0 > 0$ is so small that

$$d(x, V^+) = d(x, V^{t,+}) \quad \text{and} \quad d(x, V^-) = d(x, V^{t,-})$$

for all $x$ in a neighborhood of $V^n_0$ $(1 \leq t \leq l)$ and

$$V^n_0 \cap V^n_{t'} = \emptyset$$

if $t \neq t'$ (see [H1] Remark 2.2 for the details). Denote by $S^n_t$ the set of points $x \in V^n_0$ that can be written as $x = f^m(y)$, $m \in \mathbb{Z}$, with $y \in V^n_t$ and $f^j(y) \in V^n_t$ for all $0 \leq j \leq m$ if $m \geq 0$, or for all $m \leq j \leq 0$ if $m \leq 0$. Then, define

$$V_n = \bigcup_{t=1}^{l} V^n_t \quad \text{and} \quad S_n = \bigcup_{t=1}^{l} S^n_t.$$
Mañé proved that if $\mu = \lim_{k \to +\infty} \mu(x_k, m_k)$ and $\mu(\Lambda) > 0$, then for all $n_0 > 0$ one of the following properties holds (\cite[Lemma 4]{M1}):

(i) There exist $n \geq n_0$, $k \geq 1$ and a $k$-string $\sigma_1 \subset S_{n+1}$ such that $\sigma \cap (S_n \setminus S_{n+1}) = \emptyset$ for every $k$-string $\sigma < \sigma_1$.

(ii) There exist $n \geq n_0$, $k \geq 1$ and two $k$-strings $\sigma_1 < \sigma_2 \subset S_{n+1}$ such that $\sigma \cap (S_n \setminus S_{n+1}) = \emptyset$ for every $k$-string $\sigma_1 < \sigma < \sigma_2$.

As proved in \cite{M1}, properties (i) and (ii) lead to properties (I) and (II) in Mañé’s $C^2$ Connecting Lemma, respectively. For $\mathcal{U}$ given in the $C^2$ Connecting Lemma, the neighborhood $V$ of $\Lambda$ in properties (I) and (II) is given by $V = V_n \setminus B_n \setminus f^{-1}(B_n)$, where $B_n$ is the union of two balls at some points in $V^-$ and $V^+$ with radius $r_{1+\beta/3}$ for some $n$ (taken sufficiently large according to the given neighborhood $\mathcal{U}$ as in \cite[p. 152]{M1}) and $\beta > 0$ (see \cite[p. 150]{M1} for the choice of $\beta$). Figures 1 and 2 are conceptual figures of properties (i) and (ii), respectively. By the absence of $k$-strings in $S_n \setminus S_{n+1}$, we can find a diffeomorphism $h$ supported in $B_n$ and close to the identity by which, for $q_i$ and $y_i$ ($i = 1, 2$) in the figures, $q_2 \in \sigma_2$ is pushed to $y_2 \in V^+$, $y_1 \in V^-$ is pushed to $q_1 \in \sigma_1$, and such that $g = h \circ f$ satisfies properties (I) or (II) according to properties (i) or (ii). Define, for $1 \leq t \leq l$,

$$V^t = V_n^t \setminus B_n^t \setminus f^{-1}(B_n^t),$$

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where $B^t_n$ is the union of two balls at some points in $V^{t,-}$ and $V^{t,+}$ with the same radius as above. Now we provide the component-wise versions of (i) and (ii) as (i') and (ii'), respectively:

(i') There exist $n \geq n_0$, $k \geq 1$, $1 \leq t \leq l$ and a $k$-string $\sigma_1 \subset S^t_{n+1}$ such that $\sigma \cap (S^t_n \setminus S^t_{n+1}) = \emptyset$ for every $k$-string $\sigma < \sigma_1$.

(ii') There exist $n_1, n_2 \geq n_0$, $k \geq 1$, $1 \leq t_1, t_2 \leq l$ and two $k$-strings $\sigma_1 < \sigma_2$ with $\sigma_1 \subset S^t_{n_1+1}$ and $\sigma_2 \subset S^t_{n_2+1}$ such that $\sigma \cap ((S^t_{n_1} \setminus S^t_{n_1+1}) \cup (S^t_{n_2} \setminus S^t_{n_2+1})) = \emptyset$ for every $k$-string $\sigma_1 < \sigma < \sigma_2$.

Then, by applying the argument for proving property (I) (resp. property (II)) from property (i) (resp. property (ii)), it is easy to check that property (I*) (resp. property (II*) with $V$ in (b) and (c)) replaced by $V^t$ (resp. $V^{t_1}$ and $V^{t_2}$, respectively, which is called property II-$(t_1, t_2)$), and then we have

$$x_k \in W^s(\Lambda_t, g)$$

(resp. $W^u(\Lambda_{t_1}, g) \cap W^s(\Lambda_{t_2}, g) \setminus \Lambda \neq \emptyset$).

For every $1 \leq t \leq l$, we call property (ii') with $n = n_1 = n_2$ and $t = t_1 = t_2$ property (ii-t), which leads to property II-(t, t), that is, the creation of a homoclinic
point associated to a basic component $\Lambda_t$ of $\Lambda$. The following lemma is a refinement of [HI] Lemma 4:

**Lemma I.1 (HI Lemma 3.3).** Let $\mu \in \mathcal{M}_f(M)$ be the limit point of $\{\mu(x_k, m_k)\}$ and let $N_k(t)$ be the minimum positive integer $n$ such that the number of $0$-strings containing a point of $(x_k, \ell^m(x_k)) \cap S_{n}^k$ (i.e., the number of $k$-strings in $S_n^k$) is $\leq 1$. Suppose that property (ii-t) does not hold for all $1 \leq t \leq l$ and the following properties are satisfied for some $1 \leq t_0 \leq l$:

(a) there exists $L \in \mathbb{Z}^+$ (independent of $k$) such that for all $k$ sufficiently large,

$$S_{N_k(t_0)+L} \cap (x_k, \ell^m(x_k)) = \emptyset;$$

(b) $\lim_{k \to +\infty} N_k(t_0) = 0$.

Then $\mu(\Lambda) = 0$.

Let us observe from Lemma I.1 that if $\mu(\Lambda) > 0$, then either property (ii-t) for some $1 \leq t \leq l$ or property (i') holds. We claim this by showing property (i') under the assumption that property (ii-t) does not hold for all $1 \leq t \leq l$. By Lemma I.1, the assumption and $\mu(\Lambda) > 0$ imply that for all $1 \leq t_0 \leq l$ either property (a) or property (b) is not satisfied. If the latter case occurs for $t_0$ with $\mu(\Lambda_{t_0}) > 0$, there is a subsequence $k_j$, $j \geq 1$, of $k = 1, 2, \ldots$ such that, setting $\gamma_j = (x_{k_j}, f^{m_{k_j}}(x_{k_j}))$, we have a sequence $\gamma_j$, $j \geq 1$, satisfying the following property: there is $n > 0$ such that, if $n \geq n$, then for all $j \geq j_0$ with some $j_0 = j_0(n)$, there exists only one 0-string $\gamma_j^0$ containing a point of $\gamma_j \cap S^k_n$, and $\gamma_j \setminus \gamma_j^0$, $j \geq 1$, does not accumulate on $\Lambda_{t_0}$. This implies property (i'). On the other hand, for $t_0$ with $\mu(\Lambda_{t_0}) > 0$ again, if property (b) is satisfied and property (a) is not satisfied, then for any large $L \in \mathbb{Z}^+$ there is an arbitrarily large $k$ such that $S_{N_k(t_0)+L} \cap (x_k, \ell^m(x_k)) \neq \emptyset$ and $\lim_{k \to +\infty} N_k(t_0) = \infty$, which also implies property (i'). Indeed, the above $S_{N_k(t_0)+L}$ can be replaced by $S_{N_k(t_0)+L}^L$, for otherwise the hypothesis of Lemma I.1 with $\Lambda = \Lambda_{t_0}$ is satisfied, implying $\mu(\Lambda_{t_0}) = 0$, a contradiction.

In particular, let $\sigma_1'$ be the $k$-string $\sigma_1$ in property (i') just obtained through this observation; that is, $\sigma_1'$ is either the $k$-string in $\sigma_j^0$ or the $k$-string in $S_{N_k(t_0)+L}^L$ with large $L$ and $k$. Applying this observation to a special case where $x_k \in \text{Per}(f) \setminus \Lambda$ with period $\ell_k$ and $m_k = \ell_k$, we see that the existence of $\sigma_1'$ also leads to property (ii-t). In fact, the $k$-string $\sigma_1'$ may be thought of as if it were two $k$-strings such that $\sigma_1' = \sigma_1' < \sigma_2 = \sigma_1' \subset S_{n+1}^0$ and hence

$$\sigma \cap (S_n^0 \setminus S_{n+1}^0) = \emptyset$$

for every $k$-string $\sigma_1' < \sigma < \sigma_2$. Thus, we always have property (ii-t) for some $1 \leq t \leq l$, which leads to property II-(t,t), and hence it is possible to create a homoclinic point associated to a basic component $\Lambda_t$ of $\Lambda$ by an arbitrarily small perturbation of $f$, coinciding with $f$ in a neighborhood of $\Lambda$. This concludes the proof of Theorem A.

II. Proof of Theorem B

In this section, we prove Theorem B using the key lemma below. This lemma corresponds to the part containing the gap of the proof of [HI] Theorem D mentioned in the Remark of the Introduction. When $\Lambda$ is a single basic set, Mañé’s argument is correct. For the case where the number of basic components of $\Lambda$ is more than one, Mañé proceeds with his argument as if $S_n = S_n^0$ ($n \geq 1$) only
from the fact that some basic component $\Lambda_{t_0}$ of $\Lambda$ has a positive $\mu$-measure for some $\mu \in \mathcal{M}(x,f)$ (by which we have either property (ii-t0) or property (ii′) with $t = t_0$), and concludes that property (ii-t0) and hence property II-(t0,t0) always hold. However, the two 0-strings $\sigma_1 < \sigma_2$ of property (ii) which Mañé obtained in his proof of [M1 Theorem D] may be just those of property (ii′) with $t_1 = t_0$. Moreover, when the first 0-string $\sigma_3$ in $S_n^{t_0}$ after $\sigma_1$ appears, there is no guarantee that $\sigma \cap (S_n^{t_0} \setminus S_n^{t_0+1}) = \emptyset$ for every 0-string $\sigma_1 < \sigma < \sigma_3$ because it may happen that $\nu(\Lambda_{t_0}) = 0$ for some other $\nu \in \mathcal{M}(x,f)$.

In the general case, it seems to this author that only the following statement is possible without nontrivial improvement.

**Lemma II.1.** Let $f \in \text{Diff}^2(M)$ and let $\Lambda = \bigcup_{i=1}^l \Lambda_i$ be a finite union of disjoint basic sets $\Lambda_1, \ldots, \Lambda_l$. Suppose that $\mu(\Lambda) > 0$ for all $\mu \in \mathcal{M}(x,f)$ with some $x \notin W^s(\Lambda, f)$. Then, there exists a reordered subsequence $\hat{\Lambda}_1, \ldots, \hat{\Lambda}_s$ of $\Lambda_1, \ldots, \Lambda_l$ such that for every neighborhood $U$ of $f$ and every $1 \leq i \leq s$, there exists $g_i \in U$ coinciding with $f$ in a neighborhood of $\Lambda$ and satisfying
\[
W^u(\hat{\Lambda}_i, g_i) \cap W^s(\hat{\Lambda}_{i+1}, g_i) \setminus \Lambda \neq \emptyset
\]
with $\hat{\Lambda}_{s+1} = \hat{\Lambda}_1$.

**Proof.** First, we can suppose that property (ii-t) does not hold for all $1 \leq t \leq l$. Indeed, property (ii-t) (leading to property II-(t,t)) implies the existence of the required basic set as $\hat{\Lambda}_1 = \Lambda_t$ and $s = 1$.

Since $x \notin W^s(\Lambda, f)$ and $w_1(x) \cap \Lambda \neq \emptyset$, there exist $N_1 > N$ such that $f^{N_1}(x) \notin \mathcal{V}_0$ and $f^{N_1}(x) \in \mathcal{V}_0$. If $f > N_1$, we can define
\[
A(j) = \max\{n \geq 0 : S_n \cap \{f^N(x), \ldots, f^j(x)\} \neq \emptyset\}.
\]
Let $a_1 < a_2 < \ldots$ be the set of values of $A(j)$ and define
\[
\hat{\ell}_n = \max\{j > N_1 : A(j) = a_n\}.
\]
Note that $A(\hat{\ell}_n + 1) = a_{n+1}$. Take a subsequence $\hat{n}_1 < \hat{n}_2 < \ldots$ so that, setting $\ell_k = \hat{n}_{k+1}$ and $\bar{\ell}_k = \hat{n}_{k-1}$, we have $\mu, \bar{\mu} \in \mathcal{M}(x,f)$ with
\[
\lim_{k \to +\infty} \mu(x, \ell_k) = \mu \quad \text{and} \quad \lim_{k \to +\infty} \mu(x, \bar{\ell}_k) = \bar{\mu}.
\]
For $k \geq 1$ let $1 \leq t_k, \bar{\ell}_k \leq l$ be such that
\[
d(\Lambda_{t_k}, f^{\ell_k+1}(x)) = d(\Lambda, f^{\ell_k+1}(x)) \quad \text{and} \quad d(\Lambda_{t_k}, f^{\bar{\ell}_k+1}(x)) = d(\Lambda, f^{\bar{\ell}_k+1}(x)).
\]
Then, there exist $k$-strings (for $\{x_k = x\}$ and $\{m_k = \ell_k + 1\}$) $\sigma_k$ and $\bar{\sigma}_k$ with
\[
\sigma_k \subset S_{\Lambda(t_k+1)}^{t_k} \quad \text{and} \quad \bar{\sigma}_k \subset S_{\Lambda(t_k+1)}^{\ell_k}.
\]
Let us claim that
\[
(S_{\Lambda(t_k+1)}^{t_k} \setminus S_{\Lambda(t_k+1)}^{t_k}) \cap (x, f^{\ell_k}(x)) = \emptyset
\]
for arbitrarily large $k$. If this is false, we may assume that $t_k = \tau$ for all $k$ by taking subsequences of $\{t_k\}$ and $\{\ell_k\}$ if necessary to have
\[
(S_{\Lambda(t_k+1)}^{t_k} \setminus S_{\Lambda(t_k+1)}^{\ell_k}) \cap (x, f^{\ell_k}(x)) \neq \emptyset
\]
for all \( k \) sufficiently large. Then, by (1) and (3), we can apply Lemma I.1 to \( \{ \mu(x, \ell_k + 1) \} \), satisfying properties (a) and (b) for \( t_0 = \tau, N_k(\tau) = A(\ell_k + 1) = a_{n_k + 1} \) and \( L = 1 \). Therefore \( \mu(\Lambda) = 0 \) because
\[
\mu = \lim_{k \to +\infty} \mu(x, \ell_k) = \lim_{k \to +\infty} \mu(x, \ell_k + 1),
\]
contradicting our hypothesis. In order to obtain a reordered subsequence \( \tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_s \) of \( \Lambda_1, \ldots, \Lambda_t \) required in this lemma, it is enough to show that
\[
\text{(4)} \quad (S^{\ell_k}_{A(\ell_k+1)-1} \setminus S^{\ell_k}_{A(\ell_k+1)}) \cap (x, f^{\ell_k}(x)) = \emptyset
\]
for arbitrarily large \( k \) satisfying (2). In fact, if (4) were true, then (4) with \( \ell_k \) replaced by \( \ell_k \) (\(< \ell_k \)) would hold, and therefore, taking a subsequence of \( \{ n_k \} \) if necessary, (4) for \( \bar{\mu} \) could be used as (2) for \( \mu \) to obtain inductive choices of pairs \( (t_k, \ell_k) \) and \( (t_k, \tilde{t}_k) \). Hence, by the finiteness of basic components, after at most \( l \) times such inductive choices, we can find a reordered subsequence \( \{ \tilde{\Lambda}_i(k) \}_{i=1}^{s(k)} \) of \( \Lambda_1, \ldots, \Lambda_t \) chosen as
\[
\tilde{\Lambda}_{s(k)}(k) = \Lambda_{t_k}, \quad \tilde{\Lambda}_{s(k)-1}(k) = \Lambda_{t_k}, \quad \tilde{\Lambda}_{s(k)-2}(k) = \Lambda_{t_k}, \ldots
\]
and finally returning to \( \Lambda_{t_k} \). Note that we can suppose that \( s(k) \) and each \( \tilde{\Lambda}_i(k) \), \( 1 \leq i \leq s(k) \), do not depend on \( k \) by taking a subsequence of infinitely many choices of such a \( k \). Then, setting \( s(k) = s \) and \( \tilde{\Lambda}_i(k) = \tilde{\Lambda}_i \) for all \( 1 \leq i \leq s \), we obtain the required reordered subsequence because property (ii') with respect to \( (\tilde{\Lambda}_1, \tilde{\Lambda}_2) = (\tilde{\Lambda}_0, \tilde{\Lambda}_0') \) (and therefore property II-(\( t_1, t_2 \))) holds for every pair of consecutive basic components:
\[
(\tilde{\Lambda}_0, \tilde{\Lambda}_0') \in \{(\tilde{\Lambda}_i, \tilde{\Lambda}_{i+1}), (\tilde{\Lambda}_s, \tilde{\Lambda}_1) : 1 \leq i < s\}.
\]

Now the proof of (4) is similar to that of (2). If property (4) were false, we could assume that \( t_k = \tau \) and
\[
\text{(5)} \quad (S^\tau_{A(\ell_k+1)-1} \setminus S^\tau_{A(\ell_k+1)}) \cap (x, f^\tau(x)) \neq \emptyset
\]
for all \( k \) sufficiently large. Note that
\[
a_{n_k} = A(\bar{\ell}_k + 1) = A(\ell_k).
\]
Then, by (1) and (5), we could apply Lemma I.1 to \( \{ \mu(x, \ell_k) \} \), satisfying properties a) and b) for \( t_0 = \bar{\tau}, N_k(\bar{\tau}) \in \{ A(\ell_k), A(\ell_k) + 1 \} \) (according to whether \( \bar{\sigma}_k \) given in (1) is the only \( k \)-string for \( \{ x_k = x \} \) and \( \{ m_k = \ell_k \} \) contained in \( S^\tau_{a_{n_k}}(x) \) or not) and \( L = 1 \). Therefore \( \mu(\Lambda) = 0 \) again. This contradicts our hypothesis and completes the proof of Lemma II.1.

We say that a reordered subsequence \( \tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_s \) of \( \Lambda_1, \ldots, \Lambda_t \) is a pseudo-cycle on \( \Lambda \) if for every neighborhood \( \mathcal{U} \) of \( f \) and every \( 1 \leq i \leq s \), there exists \( g_i \in \mathcal{U} \) coinciding with \( f \) in a neighborhood of \( \Lambda \) and satisfying
\[
W^u(\tilde{\Lambda}_1, g_i) \cap W^s(\tilde{\Lambda}_{s+1}, g_i) \setminus \Lambda \neq \emptyset
\]
with \( \tilde{\Lambda}_{s+1} = \tilde{\Lambda}_1 \), where \( s \in \mathbb{Z}^+ \) is called the length of the pseudo-cycle. Lemma II.1 shows a sufficient condition for the existence of a pseudo-cycle on \( \Lambda \).

Let \( \mathcal{R}_0 \) be the residual subset of \( \text{Diff}^2(M) \) defined by the set of continuous points of the following set-valued function on \( \text{Diff}^2(M) \):
\[
f \mapsto \overline{\{ x \in W^s(p, f) : p \in \text{Per}_k(f) \}},
\]
where $W^s(p, f) \cap W^u(p, f)$ is the set of $x \in W^s(p, f) \cap W^u(p, f)$ at which $W^s(p, f)$ and $W^u(p, f)$ intersect transversally and $\text{Per}_h(f)$ is that of hyperbolic periodic points of $f$. Let $\mathcal{U}$ be the neighborhood of $f$ and $\mathcal{R}(\mathcal{U})$ the residual subset of $\mathcal{U}$ given in the hypothesis of Theorem B, and let $f_0 \in \mathcal{R}(\mathcal{U}) \cap \mathcal{R}_0$. Then, we can write $\Lambda(f_0)$ as a finite disjoint union of hyperbolic homoclinic classes:

$$\Lambda(f_0) = \bigcup_{i=1}^l \Lambda_0^i(h).$$

By the definition of $\mathcal{R}_0$, there exist neighborhoods $\mathcal{U}_0$ of $f_0 \in \text{Diff}^2(M)$ and $U_i^0$ of $\Lambda_0^i$ such that if $h \in \mathcal{U}_0$, then the number of basic components of $\Lambda(h)$ is constant, the continuation $\Lambda_0^i(h)$ of $\Lambda_0^i$ for $h$ has an isolating block $U_i^0$ (see [S]), and

$$\Lambda(h) = \bigcup_{i=1}^l \Lambda_0^i(h)$$

is the basic decomposition of $\Lambda(h)$ by homoclinic classes. Set $U = \bigcup_{i=1}^l U_i^0$. Shrinking $U$ first and then $\mathcal{U}_0$ if necessary, we have

$$W^u(\Lambda(h), h) \setminus h^{-1}(W^u(\Lambda(h), h)) \cap U = \emptyset$$

for some small $\varepsilon > 0$, and

$$\Lambda(h) \subset U$$

for all $h \in \mathcal{U}_0$. Define a partial ordering $\leq_h$ on $\{\Lambda_0^0(h), \ldots, \Lambda_0^l(h) : h \in \mathcal{U}_0\}$ by

$$\Lambda_0 \leq_h \Lambda_0'$$

for $\Lambda_0, \Lambda_0' \in \{\Lambda_0^0(h), \ldots, \Lambda_0^l(h)\}$ if $W^u(x, h)$ and $W^s(x', h)$ have a nonempty transversal intersection for some $x \in \Lambda_0$ and $x' \in \Lambda_0'$. Note that all the orderings are preserved by any small perturbation of $h$ coinciding with $h$ in a neighborhood of $\Lambda(h)$. Since the number of basic components $l$ is constant on $\mathcal{U}_0$, given a neighborhood $\mathcal{U}_1(\subset \mathcal{U}_0)$ of $f_0$, there exists $f_1 \in \mathcal{U}_1$ with the maximal number of such orderings between two basic components among diffeomorphisms in $\mathcal{U}_1$. Then $f_1$ has a neighborhood $V(f_1)(\subset \mathcal{U}_1)$ in which the number of the orderings is constant. Let $\mathcal{O}(\mathcal{U})$ be the subset of $\mathcal{U}$ defined by the set of diffeomorphisms with this open property of $f_1$, collecting all diffeomorphisms corresponding to $f_1$ when the choices of $f_0$ and its neighborhoods $\mathcal{U}_1 \subset \mathcal{U}_0$ vary. Then, $\mathcal{O}(\mathcal{U})$ is open in $\mathcal{U}$ by definition and dense in $\mathcal{U}$ because $f_1$ can be arbitrarily close to $f_0 \in \mathcal{R}(\mathcal{U}) \cap \mathcal{R}_0$. Hence $\mathcal{O}(\mathcal{U})$ is open and dense in $\mathcal{U}$.

Let us prove that $\mathcal{O}(\mathcal{U})$ works as the required open and dense subset in Theorem B. Let $g \in \mathcal{O}(\mathcal{U})$. Identifying $g$ with $f_1$, we can suppose that properties (7) and (8) hold for $h$ sufficiently close to $g$. Then, for the proof of Theorem B, it is enough to exhibit a contradiction assuming that there exists $x \notin W^s(\Lambda(g), g)$ such that $\mu(\Lambda(g)) > 0$ for all $\mu \in \mathcal{M}(x, g)$. Under the assumption, using Lemma II.1, we can find a pseudo-cycle $\Lambda_1(g), \ldots, \Lambda_l(g)$ on $\Lambda(g)$ with length $l(g)$. As in [P], there is $1 \leq i \leq l(g)$ such that

$$\dim W^u(\Lambda_i(g)) + \dim W^s(\Lambda_{i+1}(g)) \geq \dim M$$

with $\Lambda_{i+1}(g) = \Lambda_1(g)$. Without loss of generality, we may assume that $i = 1$. Then, by the definition of pseudo-cycles, there exists $h$ arbitrarily close to $g$. 

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coinciding with \( g \) in a neighborhood of \( \Lambda(g) \) and such that
\[
\hat{\Lambda}_1(g) \leq_h \hat{\Lambda}_2(g).
\]
Since all the orderings between two basic sets for \( g \) are preserved for \( h \) sufficiently close to \( g \), we have
\[
\hat{\Lambda}_1(g) \leq_g \hat{\Lambda}_2(g);
\]
otherwise (9) would be a new ordering, contradicting the choice of \( g \). Then, we apply Palis’ perturbation in [P] to prove that if \( l(g) > 1 \), then there is a pseudo-cycle with length \( l(g) - 1 \). In fact, by the definition of pseudo-cycles, there exists \( h_1 \) arbitrarily close to \( g \) coinciding with \( g \) in a neighborhood of \( \Lambda(g) \) and such that
\[
W^u(\hat{\Lambda}_2(g), h_1) \cap W^s(\hat{\Lambda}_3(g), h_1) \setminus \Lambda(g) \neq \emptyset.
\]
By (10), if \( h_1 \) is sufficiently close to \( g \), then \( \hat{\Lambda}_1(g) \leq_{h_1} \hat{\Lambda}_2(g) \). Therefore, \( W^u(\hat{\Lambda}_1(g), h_1) \) accumulates on \( W^u(\hat{\Lambda}_2(g), h_1) \) by the \( \lambda \)-lemma (see [PdM]), which together with (11) makes it possible for us to apply Palis’ perturbation to get \( \hat{g} \) arbitrarily close to \( h_1 \) (and therefore arbitrarily close to \( g \)) coinciding with \( g \) in a neighborhood of \( \Lambda(g) \) and such that
\[
W^u(\hat{\Lambda}_1(g), \hat{g}) \cap W^s(\hat{\Lambda}_3(g), \hat{g}) \setminus \Lambda(g) \neq \emptyset.
\]
This means that \( \hat{\Lambda}_1(g), \ldots, \hat{\Lambda}_{l(g)-1}(g) \) with \( \hat{\Lambda}_1(g) = \hat{\Lambda}_1(g) \) and \( \hat{\Lambda}_i(g) = \hat{\Lambda}_{i+1}(g) \) for all \( 2 \leq i \leq l(g) - 1 \) is a pseudo-cycle on \( \Lambda(g) \) with length \( l(g) - 1 \). Repeating this process a finite number of times, we finally obtain a pseudo-cycle on \( \Lambda(g) \) with length \( l(g) = 1 \). Then, \( g \) can be approximated by some \( h \in U_0 \) coinciding with \( g \) in a neighborhood of \( \Lambda(g) \) and exhibiting a transversal homoclinic point associated to a basic component of \( \Lambda(g) \). Since the basic component is contained in \( \Lambda(h) \), properties (6) and (7) imply that it is some of \( \{\Lambda^t(h) : 1 \leq t \leq l\} \) and hence there exists a transversal homoclinic point in \( U^c \) associated to some \( \Lambda^t(h) \). But this contradicts (8) because \( \Lambda(h) \) contains all the homoclinic classes associated to periodic orbits in \( \Lambda(h) \). Thus, the proof of Theorem B is complete.

III. Proof of Theorem C

In this last section, we prove Theorem C using Theorems A and B. Let \( R_0 \) be the residual subset of \( \text{Diff}^2(M) \) given in Section II, and let \( R_1 \) be the set of measurably hyperbolic diffeomorphisms in \( R_0 \). For \( f \in R_1 \) let \( S(f) = \bigcup_{k \geq 1} \Lambda_k \) be a countable union of disjoint basic sets given by the definition of measurably hyperbolic diffeomorphisms. Then, every \( \Lambda_k, k \geq 1 \), is a hyperbolic homoclinic class. In fact, since
\[
W^s(\Lambda_k, f) \cap W^u(\Lambda_k, f) \setminus \Lambda_k \subset \overline{\text{Per}_h(f)} \subset S(f),
\]
if \( y \in W^s(\Lambda_k, f) \cap W^u(\Lambda_k, f) \setminus \Lambda_k \), then \( y \in \Lambda_{k'} \) for some \( k' \neq k \), which contradicts that \( \Lambda_k \cap \Lambda_{k'} = \emptyset \). By the definition of \( R_0 \), for every \( k \geq 1 \), there exist neighborhoods \( U_k \) of \( f \) and \( U_k \) of \( \Lambda_k \) such that the continuation \( \Lambda_k(h) \) of \( \Lambda_k \) for \( h \in U_k \) is still a homoclinic class, satisfying that
\[
W^s_{g_k}(\Lambda_k(h), h) \setminus h^{-1}(W^u_{g_k}(\Lambda_k(h), h)) \cap U_k = \emptyset
\]
for some small \( \varepsilon_k > 0 \), and
\[
\Lambda_k(h) \subset U_k
\]
with an isolating block \( U_k \) for all \( h \in U_k \). We first prove the following lemma:
Lemma III.1. If \( f \in \mathcal{R}_1 \), then \( S(f) = \bigcup_{k=1}^{s_f} \Lambda_k \) for some disjoint basic sets \( \Lambda_k \), \( 1 \leq k \leq s_f \).

Proof. Suppose that \( S(f) \) has infinitely many basic components \( \Lambda_k \), \( k \geq 1 \). Using the Anosov Closing Lemma (see \([S]\)), we can find a sequence of periodic points \( p_k \), \( k \geq 1 \), such that the Hausdorff distance between \( \mathcal{O}_f(p_k) \) and \( \Lambda_k \) is less than \( 1/k \). Let \( \ell_k \geq 1 \) be the period of \( p_k \). For the sequence of probabilities \( \mu(p_k, \ell_k) \), \( k \geq 1 \), take an accumulation point \( \mu \in \mathcal{M}_f(M) \). Since \( \mu(S(f)) = 1 \), there is \( k_0 > 0 \) such that \( \mu(\Lambda_{k_0}) > 0 \). Then, it is possible to apply Theorem A provided that \( \mathcal{O}_f(p_k) \) is not contained in \( \Lambda_{k_0} \) for sufficiently large \( k \). As to the last condition, otherwise, there exists \( k > k_0 \) such that \( \Lambda_k \) is contained in the isolating block \( \overline{U}_{k_0} \) of \( \Lambda_{k_0} \). But then \( \Lambda_k \subseteq \Lambda_{k_0} \), contradicting that \( \Lambda_{k_0} \cap \Lambda_k = \emptyset \). Applying Theorem A, we obtain a transversal homoclinic point associated to \( \Lambda_{k_0} \) for some \( g \in \mathcal{U}_{k_0} \) coinciding with \( f \) in a neighborhood of \( \Lambda_{k_0} \). But this contradicts (1) and (2) with \( k = k_0 \) in a manner similar to the last argument in Section II.

By Lemma III.1, if \( f \in \mathcal{R}_1 \), then there exists a neighborhood \( \mathcal{V}_f \) in \( \bigcap_{k=1}^{s_f} \mathcal{U}_k \) of \( f \) such that, for every \( h \in \mathcal{V}_f \), the continuation \( \Lambda_k(h) \) of \( \Lambda_k \) is defined for all \( 1 \leq k \leq s_f \). Then, we have the following properties for all \( h \in \mathcal{V}_f \):

(a) \( \Lambda_k(h) \), \( 1 \leq k \leq s_f \), are disjoint hyperbolic homoclinic classes satisfying (1) and (2):

(b) \( h \) does not exhibit any transversal homoclinic point associated to \( \Lambda_k(h) \) for all \( 1 \leq k \leq s_f \).

Then, by Theorem B, there exists an open and dense subset \( \mathcal{O}(\mathcal{V}_f) \) of \( \mathcal{V}_f \) such that every \( g \in \mathcal{O}(\mathcal{V}_f) \) satisfies that:

(c) Given \( x \in M \), if \( \mu(\bigcup_{k=1}^{s_f} \Lambda_k(g)) > 0 \) for all \( \mu \in \mathcal{M}(x,g) \), then \( x \in W^s(\bigcup_{k=1}^{s_f} \Lambda_k(g), g) \).

Now, since \( \bigcup_{f \in \mathcal{R}_1} \mathcal{O}(\mathcal{V}_f) \) is an open and dense subset of \( \bigcup_{f \in \mathcal{R}_1} \mathcal{V}_f \), we can define another residual subset \( \mathcal{R} \) of \( \text{Diff}^2(M) \) by

\[
\mathcal{R} = \bigcup_{f \in \mathcal{R}_1} \mathcal{O}(\mathcal{V}_f) \cup (\mathcal{R}_0 \setminus \overline{\mathcal{R}_1}).
\]

In order to prove Theorem C, it suffices to show that if \( g \in \mathcal{R} \) is measurably hyperbolic, then \( g \) is Axiom A with no cycles. By the definition of \( \mathcal{R}_1 \), if \( g \in \mathcal{R} \) is measurably hyperbolic, then \( g \notin \mathcal{R}_0 \setminus \overline{\mathcal{R}_1} \) and hence \( g \in \mathcal{O}(\mathcal{V}_f) \) for some \( f \in \mathcal{R}_1 \), which implies that

\[
S(g) = \bigcup_{k=1}^{s_f} \Lambda_k(g).
\]

In fact, since \( g \in \mathcal{V}_f \) with \( f \in \mathcal{R}_1 \), if \( \mathcal{V}_f \) has been taken sufficiently small, every basic component of \( S(g) \) (which is also a hyperbolic homoclinic class) is contained in \( \bigcup_{k=1}^{s_f} \mathcal{U}_k \) and therefore, by property (a), it must be some of \( \{\Lambda_k(g) : 1 \leq k \leq s_f\} \), implying \( S(g) \subset \bigcup_{k=1}^{s_f} \Lambda_k(g) \). The opposite inclusion is trivial from the density of periodic points in \( \Lambda_k(g) \) by property (a).

Suppose that there exists \( x_0 \in \Omega(g) \setminus S(g) \) for \( g \in \mathcal{O}(\mathcal{V}_f) \). Then, by the definition of the nonwandering set \( \Omega(g) \), there exists a sequence of strings \( \gamma_j = (y_j, g^{n_j}(y_j)) \), \( j \geq 1 \), outside \( S(g) \) such that \( \lim_{j \to +\infty} y_j = \lim_{j \to +\infty} g^{n_j}(y_j) = x_0 \). Let \( \Gamma \) be the
set of accumulation points of \( \{ \gamma \} \). Since \( \omega_g(x_0) \cap S(g) \neq \emptyset \) with \( S(g) \) satisfying (3), we have \( 1 \leq k_1 \leq s_f \) such that \( \omega_g(x_0) \cap \Lambda_{k_1}(g) \neq \emptyset \), and then

\[
\left( W^u(\Lambda_{k_1}(g), g) \setminus S(g) \right) \cap \Gamma \neq \emptyset.
\]

Take

\[ x_1 \in \left( W^u(\Lambda_{k_1}(g), g) \setminus S(g) \right) \cap \Gamma. \]

Since \( \mu(S(g)) = 1 \) for all \( \mu \in \mathcal{M}(x_1, g) \), property (c) and (3) imply that \( x_1 \in W^s(S(g), g) \). Let \( 1 \leq k_2 \leq s \) be such that

\[ x_1 \in W^s(\Lambda_{k_2}(g), g). \]

Then, we can choose

\[ x_2 \in \left( W^u(\Lambda_{k_2}(g), g) \setminus S(g) \right) \cap \Gamma \]

similarly to the choice of \( x_1 \). Repeating this a finite number of times and changing the indexing of \( \{ x_i \} \) if necessary, we obtain \( \{ x_i : 1 \leq i \leq \ell \} \) and \( \{ k_i : 1 \leq i \leq \ell \} \) with some \( 1 \leq \ell \leq s_f \) such that

\[ x_i \in W^u(\Lambda_{k_i}(g), g) \cap W^s(\Lambda_{k_{i+1}}(g), g) \setminus S(g) \]

for all \( 1 \leq i \leq \ell \), where \( \Lambda_{k_{i+1}}(g) = \Lambda_{k_i}(g) \). This \( \Lambda_{k_1}(g), \ldots, \Lambda_{k_\ell}(g) \) is an \( \ell \)-cycle on \( S(g) \) and hence, using Palis’ perturbation in [P] decreasing the lengths of cycles again, we can create a 1-cycle; that is, a transversal homoclinic point associated to some \( \Lambda_k(g) \) can be created for some \( h \in \mathcal{V}_f \) coinciding with \( g \) in a neighborhood of \( S(g) \). Since \( \Lambda_k(g) = \Lambda_k(h) \) by property (a), this contradicts property (b). Thus, we have proved that \( S(g) = \Omega(g) \) in which periodic points are dense by property (a) and (3), and \( \Omega(g) = \bigcup_{k=1}^{\ell} \Lambda_k(g) \) has no cycles, completing the proof of Theorem C.

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References


[Pu] E. Pujals, Some simple questions related to the \( C^r \) stability conjecture, Nonlinearity 21 (2008), T233-T237. MR2448223
Applications of Mañé’s $C^2$ Connecting Lemma


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