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APPLICATIONS OF MAÑÉ'S C^2 CONNECTING LEMMA

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Abstract. We consider a few applications of Mañé's C^2 Connecting Lemma. These are the C^2 creation of homoclinic points associated to a basic set (i.e., isolated transitive hyperbolic set), a C^2 locally generic criterion to know whether a given point belongs to the stable set of hyperbolic homoclinic classes, and that measurably hyperbolic diffeomorphisms (i.e., having the closure of supports of all invariant measures as a countable union of disjoint basic sets) are C^2 generically uniformly hyperbolic diffeomorphisms.

Introduction

Contrary to developed structure theories of individual C^2 diffeomorphisms, we have only a few C^2 perturbation results, which is a big obstacle for the understanding of global dynamics in the space of C^2 diffeomorphisms on a compact manifold (see [Pu] for instance). So, C^2 generic properties are quite valuable even for very special cases. In [M1], Mañé developed C^2 perturbation techniques to connect the stable and unstable sets of isolated hyperbolic sets. In his C^2 Connecting Lemma, Mañé assumed a condition that some type of invariant measures have positive value on an isolated hyperbolic set, but surprisingly it works for general isolated hyperbolic sets and compact manifolds. (For other results on the C^2 creation of homoclinic points on specific manifolds, see [Pi] and [R] for the two-sphere, and [O] for the two-torus.)

In particular, when an isolated hyperbolic set can be decomposed by basic sets, some problems (the C^1 Stability Conjecture for instance [M2]) are reduced to creating homoclinic points associated to a basic component of the isolated hyperbolic set. In the C^1 case, we have the C^1 Connecting Lemma ([H2]) to solve such problems, and its C^2 version is expected to play a similar role in the C^2 case even if it requires some assumptions. The first theorem, just a modification of Mañé's result, deals with the C^2 creation of homoclinic points associated to a basic component of an isolated hyperbolic set. The second theorem is a C^2 locally generic criterion to know whether a given point belongs to the stable set of hyperbolic homoclinic classes. To prove the theorem, we need a key lemma concerning the existence of a so-called pseudo-cycle. Using these theorems, we prove the third theorem: C^2 generically, measurably hyperbolic diffeomorphisms (i.e., having the closure of

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supports of all invariant measures as a countable union of disjoint basic sets) are uniformly hyperbolic ones (Axiom A with no cycles). More presicely, in a C^2 residual subset at which every homoclinic class varies continuously with respect to C^2 perturbations, we apply the first theorem to show that the closure of supports of all invariant measures for every measurably hyperbolic diffeomorphism is a finite union of disjoint basic sets that are also homoclinic classes. Then, we use the second theorem in order to extend the hyperbolicity to the whole nonwandering set for every measurably hyperbolic diffeomorphism in some other C^2 residual subset. In the proofs of the last two theorems, another ingredient is Palis' perturbation ([P]) decreasing the lengths of cycles of basic sets. Applying his perturbation, we finally create a homoclinic point associated to a basic set (1-cycle) under the assumption that the theorems are not true, which contradicts the choices of the residual subsets above. As a consequence, we see that, in the C^2 generic viewpoint, it is sufficient to find a measurably hyperbolic diffeomorphism in a residual subset to obtain a uniformly hyperbolic diffeomorphism.

Let M be a smooth compact manifold without boundary, and let $\mathrm{Diff}^2(M)$ be the space of C^2 diffeomorphisms with the C^2 topology. Denote by $\mathcal{M}(M)$ the set of probabilities on the Borel σ-algebra $\mathcal{B}$ of M endowed with its usual topology ($\mu_n \to \mu$ if and only if $\int \varphi d\mu_n \to \int \varphi d\mu$ for every continuous function $\varphi\colon M \to \mathbb{R}$). For $f \in \mathrm{Diff}^2(M)$, let $\mathcal{M}_f(M)$ be the set of f-invariant elements of $\mathcal{M}(M)$. For $x \in M$ and $n \in \mathbb{Z}^+$, define a probability $\mu(x,n) \in \mathcal{M}(M)$ by

$$\mu(x,n) = \frac{1}{n}\sum_{j=1}^{n} \delta_{f^j(x)}.$$

Denote by $\mathcal{M}(x,f)$ the set of $\mu \in \mathcal{M}(M)$ such that there exists a sequence of positive integers $n_1 < n_2 < \dots$ satisfying

$$\mu = \lim_{i\to+\infty} \mu(x,n_i).$$

Note that $\mathcal{M}(x,f) \subset \mathcal{M}_f(M)$.

We say that $\Lambda \subset M$ is a *hyperbolic set* of f if it is compact f-invariant and there exist a Df-invariant continuous splitting $TM|\Lambda = E^s \oplus E^u$ and constants $C > 0$, $0 < \lambda < 1$ such that

$$\|(Df^n)|E^s(x)\| \le C\lambda^n$$

and

$$\|(Df^{-n})|E^u(x)\| \le C\lambda^n$$

for all $x \in \Lambda$, $n \ge 1$. In particular, a hyperbolic set Λ is *isolated* if there exists an *isolating block* U of Λ, that is, a compact neighborhood U of Λ such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$. For an isolated hyperbolic set Λ, we define the local stable and unstable sets of Λ by

$$W^s_\varepsilon(\Lambda,f) = \bigcup_{x\in\Lambda} W^s_\varepsilon(x,f),$$
$$W^u_\varepsilon(\Lambda,f) = \bigcup_{x\in\Lambda} W^u_\varepsilon(x,f),$$

where $W^s_\varepsilon(x,f)$ and $W^u_\varepsilon(x,f)$ are the local stable and unstable manifolds at $x \in \Lambda$ with size ε for some small $\varepsilon > 0$. Then,

$$W^s(\Lambda, f) = \bigcup_{n\geq 0} f^{-n}(W^s_\varepsilon(\Lambda, f)),$$
$$W^u(\Lambda, f) = \bigcup_{n\geq 0} f^{n}(W^u_\varepsilon(\Lambda, f)),$$

where $W^s(\Lambda, f)$ and $W^u(\Lambda, f)$ are the stable and unstable sets of Λ defined by

$$W^s(\Lambda, f) = \{y \in M : \lim_{n\to+\infty} d(f^n(y), \Lambda) = 0\},$$
$$W^u(\Lambda, f) = \{y \in M : \lim_{n\to-\infty} d(f^n(y), \Lambda) = 0\}.$$

It is well known that if Λ is an isolated hyperbolic set of f with the nonwandering set $\Omega(f|\Lambda) = \Lambda$, then Λ can be uniquely decomposed by disjoint finite *basic sets* (i.e., isolated transitive hyperbolic sets) as:

$$\Lambda = \bigcup_{t=1}^{l} \Lambda_t,$$

which is called the *basic decomposition* of Λ, and each Λ_t is called a *basic component* of Λ. When

$$W^s(\Lambda_t, f) \cap W^u(\Lambda_t, f) \setminus \Lambda_t \neq \emptyset,$$

we say that f exhibits a *homoclinic point associated to a basic component* Λ_t of Λ. We call a sequence $\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_\ell$ of basic sets a *reordered subsequence* of $\Lambda_1, \ldots, \Lambda_l$ if $\tilde{\Lambda}_i = \Lambda_{j_i}$ $(1 \leq i \leq \ell)$ for some $\{j_1, \ldots, j_\ell\} \subset \{1, \ldots, l\}$ with $j_i \neq j_{i'}$ when $1 \leq i \neq i' \leq \ell$. An *$\ell$-cycle* on Λ is a reordered subsequence $\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_\ell$ of $\Lambda_1, \ldots, \Lambda_l$ such that

$$W^u(\tilde{\Lambda}_i, f) \cap W^s(\tilde{\Lambda}_{i+1}, f) \setminus \Lambda \neq \emptyset$$

for all $1 \leq i \leq \ell$ with $\tilde{\Lambda}_{\ell+1} = \tilde{\Lambda}_1$. Here, ℓ is called the *length* of the ℓ-cycle. Note that the existence of a 1-cycle implies that of a homoclinic point associated to a basic component of Λ. Now let us recall a C^2 Connecting Lemma ([M2, Theorem I.1]) by Mañé:

Mañé's C^2 Connecting Lemma. Let $f \in \mathrm{Diff}^2(M)$ and let Λ be an isolated hyperbolic set of f. Let $\bar{x} \notin \Lambda$ be a point such that there exist a sequence $\{x_k\} \subset M$ converging to $\bar{x}$ and a sequence of integers $0 < m_1 < m_2 < \ldots$ such that the sequence of probabilities $\mu(x_k, m_k)$, $k \geq 1$, converges to a probability μ that satisfies $\mu(\Lambda) > 0$. Then, for given neighborhoods $\mathcal{U}$ of f in $\mathrm{Diff}^2(M)$ and U of Λ in M, there exist $g \in \mathcal{U}$ and a neighborhood $V \subset U$ of Λ such that

$$f\,|(V \cup U^c) = g\,|(V \cup U^c),$$
$$f^{-1}|(V \cup U^c) = g^{-1}|(V \cup U^c)$$

and moreover they satisfy one of the following properties:

(I) There exist $k \geq 1$ and $0 < s < m_k$ such that

$$g^j(x_k) = f^j(x_k)$$

for all $0 \leq j < s$ and

$$g^{s+1}(x_k) \in \bigcap_{n\geq 0} g^{-n}(V) = \bigcap_{n\geq 0} f^{-n}(V).$$

(II) There exist $k \geq 1$ and $0 < s_0 < s_1 < m_k$ such that
  (a) $g^j(f^{s_0}(x_k)) = f^j(f^{s_0}(x_k))$ for $0 \leq j < s_1 - s_0$,
  (b) $g^{-2}(f^{s_0}(x_k)) \in \bigcap_{n\geq 0} g^n(V) = \bigcap_{n\geq 0} f^n(V)$,
  (c) $g^{(s_1-s_0)+1}(f^{s_0}(x_k)) \in \bigcap_{n\geq 0} g^{-n}(V) = \bigcap_{n\geq 0} f^{-n}(V)$.

This statement includes connecting an orbit with the stable set $W^s(\Lambda, f)$ or the unstable set $W^u(\Lambda, f)$ of Λ by an arbitrarily small C^2 perturbation without changing f in a neighborhood of Λ. In fact, property (I) implies $x_k \in W^s(\Lambda, g)$ and property (II) implies $W^s(\Lambda, g) \cap W^u(\Lambda, g) \setminus \Lambda \neq \emptyset$. Note that, when $\Omega(f|\Lambda) = \Lambda$, unless Λ is a single basic set, property (II) does not always imply the creation of a homoclinic point associated to a basic component of Λ. In order to always get such a homoclinic point from the hypothesis of Mañé's C^2 Connecting Lemma, we apply this to a special case where $\{x_k\}$ is a sequence of periodic points of f not contained in Λ and m_k is the period of x_k. Note that the convergence $\lim_{k\to+\infty} x_k = \bar{x} \notin \Lambda$ can be assumed by replacing x_k by some periodic point on the same periodic orbit with x_k and taking a subsequence of $k = 1, 2, \ldots$ if necessary. Consequently, we obtain the following theorem:

Theorem A. *Let $f \in \mathrm{Diff}^2(M)$ and let Λ be an isolated hyperbolic set of f with $\Omega(f|\Lambda) = \Lambda$. Suppose that there exists a sequence $\{p_k\}$ of periodic points of f with $p_k \notin \Lambda$ such that the probabilities $\mu(p_k, \ell_k)$ with ℓ_k, the period of p_k, $k \geq 1$, converge to a probability μ that satisfies $\mu(\Lambda) > 0$. Then, for every neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^2(M)$, there exists $g \in \mathcal{U}$ coinciding with f in a neighborhood of Λ and exhibiting a homoclinic point associated to a basic component of Λ.*

A *homoclinic class* is the closure of the transversal intersection between the stable and unstable manifolds of a hyperbolic periodic orbit. It is well-known that every hyperbolic homoclinic class is a basic set. The following theorem is obtained as applications of Mañé's C^2 Connecting Lemma and Palis' perturbation decreasing the lengths of cycles of basic sets:

Theorem B. *Let $f \in \mathrm{Diff}^2(M)$ have a hyperbolic set Λ and let $\mathcal{U}$ be a neighborhood of f such that the continuation $\Lambda(g)$ of Λ is defined for all $g \in \mathcal{U}$. Then, when $\Lambda(g)$ is a finite union of homoclinic classes for all g in a residual subset of $\mathcal{U}$, there exists an open and dense subset $\mathcal{O}(\mathcal{U})$ of $\mathcal{U}$ such that given $x \in M$, if $\mu(\Lambda(g)) > 0$ for all $\mu \in \mathcal{M}(x, g)$ with $g \in \mathcal{O}(\mathcal{U})$, then $x \in W^s(\Lambda(g), g)$.*

Remark. Mañé used the conclusion of this theorem (for the C^1 case) without the generic condition in his proof of the C^1 Stability Conjecture [M2, Lemma V.5]. In this author's opinion, his proof without the generic condition contains a gap coming from that of the proof of [M1, Theorem D]. But the statement of [M1, Theorem D] for the C^1 case itself is true by the C^1 Connecting Lemma [H2].

We say that the diffeomorphism f is *measurably hyperbolic* if the closure of supports of all f-invariant measures

$$S(f) = \overline{\{x \in \mathrm{supp}\,(\mu) : \mu \in \mathcal{M}_f(M)\}}$$

is a countable union of disjoint basic sets. The following theorem shows that: C^2 generically, every measurably hyperbolic diffeomorphism is a uniformly hyperbolic one.

Theorem C. *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^2(M)$ such that if $f \in \mathcal{R}$ is measurably hyperbolic, then f is an Axiom A diffeomorphism with no cycles.*

In Section I, Theorem A is proved following Mañé's perturbation framework. In Section II, Theorem B is proved after the proof of the key lemma. Finally, in Section III, Theorem C is proved using Theorems A and B. All the perturbations we made in this paper are those with respect to the C^2 topology.

I. Proof of Theorem A

Let us recall Mañé's perturbation framework. For f, Λ and U given in Theorem A, define

$$V^+ = W^s_\varepsilon(\Lambda, f) \quad \text{and} \quad V^- = W^u_\varepsilon(\Lambda, f),$$

where $\varepsilon > 0$ is so small that

$$W^u_\varepsilon(\Lambda, f) \cup W^s_\varepsilon(\Lambda, f) \subset U_0$$

for an isolating block U_0 of Λ contained in U. Then, $f(V^+) \subset V^+$, $f^{-1}(V^-) \subset V^-$ and

$$\Lambda = \bigcup_{t=1}^{l} \Lambda_t = V^+ \cap V^-$$

for some $l \in \mathbb{Z}^+$. To obtain a perturbation exhibiting a homoclinic point associated to a basic component of Λ, we need to take a component-wise perturbation framework as considered in [H1]. Restricting this to each basic component Λ_t, we have

$$\Lambda_t = V^{t,+} \cap V^{t,-},$$

where

$$V^{t,+} = W^s_\varepsilon(\Lambda_t, f) \quad \text{and} \quad V^{t,-} = W^u_\varepsilon(\Lambda_t, f).$$

Let

$$V^t_n = \{x \in M : d(x, V^{t,+}) \le r_n, \quad d(x, V^{t,-}) \le r_n\}$$

with $r_n = r_{n-1}^{1+\delta}$ $(n = 1, 2, \dots)$ for some $0 < \delta < 1$, where $r_0 > 0$ is a sufficiently small positive number. We can suppose $r_0 > 0$ is so small that

$$d(x, V^+) = d(x, V^{t,+}) \quad \text{and} \quad d(x, V^-) = d(x, V^{t,-})$$

for all x in a neighborhood of V^t_0 $(1 \le t \le l)$ and

$$V^t_0 \cap V^{t'}_0 = \emptyset$$

if $t \ne t'$ (see [H1, Remark 2.2] for the details). Denote by S^t_n the set of points $x \in V^t_0$ that can be written as $x = f^m(y)$, $m \in \mathbb{Z}$, with $y \in V^t_n$ and $f^j(y) \in V^t_0$ for all $0 \le j \le m$ if $m \ge 0$, or for all $m \le j \le 0$ if $m \le 0$. Then, define

$$V_n = \bigcup_{t=1}^{l} V^t_n \quad \text{and} \quad S_n = \bigcup_{t=1}^{l} S^t_n.$$

We denote by $\mathcal{O}_f(x)$ the full orbit of x, and by (x, y) with $y = f^j(x)$ $(j \ge 0)$ a finite part of the forward orbit of x written as $\{f^i(x) : 0 \le i \le j\}$, which is called a *string*. A string of the form $\sigma = \{x, \dots, f^j(x)\} \subset V_0$ such that $f^{-1}(x) \notin V_0$ and $f^{j+1}(x) \notin V_0$ is called a 0-*string*. For $\{x_k\}$ and $\{m_k\}$ given in Mañé's C^2 Connecting Lemma, a k-*string* is a string of the form $\sigma = (f^i(x_k), f^{i+j}(x_k)) \subset V_0$ with $1 \le i \le i + j \le m_k$ such that $f^{i-1}(x_k) \notin V_0$ and $f^{i+j+1}(x_k) \notin V_0$ if $(f^i(x_k), f^{m_k}(x_k)) \cap V^c_0 \ne \emptyset$; otherwise $i + j = m_k$. That is, the k-string $(f^i(x_k), f^{i+j}(x_k))$ is the intersection of $(x_k, f^{m_k}(x_k))$ and the 0-string with $f^i(x_k)$. For strings σ_1 and σ_2, we write $\sigma_1 < \sigma_2$ if the first element of σ_2 is a strictly positive iterate of the last element of σ_1.

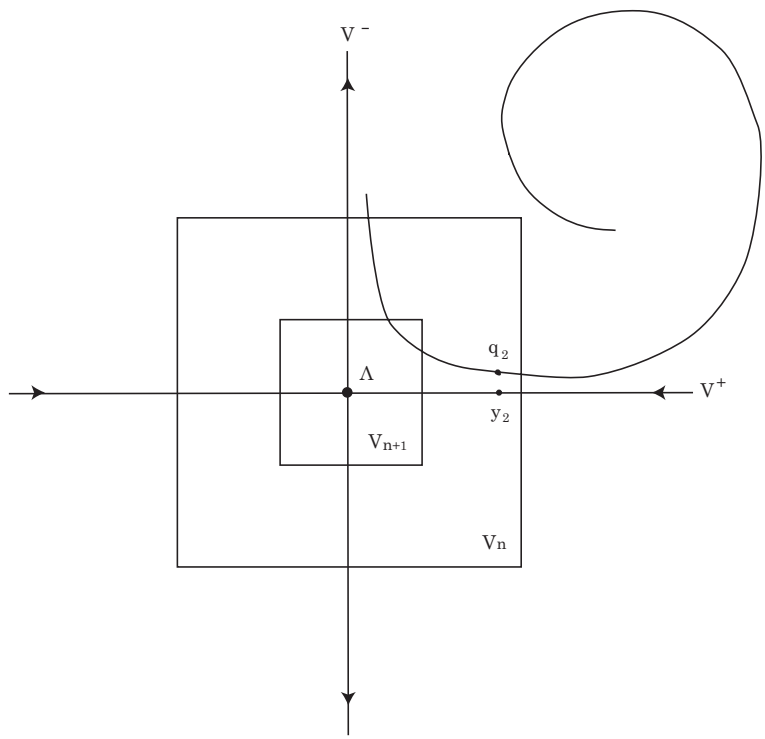

FIGURE 1. Property (i)

Mañé proved that if $\mu = \lim_{k\to+\infty} \mu(x_k, m_k)$ and $\mu(\Lambda) > 0$, then for all $n_0 > 0$ one of the following properties holds ([M1, Lemma 4]):

(i) There exist $n \geq n_0$, $k \geq 1$ and a k-string $\sigma_1 \subset S_{n+1}$ such that $\sigma \cap (S_n \setminus S_{n+1}) = \emptyset$ for every k-string $\sigma < \sigma_1$.
(ii) There exist $n \geq n_0$, $k \geq 1$ and two k-strings $\sigma_1 < \sigma_2 \subset S_{n+1}$ such that $\sigma \cap (S_n \setminus S_{n+1}) = \emptyset$ for every k-string $\sigma_1 < \sigma < \sigma_2$.

As proved in [M1], properties (i) and (ii) lead to properties (I) and (II) in Mañé's C^2 Connecting Lemma, respectively. For $\mathcal{U}$ given in the C^2 Connecting Lemma, the neighborhood V of Λ in properties (I) and (II) is given by $V = V_n \setminus B_n \setminus f^{-1}(B_n)$, where B_n is the union of two balls at some points in V^- and V^+ with radius $r_n^{1+(\beta/3)}$ for some n (taken sufficiently large according to the given neighborhood $\mathcal{U}$ as in [M1, p. 152]) and $\beta > 0$ (see [M1, p. 150] for the choice of β). Figures 1 and 2 are conceptual figures of properties (i) and (ii), respectively. By the absence of k-strings in $S_n \setminus S_{n+1}$, we can find a diffeomorphism h supported in B_n and close to the identity by which, for q_i and y_i $(i = 1, 2)$ in the figures, $q_2 \in \sigma_2$ is pushed to $y_2 \in V^+$, $y_1 \in V^-$ is pushed to $q_1 \in \sigma_1$, and such that $g = h \circ f$ satisfies properties (I) or (II) according to properties (i) or (ii). Define, for $1 \leq t \leq l$,

$$V^t = V_n^t \setminus B_n^t \setminus f^{-1}(B_n^t),$$

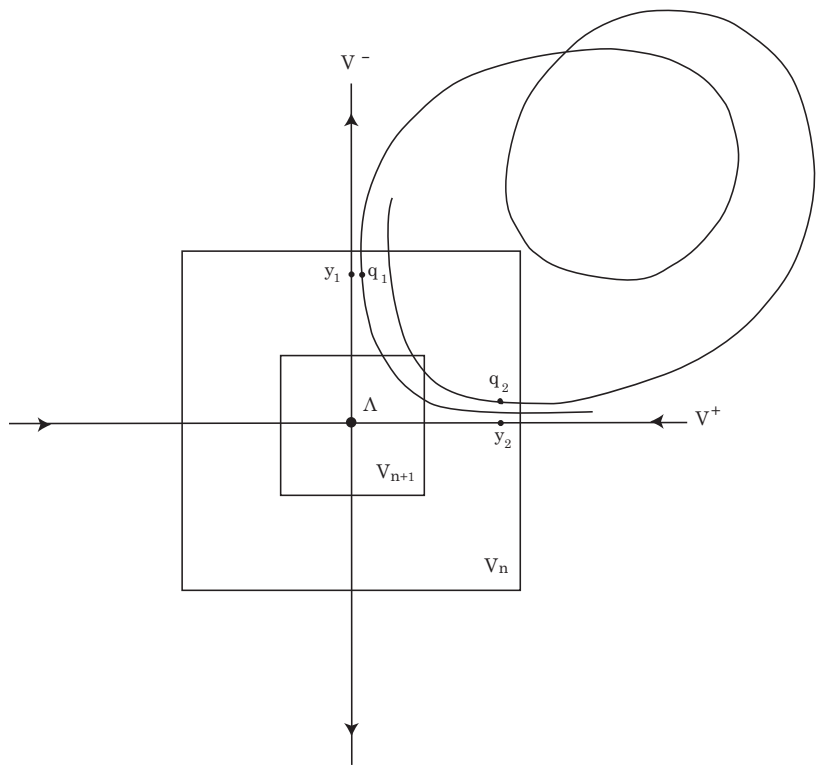

FIGURE 2. Property (ii)

where B_n^t is the union of two balls at some points in $V^{t,-}$ and $V^{t,+}$ with the same radius as above. Now we provide the component-wise versions of (i) and (ii) as (i′) and (ii′), respectively:

(i′) There exist $n \geq n_0$, $k \geq 1$, $1 \leq t \leq l$ and a k-string $\sigma_1 \subset S_{n+1}^t$ such that $\sigma \cap (S_n^t \setminus S_{n+1}^t) = \emptyset$ for every k-string $\sigma < \sigma_1$.

(ii′) There exist n_1, $n_2 \geq n_0$, $k \geq 1$, $1 \leq t_1, t_2 \leq l$ and two k-strings $\sigma_1 < \sigma_2$ with $\sigma_1 \subset S_{n_1+1}^{t_1}$ and $\sigma_2 \subset S_{n_2+1}^{t_2}$ such that $\sigma \cap \big((S_{n_1}^{t_1} \setminus S_{n_1+1}^{t_1}) \cup (S_{n_2}^{t_2} \setminus S_{n_2+1}^{t_2})\big) = \emptyset$ for every k-string $\sigma_1 < \sigma < \sigma_2$.

Then, by applying the argument for proving property (I) (resp. property (II)) from property (i) (resp. property (ii)), it is easy to check that property (i′) (resp. property (ii′)) leads to property (I) with V (resp. property (II) with V in (b) and (c)) replaced by V^t (resp. V^{t_1} and V^{t_2}, respectively, which is called *property II-(t_1, t_2)*), and then we have

$$x_k \in W^s(\Lambda_t, g)$$
$$\big(\text{resp.} \quad W^u(\Lambda_{t_1}, g) \cap W^s(\Lambda_{t_2}, g) \setminus \Lambda \neq \emptyset\big).$$

For every $1 \leq t \leq l$, we call property (ii′) with $n = n_1 = n_2$ and $t = t_1 = t_2$ *property (ii-t)*, which leads to property II-(t, t), that is, the creation of a homoclinic

point associated to a basic component Λ_t of Λ. The following lemma is a refinement of [M1, Lemma 4]:

Lemma I.1 ([H1, Lemma 3.3]). *Let $\mu \in \mathcal{M}_f(M)$ be the limit point of $\{\mu(x_k, m_k)\}$ and let $N_k(t)$ be the minimum positive integer n such that the number of 0-strings containing a point of $(x_k, f^{m_k}(x_k)) \cap S_n^t$ (i.e., the number of k-strings in S_n^t) is ≤ 1. Suppose that property (ii-t) does not hold for all $1 \leq t \leq l$ and the following properties are satisfied for some $1 \leq t_0 \leq l$:*

(a) *there exists $L \in \mathbb{Z}^+$ (independent of k) such that for all k sufficiently large,*

$$S_{N_k(t_0)+L} \cap (x_k, f^{m_k}(x_k)) = \emptyset;$$

(b) $\lim_{k\to+\infty} N_k(t_0) = \infty$.

Then $\mu(\Lambda) = 0$.

Let us observe from Lemma I.1 that if $\mu(\Lambda) > 0$, then either property (ii-t) for some $1 \leq t \leq l$ or property (i$'$) holds. We claim this by showing property (i$'$) under the assumption that property (ii-t) does not hold for all $1 \leq t \leq l$. By Lemma I.1, the assumption and $\mu(\Lambda) > 0$ imply that for all $1 \leq t_0 \leq l$ either property (a) or property (b) is not satisfied. If the latter case occurs for t_0 with $\mu(\Lambda_{t_0}) > 0$, there is a subsequence k_j, $j \geq 1$, of $k = 1, 2, \dots$ such that, setting $\gamma_j = (x_{k_j}, f^{m_{k_j}}(x_{k_j}))$, we have a sequence γ_j, $j \geq 1$, satisfying the following property: there is $\bar{n} > 0$ such that, if $n \geq \bar{n}$, then for all $j \geq j_0$ with some $j_0 = j_0(n)$, there exists only one 0-string $\sigma_j^{t_0}$ containing a point of $\gamma_j \cap S_n^{t_0}$, and $\gamma_j \setminus \sigma_j^{t_0}$, $j \geq 1$, does not accumulate on Λ_{t_0}. This implies property (i$'$). On the other hand, for t_0 with $\mu(\Lambda_{t_0}) > 0$ again, if property (b) is satisfied and property (a) is not satisfied, then for any large $L \in \mathbb{Z}^+$ there is an arbitrarily large k such that $S_{N_k(t_0)+L} \cap (x_k, f^{m_k}(x_k)) \neq \emptyset$ and $\lim_{k\to+\infty} N_k(t_0) = \infty$, which also implies property (i$'$). Indeed, the above $S_{N_k(t_0)+L}$ can be replaced by $S^{t_0}_{N_k(t_0)+L}$, for otherwise the hypothesis of Lemma I.1 with $\Lambda = \Lambda_{t_0}$ is satisfied, implying $\mu(\Lambda_{t_0}) = 0$, a contradiction.

In particular, let σ_1' be the k-string σ_1 in property (i$'$) just obtained through this observation; that is, σ_1' is either the k-string in $\sigma_j^{t_0}$ or the k-string in $S^{t_0}_{N_k(t_0)+L}$ with large L and k. Applying this observation to a special case where $x_k \in \mathrm{Per}(f) \setminus \Lambda$ with period ℓ_k and $m_k = \ell_k$, we see that the existence of σ_1' also leads to property (ii-t_0). In fact, the k-string σ_1' may be thought of as if it were two k-strings such that $\sigma_1' = \tilde{\sigma}_1 < \tilde{\sigma}_2 = \sigma_1' \subset S_{n+1}^{t_0}$ and hence

$$\sigma \cap (S_n^{t_0} \setminus S_{n+1}^{t_0}) = \emptyset$$

for every k-string $\tilde{\sigma}_1 < \sigma < \tilde{\sigma}_2$. Thus, we always have property (ii-t) for some $1 \leq t \leq l$, which leads to property II-(t, t), and hence it is possible to create a homoclinic point associated to a basic component Λ_t of Λ by an arbitrarily small perturbation of f, coinciding with f in a neighborhood of Λ. This concludes the proof of Theorem A.

II. Proof of Theorem B

In this section, we prove Theorem B using the key lemma below. This lemma corresponds to the part containing the gap of the proof of [M1, Theorem D] mentioned in the Remark of the Introduction. When Λ is a single basic set, Mañé's argument is correct. For the case where the number of basic components of Λ is more than one, Mañé proceeds with his argument as if S_n were $S_n^{t_0}$ $(n \geq 1)$ only

from the fact that some basic component Λ_{t_0} of Λ has a positive μ-measure for some $\mu \in \mathcal{M}(x,f)$ (by which we have either property (ii-t_0) or property (i$'$) with $t = t_0$), and concludes that property (ii-t_0) and hence property II-(t_0,t_0) always hold. However, the two 0-strings $\sigma_1 < \sigma_2$ of property (ii) which Mañé obtained in his proof of [M1, Theorem D] may be just those of property (ii$'$) with $t_1 = t_0$. Moreover, when the first 0-string σ_3 in $S^{t_0}_{n+1}$ after σ_1 appears, there is no guarantee that $\sigma \cap (S^{t_0}_n \setminus S^{t_0}_{n+1}) = \emptyset$ for every 0-string $\sigma_1 < \sigma < \sigma_3$ because it may happen that $\nu(\Lambda_{t_0}) = 0$ for some other $\nu \in \mathcal{M}(x,f)$.

In the general case, it seems to this author that only the following statement is possible without nontrivial improvement.

Lemma II.1. *Let $f \in \mathrm{Diff}^2(M)$ and let $\Lambda = \bigcup_{t=1}^{l} \Lambda_t$ be a finite union of disjoint basic sets $\Lambda_1, \dots, \Lambda_l$. Suppose that $\mu(\Lambda) > 0$ for all $\mu \in \mathcal{M}(x,f)$ with some $x \notin W^s(\Lambda,f)$. Then, there exists a reordered subsequence $\tilde\Lambda_1, \dots, \tilde\Lambda_s$ of $\Lambda_1, \dots, \Lambda_l$ such that for every neighborhood $\mathcal{U}$ of f and every $1 \le i \le s$, there exists $g_i \in \mathcal{U}$ coinciding with f in a neighborhood of Λ and satisfying*

$$W^u(\tilde\Lambda_i, g_i) \cap W^s(\tilde\Lambda_{i+1}, g_i) \setminus \Lambda \neq \emptyset$$

with $\tilde\Lambda_{s+1} = \tilde\Lambda_1$.

Proof. First, we can suppose that property (ii-t) does not hold for all $1 \le t \le l$. Indeed, property (ii-t) (leading to property II-(t,t)) implies the existence of the required basic set as $\tilde\Lambda_1 = \Lambda_t$ and $s = 1$.

Since $x \notin W^s(\Lambda,f)$ and $\omega_f(x) \cap \Lambda \neq \emptyset$, there exist $N_1 > N$ such that $f^N(x) \notin V_0$ and $f^{N_1}(x) \in V_0$. If $j > N_1$, we can define

$$A(j) = \max\{n \ge 0 : S_n \cap \{f^N(x), \dots, f^j(x)\} \neq \emptyset\}.$$

Let $a_1 < a_2 < \dots$ be the set of values of $A(j)$ and define

$$\tilde\ell_n = \max\{j > N_1 : A(j) = a_n\}.$$

Note that $A(\tilde\ell_n + 1) = a_{n+1}$. Take a subsequence $\tilde\ell_{n_1} < \tilde\ell_{n_2} < \dots$ so that, setting $\ell_k = \tilde\ell_{n_k}$ and $\bar\ell_k = \tilde\ell_{n_k - 1}$, we have μ, $\bar\mu \in \mathcal{M}(x,f)$ with

$$\lim_{k\to+\infty} \mu(x,\ell_k) = \mu \quad \text{and} \quad \lim_{k\to+\infty} \mu(x,\bar\ell_k) = \bar\mu.$$

For $k \ge 1$ let $1 \le t_k, \bar t_k \le l$ be such that

$$d(\Lambda_{t_k}, f^{\ell_k+1}(x)) = d(\Lambda, f^{\ell_k+1}(x)) \quad \text{and} \quad d(\Lambda_{\bar t_k}, f^{\bar\ell_k+1}(x)) = d(\Lambda, f^{\bar\ell_k+1}(x)).$$

Then, there exist k-strings (for $\{x_k = x\}$ and $\{m_k = \ell_k + 1\}$) σ_k and $\bar\sigma_k$ with

$$\sigma_k \subset S^{t_k}_{A(\ell_k+1)} \quad \text{and} \quad \bar\sigma_k \subset S^{\bar t_k}_{A(\bar\ell_k+1)}. \tag{1}$$

Let us claim that

$$\big(S^{t_k}_{A(\ell_k+1)-1} \setminus S^{t_k}_{A(\ell_k+1)}\big) \cap (x, f^{\ell_k}(x)) = \emptyset \tag{2}$$

for arbitrarily large k. If this is false, we may assume that $t_k = \tau$ for all k by taking subsequences of $\{t_k\}$ and $\{\ell_k\}$ if necessary to have

$$\big(S^{\tau}_{A(\ell_k+1)-1} \setminus S^{\tau}_{A(\ell_k+1)}\big) \cap (x, f^{\ell_k}(x)) \neq \emptyset \tag{3}$$

for all k sufficiently large. Then, by (1) and (3), we can apply Lemma I.1 to $\{\mu(x, \ell_k + 1)\}$, satisfying properties (a) and (b) for $t_0 = \tau$, $N_k(\tau) = A(\ell_k + 1) = a_{n_k+1}$ and $L = 1$. Therefore $\mu(\Lambda) = 0$ because

$$\mu = \lim_{k\to+\infty} \mu(x, \ell_k) = \lim_{k\to+\infty} \mu(x, \ell_k + 1),$$

contradicting our hypothesis. In order to obtain a reordered subsequence $\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_s$ of $\Lambda_1, \dots, \Lambda_l$ required in this lemma, it is enough to show that

$$\big(S^{\bar{t}_k}_{A(\bar{\ell}_k+1)-1} \setminus S^{\bar{t}_k}_{A(\bar{\ell}_k+1)}\big) \cap (x, f^{\ell_k}(x)) = \emptyset \tag{4}$$

for arbitrarily large k satisfying (2). In fact, if (4) were true, then (4) with ℓ_k replaced by $\bar{\ell}_k$ ($< \ell_k$) would hold, and therefore, taking a subsequence of $\{n_k\}$ if necessary, (4) for $\bar{\mu}$ could be used as (2) for μ to obtain inductive choices of pairs $(t_k, \bar{t}_k)$ and $(\bar{t}_k, \bar{\bar{t}}_k)$. Hence, by the finiteness of basic components, after at most l times such inductive choices, we can find a reordered subsequence $\{\tilde{\Lambda}_i(k)\}_{i=1}^{s(k)}$ of $\Lambda_1, \dots, \Lambda_l$ chosen as

$$\tilde{\Lambda}_{s(k)}(k) = \Lambda_{t_k},\ \tilde{\Lambda}_{s(k)-1}(k) = \Lambda_{\bar{t}_k},\ \tilde{\Lambda}_{s(k)-2}(k) = \Lambda_{\bar{\bar{t}}_k}, \dots,$$

and finally returning to Λ_{t_k}. Note that we can suppose that $s(k)$ and each $\tilde{\Lambda}_i(k)$, $1 \le i \le s(k)$, do not depend on k by taking a subsequence of infinitely many choices of such a k. Then, setting $s(k) = s$ and $\tilde{\Lambda}_i(k) = \tilde{\Lambda}_i$ for all $1 \le i \le s$, we obtain the required reordered subsequence because property (ii′) with respect to $(\Lambda_{t_1}, \Lambda_{t_2}) = (\tilde{\Lambda}_0, \tilde{\Lambda}'_0)$ (and therefore property II-(t_1, t_2)) holds for every pair of consecutive basic components:

$$(\tilde{\Lambda}_0, \tilde{\Lambda}'_0) \in \{(\tilde{\Lambda}_i, \tilde{\Lambda}_{i+1}), (\tilde{\Lambda}_s, \tilde{\Lambda}_1) : 1 \le i < s\}.$$

Now the proof of (4) is similar to that of (2). If property (4) were false, we could assume that $\bar{t}_k = \bar{\tau}$ and

$$\big(S^{\bar{\tau}}_{A(\bar{\ell}_k+1)-1} \setminus S^{\bar{\tau}}_{A(\bar{\ell}_k+1)}\big) \cap (x, f^{\ell_k}(x)) \neq \emptyset \tag{5}$$

for all k sufficiently large. Note that

$$a_{n_k} = A(\bar{\ell}_k + 1) = A(\ell_k).$$

Then, by (1) and (5), we could apply Lemma I.1 to $\{\mu(x, \ell_k)\}$, satisfying properties a) and b) for $t_0 = \bar{\tau}$, $N_k(\bar{\tau}) \in \{A(\ell_k), A(\ell_k) + 1\}$ (according to whether $\bar{\sigma}_k$ given in (1) is the only k-string for $\{x_k = x\}$ and $\{m_k = \ell_k\}$ contained in $S^{\bar{\tau}}_{a_{n_k}}$ or not) and $L = 1$. Therefore $\mu(\Lambda) = 0$ again. This contradicts our hypothesis and completes the proof of Lemma II.1. □

We say that a reordered subsequence $\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_s$ of $\Lambda_1, \dots, \Lambda_l$ is a *pseudo-cycle* on Λ if for every neighborhood $\mathcal{U}$ of f and every $1 \le i \le s$, there exists $g_i \in \mathcal{U}$ coinciding with f in a neighborhood of Λ and satisfying

$$W^u(\tilde{\Lambda}_i, g_i) \cap W^s(\tilde{\Lambda}_{i+1}, g_i) \setminus \Lambda \neq \emptyset$$

with $\tilde{\Lambda}_{s+1} = \tilde{\Lambda}_1$, where $s \in \mathbb{Z}^+$ is called the *length* of the pseudo-cycle. Lemma II.1 shows a sufficient condition for the existence of a pseudo-cycle on Λ.

Let $\mathcal{R}_0$ be the residual subset of $\mathrm{Diff}^2(M)$ defined by the set of continuous points of the following set-valued function on $\mathrm{Diff}^2(M)$:

$$f \mapsto \overline{\{x \in W^s(p, f) \pitchfork W^u(p, f) : p \in \mathrm{Per}_h(f)\}},$$

where $W^s(p,f) \pitchfork W^u(p,f)$ is the set of $x \in W^s(p,f) \cap W^u(p,f)$ at which $W^s(p,f)$ and $W^u(p,f)$ intersect transversally and $\mathrm{Per}_h(f)$ is that of hyperbolic periodic points of f. Let $\mathcal{U}$ be the neighborhood of f and $\mathcal{R}(\mathcal{U})$ the residual subset of $\mathcal{U}$ given in the hypothesis of Theorem B, and let $f_0 \in \mathcal{R}(\mathcal{U}) \cap \mathcal{R}_0$. Then, we can write $\Lambda(f_0)$ as a finite disjoint union of hyperbolic homoclinic classes:

$$\Lambda(f_0) = \bigcup_{t=1}^{l} \Lambda_t^0.$$

By the definition of $\mathcal{R}_0$, there exist neighborhoods $\mathcal{U}_0$ of $f_0 \in \mathrm{Diff}^2(M)$ and U_t^0 of Λ_t^0 such that if $h \in \mathcal{U}_0$, then the number of basic components of $\Lambda(h)$ is constant, the continuation $\Lambda_t^0(h)$ of Λ_t^0 for h has an isolating block $\overline{U_t^0}$ (see [S]), and

$$\Lambda(h) = \bigcup_{t=1}^{l} \Lambda_t^0(h) \tag{6}$$

is the basic decomposition of $\Lambda(h)$ by homoclinic classes. Set $U = \bigcup_{t=1}^{l} U_t^0$. Shrinking U first and then $\mathcal{U}_0$ if necessary, we have

$$\overline{W_\varepsilon^u(\Lambda(h),h) \setminus h^{-1}(W_\varepsilon^u(\Lambda(h),h))} \cap \overline{U} = \emptyset \tag{7}$$

for some small $\varepsilon > 0$, and

$$\Lambda(h) \subset U \tag{8}$$

for all $h \in \mathcal{U}_0$. Define a partial ordering $\leq_h$ on $\{\Lambda_1^0(h), \dots, \Lambda_l^0(h) : h \in \mathcal{U}_0\}$ by

$$\Lambda_0 \leq_h \Lambda_0'$$

for Λ_0, $\Lambda_0' \in \{\Lambda_1^0(h), \dots, \Lambda_l^0(h)\}$ if $W^u(x,h)$ and $W^s(x',h)$ have a nonempty transversal intersection for some $x \in \Lambda_0$ and $x' \in \Lambda_0'$. Note that all the orderings are preserved by any small perturbation of h coinciding with h in a neighborhood of $\Lambda(h)$. Since the number of basic components l is constant on $\mathcal{U}_0$, given a neighborhood $\mathcal{U}_1 (\subset \mathcal{U}_0)$ of f_0, there exists $f_1 \in \mathcal{U}_1$ with the maximal number of such orderings between two basic components among diffeomorphisms in $\mathcal{U}_1$. Then f_1 has a neighborhood $\mathcal{V}(f_1)(\subset \mathcal{U}_1)$ in which the number of the orderings is constant. Let $\mathcal{O}(\mathcal{U})$ be the subset of $\mathcal{U}$ defined by the set of diffeomorphisms with this open property of f_1, collecting all diffeomorphisms corresponding to f_1 when the choices of f_0 and its neighborhoods $\mathcal{U}_1 \subset \mathcal{U}_0$ vary. Then, $\mathcal{O}(\mathcal{U})$ is open in $\mathcal{U}$ by definition and dense in $\mathcal{U}$ because f_1 can be arbitrarily close to $f_0 \in \mathcal{R}(\mathcal{U}) \cap \mathcal{R}_0$. Hence $\mathcal{O}(\mathcal{U})$ is open and dense in $\mathcal{U}$.

Let us prove that $\mathcal{O}(\mathcal{U})$ works as the required open and dense subset in Theorem B. Let $g \in \mathcal{O}(\mathcal{U})$. Identifying g with f_1, we can suppose that properties (7) and (8) hold for h sufficiently close to g. Then, for the proof of Theorem B, it is enough to exhibit a contradiction assuming that there exists $x \notin W^s(\Lambda(g),g)$ such that $\mu(\Lambda(g)) > 0$ for all $\mu \in \mathcal{M}(x,g)$. Under the assumption, using Lemma II.1, we can find a pseudo-cycle $\tilde{\Lambda}_1(g), \dots, \tilde{\Lambda}_{l(g)}(g)$ on $\Lambda(g)$ with length $l(g)$. As in [P], there is $1 \leq i \leq l(g)$ such that

$$\dim W^u(\tilde{\Lambda}_i(g)) + \dim W^s(\tilde{\Lambda}_{i+1}(g)) \geq \dim M$$

with $\tilde{\Lambda}_{l(g)+1}(g) = \tilde{\Lambda}_1(g)$. Without loss of generality, we may assume that $i = 1$. Then, by the definition of pseudo-cycles, there exists h arbitrarily close to g,

coinciding with g in a neighborhood of $\Lambda(g)$ and such that

$$\tilde{\Lambda}_1(g) \leq_h \tilde{\Lambda}_2(g). \tag{9}$$

Since all the orderings between two basic sets for g are preserved for h sufficiently close to g, we have

$$\tilde{\Lambda}_1(g) \leq_g \tilde{\Lambda}_2(g); \tag{10}$$

otherwise (9) would be a new ordering, contradicting the choice of g. Then, we apply Palis' perturbation in [P] to prove that if $l(g) > 1$, then there is a pseudo-cycle with length $l(g) - 1$. In fact, by the definition of pseudo-cycles, there exists h_1 arbitrarily close to g coinciding with g in a neighborhood of $\Lambda(g)$ and such that

$$W^u(\tilde{\Lambda}_2(g), h_1) \cap W^s(\tilde{\Lambda}_3(g), h_1) \setminus \Lambda(g) \neq \emptyset. \tag{11}$$

By (10), if h_1 is sufficiently close to g, then $\tilde{\Lambda}_1(g) \leq_{h_1} \tilde{\Lambda}_2(g)$. Therefore, $W^u(\tilde{\Lambda}_1(g), h_1)$ accumulates on $W^u(\tilde{\Lambda}_2(g), h_1)$ by the λ-lemma (see [PdM]), which together with (11) makes it possible for us to apply Palis' perturbation to get $\tilde{g}$ arbitrarily close to h_1 (and therefore arbitrarily close to g) coinciding with g in a neighborhood of $\Lambda(g)$ and such that

$$W^u(\tilde{\Lambda}_1(g), \tilde{g}) \cap W^s(\tilde{\Lambda}_3(g), \tilde{g}) \setminus \Lambda(g) \neq \emptyset.$$

This means that $\hat{\Lambda}_1(g), \ldots, \hat{\Lambda}_{l(g)-1}(g)$ with $\hat{\Lambda}_1(g) = \tilde{\Lambda}_1(g)$ and $\hat{\Lambda}_i(g) = \tilde{\Lambda}_{i+1}(g)$ for all $2 \leq i \leq l(g) - 1$ is a pseudo-cycle on $\Lambda(g)$ with length $l(g) - 1$. Repeating this process a finite number of times, we finally obtain a pseudo-cycle on $\Lambda(g)$ with length $l(g) = 1$. Then, g can be approximated by some $h \in \mathcal{U}_0$ coinciding with g in a neighborhood of $\Lambda(g)$ and exhibiting a transversal homoclinic point associated to a basic component of $\Lambda(g)$. Since the basic component is contained in $\Lambda(h)$, properties (6) and (7) imply that it is some of $\{\Lambda_t^0(h) : 1 \leq t \leq l\}$ and hence there exists a transversal homoclinic point in U^c associated to some $\Lambda_t^0(h)$. But this contradicts (8) because $\Lambda(h)$ contains all the homoclinic classes associated to periodic orbits in $\Lambda(h)$. Thus, the proof of Theorem B is complete.

III. Proof of Theorem C

In this last section, we prove Theorem C using Theorems A and B. Let $\mathcal{R}_0$ be the residual subset of $\mathrm{Diff}^2(M)$ given in Section II, and let $\mathcal{R}_1$ be the set of measurably hyperbolic diffeomorphisms in $\mathcal{R}_0$. For $f \in \mathcal{R}_1$ let $S(f) = \bigcup_{k \geq 1} \Lambda_k$ be a countable union of disjoint basic sets given by the definition of measurably hyperbolic diffeomorphisms. Then, every Λ_k, $k \geq 1$, is a hyperbolic homoclinic class. In fact, since

$$W^s(\Lambda_k, f) \pitchfork W^u(\Lambda_k, f) \setminus \Lambda_k \subset \overline{\mathrm{Per}_h(f)} \subset S(f),$$

if $y \in W^s(\Lambda_k, f) \pitchfork W^u(\Lambda_k, f) \setminus \Lambda_k$, then $y \in \Lambda_{k'}$ for some $k' \neq k$, which contradicts that $\Lambda_k \cap \Lambda_{k'} = \emptyset$. By the definition of $\mathcal{R}_0$, for every $k \geq 1$, there exist neighborhoods $\mathcal{U}_k$ of f and U_k of Λ_k such that the continuation $\Lambda_k(h)$ of Λ_k for $h \in \mathcal{U}_k$ is still a homoclinic class, satisfying that

$$\overline{W^u_{\varepsilon_k}(\Lambda_k(h), h) \setminus h^{-1}(W^u_{\varepsilon_k}(\Lambda_k(h), h))} \cap \overline{U}_k = \emptyset \tag{1}$$

for some small $\varepsilon_k > 0$, and

$$\Lambda_k(h) \subset U_k \tag{2}$$

with an isolating block $\overline{U}_k$ for all $h \in \mathcal{U}_k$. We first prove the following lemma:

Lemma III.1. *If $f \in \mathcal{R}_1$, then $S(f) = \bigcup_{k=1}^{s_f} \Lambda_k$ for some disjoint basic sets Λ_k, $1 \le k \le s_f$.*

Proof. Suppose that $S(f)$ has infinitely many basic components Λ_k, $k \ge 1$. Using the Anosov Closing Lemma (see [S]), we can find a sequence of periodic points p_k, $k \ge 1$, such that the Hausdorff distance between $\mathcal{O}_f(p_k)$ and Λ_k is less than $1/k$. Let $\ell_k \ge 1$ be the period of p_k. For the sequence of probabilities $\mu(p_k, \ell_k)$, $k \ge 1$, take an accumulation point $\mu \in \mathcal{M}_f(M)$. Since $\mu(S(f)) = 1$, there is $k_0 > 0$ such that $\mu(\Lambda_{k_0}) > 0$. Then, it is possible to apply Theorem A provided that $\mathcal{O}_f(p_k)$ is not contained in Λ_{k_0} for sufficiently large k. As to the last condition, otherwise, there exists $\bar{k} > k_0$ such that $\Lambda_{\bar{k}}$ is contained in the isolating block $\overline{U}_{k_0}$ of Λ_{k_0}. But then $\Lambda_{\bar{k}} \subset \Lambda_{k_0}$, contradicting that $\Lambda_{k_0} \cap \Lambda_{\bar{k}} = \emptyset$. Applying Theorem A, we obtain a transversal homoclinic point associated to Λ_{k_0} for some $g \in \mathcal{U}_{k_0}$ coinciding with f in a neighborhood of Λ_{k_0}. But this contradicts (1) and (2) with $k = k_0$ in a manner similar to the last argument in Section II. □

By Lemma III.1, if $f \in \mathcal{R}_1$, then there exists a neighborhood $\mathcal{V}_f$ in $\bigcap_{k=1}^{s_f} \mathcal{U}_k$ of f such that, for every $h \in \mathcal{V}_f$, the continuation $\Lambda_k(h)$ of Λ_k is defined for all $1 \le k \le s_f$. Then, we have the following properties for all $h \in \mathcal{V}_f$:

(a) $\Lambda_k(h)$, $1 \le k \le s_f$, are disjoint hyperbolic homoclinic classes satisfying (1) and (2);
(b) h does not exhibit any transversal homoclinic point associated to $\Lambda_k(h)$ for all $1 \le k \le s_f$.

Then, by Theorem B, there exists an open and dense subset $\mathcal{O}(\mathcal{V}_f)$ of $\mathcal{V}_f$ such that every $g \in \mathcal{O}(\mathcal{V}_f)$ satisfies that:

(c) Given $x \in M$, if $\mu(\bigcup_{k=1}^{s_f} \Lambda_k(g)) > 0$ for all $\mu \in \mathcal{M}(x, g)$, then $x \in W^s(\bigcup_{k=1}^{s_f} \Lambda_k(g), g)$.

Now, since $\bigcup_{f \in \mathcal{R}_1} \mathcal{O}(\mathcal{V}_f)$ is an open and dense subset of $\bigcup_{f \in \mathcal{R}_1} \mathcal{V}_f$, we can define another residual subset $\mathcal{R}$ of $\mathrm{Diff}^2(M)$ by

$$\mathcal{R} = \bigcup_{f \in \mathcal{R}_1} \mathcal{O}(\mathcal{V}_f) \cup (\mathcal{R}_0 \setminus \overline{\mathcal{R}}_1).$$

In order to prove Theorem C, it suffices to show that if $g \in \mathcal{R}$ is measurably hyperbolic, then g is Axiom A with no cycles. By the definition of $\mathcal{R}_1$, if $g \in \mathcal{R}$ is measurably hyperbolic, then $g \notin \mathcal{R}_0 \setminus \overline{\mathcal{R}}_1$ and hence $g \in \mathcal{O}(\mathcal{V}_f)$ for some $f \in \mathcal{R}_1$, which implies that

$$S(g) = \bigcup_{k=1}^{s_f} \Lambda_k(g). \tag{3}$$

In fact, since $g \in \mathcal{V}_f$ with $f \in \mathcal{R}_1$, if $\mathcal{V}_f$ has been taken sufficiently small, every basic component of $S(g)$ (which is also a hyperbolic homoclinic class) is contained in $\bigcup_{k=1}^{s_f} U_k$ and therefore, by property (a), it must be some of $\{\Lambda_k(g) : 1 \le k \le s_f\}$, implying $S(g) \subset \bigcup_{k=1}^{s_f} \Lambda_k(g)$. The opposite inclusion is trivial from the density of periodic points in $\Lambda_k(g)$ by property (a).

Suppose that there exists $x_0 \in \Omega(g) \setminus S(g)$ for $g \in \mathcal{O}(\mathcal{V}_f)$. Then, by the definition of the nonwandering set $\Omega(g)$, there exists a sequence of strings $\gamma_j = (y_j, g^{n_j}(y_j))$, $j \ge 1$, outside $S(g)$ such that $\lim_{j \to +\infty} y_j = \lim_{j \to +\infty} g^{n_j}(y_j) = x_0$. Let Γ be the

set of accumulation points of $\{\gamma_j\}$. Since $\omega_g(x_0) \cap S(g) \neq \emptyset$ with $S(g)$ satisfying (3), we have $1 \leq k_1 \leq s_f$ such that $\omega_g(x_0) \cap \Lambda_{k_1}(g) \neq \emptyset$, and then

$$\big(W^u(\Lambda_{k_1}(g), g) \setminus S(g)\big) \cap \Gamma \neq \emptyset.$$

Take

$$x_1 \in \big(W^u(\Lambda_{k_1}(g), g) \setminus S(g)\big) \cap \Gamma.$$

Since $\mu(S(g)) = 1$ for all $\mu \in \mathcal{M}(x_1, g)$, property (c) and (3) imply that $x_1 \in W^s(S(g), g)$. Let $1 \leq k_2 \leq s$ be such that

$$x_1 \in W^s(\Lambda_{k_2}(g), g).$$

Then, we can choose

$$x_2 \in \big(W^u(\Lambda_{k_2}(g), g) \setminus S(g)\big) \cap \Gamma$$

similarly to the choice of x_1. Repeating this a finite number of times and changing the indexing of $\{x_i\}$ if necessary, we obtain $\{x_i : 1 \leq i \leq \ell\}$ and $\{k_i : 1 \leq i \leq \ell\}$ with some $1 \leq \ell \leq s_f$ such that

$$x_i \in W^u(\Lambda_{k_i}(g), g) \cap W^s(\Lambda_{k_{i+1}}(g), g) \setminus S(g)$$

for all $1 \leq i \leq \ell$, where $\Lambda_{k_{\ell+1}}(g) = \Lambda_{k_1}(g)$. This $\Lambda_{k_1}(g), \ldots, \Lambda_{k_\ell}(g)$ is an ℓ-cycle on $S(g)$ and hence, using Palis' perturbation in [P] decreasing the lengths of cycles again, we can create a 1-cycle; that is, a transversal homoclinic point associated to some $\Lambda_{k_i}(g)$ can be created for some $h \in \mathcal{V}_f$ coinciding with g in a neighborhood of $S(g)$. Since $\Lambda_{k_i}(g) = \Lambda_{k_i}(h)$ by property (a), this contradicts property (b). Thus, we have proved that $S(g) = \Omega(g)$ in which periodic points are dense by property (a) and (3), and $\Omega(g) = \bigcup_{k=1}^{s_f} \Lambda_k(g)$ has no cycles, completing the proof of Theorem C.

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