FIXED POINTS IN INDECOMPOSABLE $k$-JUNCTIONED TREE-LIKE CONTINUA

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Abstract. Let $M$ be an indecomposable $k$-junctioned tree-like continuum. Let $f$ be a map of $M$ that sends each composant of $M$ into itself. Using an argument of O. H. Hamilton, we prove that $f$ has a fixed point.

D. P. Bellamy [1] in 1979 defined a tree-like continuum that does not have the fixed-point property (also see [4], [10], [11], and [13]). In 1998, the author [6] proved that every tree-like continuum has the fixed-point property for deformations. This was accomplished by showing that every map of a tree-like continuum $M$ that sends each arc-component of $M$ into itself has a fixed point. The collection of arc-components is a partition of a continuum. Hence a map that sends each arc-component into itself is sometimes called an arc-component-preserving map. Bellamy’s continuum is indecomposable and its arc-components are composants. The collection of composants of an indecomposable continuum is a partition that is refined by the collection of arc-components. The difference between these two partitions is sometimes extreme. Each arc-component consists of one point in P. Minc’s example [11] of a hereditarily indecomposable tree-like continuum without the fixed-point property. Must every composant-preserving map of an indecomposable tree-like continuum $M$ have a fixed point? We prove the answer is yes if $M$ is $k$-junctioned.

1. Preliminaries

A space $X$ has the fixed-point property if for each map $f$ of $X$ into $X$ there is a point $x$ of $X$ such that $f(x) = x$.

A chain is a finite collection $A = \{A_1, A_2, ..., A_n\}$ of open sets such that $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The elements $A_1$ and $A_n$ are called end links of $A$. Each element of $A \setminus \{A_1, A_n\}$ is called an interior link of $A$. If $n > 2$ and $A_1$ also intersects $A_n$, the collection $A$ is called a circular chain.

A collection $\mathcal{E}$ of sets is coherent if for each nonempty proper subcollection $\mathcal{F}$ of $\mathcal{E}$ there is an element of $\mathcal{F}$ that intersects an element of $\mathcal{E} \setminus \mathcal{F}$.

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A finite coherent collection \( \mathcal{T} \) of open sets is a tree chain if no three elements of \( \mathcal{T} \) have a point in common and no subcollection of \( \mathcal{T} \) is a circular chain. An element of \( \mathcal{T} \) that intersects more than two other elements of \( \mathcal{T} \) is called a junction link of \( \mathcal{T} \). Let \( \mathcal{J} \) be the collection of junction links of \( \mathcal{T} \). Each maximal chain in \( \mathcal{T} \setminus \mathcal{J} \) is called a chain section of \( \mathcal{T} \).

A continuum is indecomposable if it is not the union of two proper subcontinua. A continuum is hereditarily indecomposable if each of its subcontinua is indecomposable.

Let \( x \) be a point of a continuum \( M \). The \( x \)-arc-component of \( M \) is the union of \( \{x\} \) and all arcs in \( M \) that contain \( x \). The \( x \)-composant of \( M \) is the union of all proper subcontinua of \( M \) that contain \( x \). If \( M \) is indecomposable, then \( M \) is the union of uncountably many dense disjoint composants.

A tree chain with mesh less than a number \( \epsilon \) is called an \( \epsilon \)-tree-chain.

A continuum \( M \) is tree-like if for each positive number \( \epsilon \) there is an \( \epsilon \)-tree-chain covering \( M \).

A tree-like continuum \( M \) is \( k \)-junctioned if \( k \) is the least integer such that for every positive number \( \epsilon \) there is an \( \epsilon \)-tree-chain covering \( M \) with \( k \) junction links, \(3, 5\).

A tree-like continuum \( M \) has the chain-section property if for each positive number \( \epsilon \) there is an \( \epsilon \)-tree-chain \( \mathcal{T} \) covering \( M \) that has a chain section \( S \) such that each point of \( M \) is within the distance \( \epsilon \) of an element of \( S \).

2. Results

**Theorem 2.1.** Suppose that \( f \) is a map of an indecomposable tree-like continuum \( M \) that sends each composant of \( M \) into itself. Suppose that \( M \) has the chain-section property. Then \( f \) has a fixed point.

**Proof.** Assume that \( f \) moves each point of \( M \). Let \( \rho \) be a metric on \( M \). Let \( \epsilon \) be a positive number such that \( \rho(x, f(x)) > \epsilon \) for each point \( x \) of \( M \).

Let \( \{ \Sigma_i : i = 1, 2, ... \} \) be a countable open base for \( M \). For each positive integer \( i \), let \( \Omega_i = \{ x \in M : x \) and \( f(x) \) belong to a continuum in \( M \setminus \Sigma_i \} \). Each \( \Omega_i \) is a closed subset of \( M \) [12, Th.58, p.23] and \( M = \bigcup \{ \Omega_i : i = 1, 2, ... \} \) since \( f \) sends each composant into itself. By the Baire Category Theorem, there is an integer \( j \) such that \( \Omega_j \) contains a nonempty open subset \( E \) of \( M \). Let \( F \) be a nonempty open subset of \( M \) whose closure is contained in \( E \).

Since \( M \) is indecomposable, the open set \( \Sigma_j \) cannot be contained in any single component of \( M \setminus F \). Hence \( M \setminus F \) is the union of two separated sets \( V \) and \( W \) that intersect \( \Sigma_j \) [12, Th. 44, p. 15]. Let \( v \) and \( w \) be points of \( V \cap \Sigma_j \) and \( W \cap \Sigma_j \), respectively. Let \( \epsilon_1 \) be a positive number less than \( \epsilon \), \( \rho(V, W), \frac{1}{\epsilon} \rho(\{v, w\}, M \setminus \Sigma_j) \), and \( \rho(F, M \setminus E) \).

Let \( T_1 \) be an \( \epsilon_1 \)-tree-chain \( \mathcal{T}_1 \) covering \( M \) with a chain section \( S_1 \) such that each point of \( M \) is within \( \epsilon_1 \) of an element of \( S_1 \). Let \( A = \{ A_1, A_2, ..., A_{n_1} \} \) be a chain consisting of interior links of \( S_1 \) such that \( \rho(v, A_1) < \epsilon_1 \) and \( \rho(w, A_{n_1}) < \epsilon_1 \). Note that \( A_1 \subset V \cap \Sigma_j \) and \( A_{n_1} \subset W \cap \Sigma_j \). Since \( F \) separates \( A_1 \) from \( A_{n_1} \) in \( M \), there is an interior link \( A_{i_1} \) of \( A \) that intersects \( F \). Note that \( A_{i_1} \subset E \subset \Omega_j \).

By the uniform continuity of \( f \), there is a positive number \( \delta \) such that for each subset \( X \) of \( M \) having diameter less than \( \delta \), both \( X \) and \( f(X) \) are contained in elements of \( T_1 \).
Let \( \hat{v} \) and \( \hat{w} \) be points of \( A_1 \) and \( A_{n_1} \), respectively.

Let \( \epsilon_2 \) be a positive number less than \( \epsilon_1, \delta, \frac{1}{2} \rho(\hat{v}, M \setminus A_1) \), and \( \frac{1}{3} \rho(\hat{w}, M \setminus A_{n_1}) \).

Let \( T_2 \) be an \( \epsilon_2 \)-tree-chain covering \( M \) that has a chain section \( S_2 \) such that each point of \( M \) is within \( \epsilon_2 \) of an element of \( S_2 \). Since no subcollection of \( T_1 \) is a circular chain, there exists a subchain \( B = \{B_1, B_2, ..., B_{n_2}\} \) of \( S_2 \) such that \( B \) runs through \( A \) (that is, \( B_1 \subset A_1, B_{n_2} \subset A_{n_1} \), and \( \bigcup B \subset \bigcup A \)). Let \( B_{i_2} \) be the first link of \( B \) contained in \( A_i \), and let \( B_{j_2} \) be the last link of \( B \) contained in \( A_j \).

Since \( M \) is indecomposable, \( M \setminus B_1 \) is the union of separated sets \( Y \) and \( Z \) that intersect \( B_{n_2} \). Let \( y \) and \( z \) be points of \( Y \cap B_{n_2} \) and \( Z \cap B_{n_2} \), respectively. Let \( \epsilon_3 \) be a positive number less than \( \epsilon_2, \rho(Y, Z) \), and \( \frac{1}{4} \rho(\{y, z\}, M \setminus B_{n_2}) \).

Let \( T_2 \) be an \( \epsilon_3 \)-tree-chain covering \( M \) that refines \( T_2 \) and has a chain section \( S_3 \) such that each point of \( M \) is within \( \epsilon_3 \) of an element of \( S_3 \). Let \( C = \{C_1, C_2, ..., C_{n_3}\} \) be a chain of interior links of \( S_3 \) such that \( C_1 \subset Y \cap B_{n_2} \) and \( C_{n_3} \subset Z \cap B_{n_2} \). Since \( T_2 \) does not contain a circular chain and \( B_1 \) separates \( Y \) from \( Z \) and hence \( C_1 \) from \( C_{n_3} \) in \( M \), there exist two subchains of \( C \) such that one subchain runs through the chain \( \{Y \cap B_{n_2}, Y \cap B_{n_2-1}, ..., Y \cap B_2, B_1\} \) and is followed by the second that runs through \( \{B_1, Z \cap B_2, ..., Z \cap B_{n_2}\} \).

Since \( C \) is comprised of interior links, the collection \( T_3 \setminus C \) consists of two coherent subcollections \( D \) and \( E \) such that \( \bigcup D \cap \bigcup E = \emptyset \), \( C_1 \cap \bigcup D \neq \emptyset \), and \( C_{n_3} \cap \bigcup E \neq \emptyset \).

Let \( C_0 = \bigcup D \) and \( C_{n_3+1} = \bigcup E \). Let \( F \) denote the chain \( \{C_0, C_1, ..., C_{n_3}, C_{n_3+1}\} \).

Let \( J \) be a subcontinuum of \( M \) in \( \bigcup C \) that intersects both \( C_1 \) and \( C_{n_3} \) [12, Th. 44, p. 15].

Let \( K = \{x \in J : \text{a link of } F \text{ containing } x \text{ follows a link of } F \text{ containing } f(x)\} \).

Let \( L = \{x \in J : \text{a link of } F \text{ containing } x \text{ precedes a link of } F \text{ containing } f(x)\} \).

Since \( \epsilon_3 < \epsilon \), \( J \subset \bigcup C \), and \( f \) is continuous, it follows that \( K \) and \( L \) are disjoint closed subsets of \( J \) and \( J = K \cup L \) [8]. Note that since \( J \) is connected, either \( K \) or \( L \) must be empty.

Since \( B_{i_2} \subset \Omega_j \) and \( A_i \cup A_{nj} \subset \Sigma_j \), for each point \( x \) of \( B_{i_2} \) there is a continuum in \( M \setminus (A_i \cup A_{nj}) \) that contains \( x \) and \( f(x) \). Consequently, since \( \epsilon_2 < \delta \), it follows that \( f(B_{i_2}) \) is contained in an interior link \( A_{j_2} \) of \( A \).

Furthermore, \( i_1 \neq j_1 \) since \( \epsilon_1 < \epsilon \).

Note \( i_1 > j_1 \). To see this, suppose \( i_1 < j_1 \). Let \( C_\alpha \) be the last link of \( F \) that is contained in \( Y \cap B_{i_2} \). Let \( C_\gamma \) be the first link of \( F \) that follows \( C_\alpha \) and lies in \( B_1 \). Note \( \bigcup \{C_\alpha, C_{\alpha+1}, ..., C_\gamma\} \subset \{B_1, B_2, ..., B_{i_2}\} \). Since \( B_{i_2} \) is the first link of \( B \) in \( A_i \), it follows that \( A_{j_1} \cap \bigcup \{C_\alpha, C_{\alpha+1}, ..., C_\gamma\} = \emptyset \). Note \( C_1 \cup C_\gamma \subset \Sigma_j \), \( C_\alpha \subset \Omega_j \), and \( f(C_\alpha) \subset A_{j_1} \). Let \( a \) be a point of \( C_\alpha \). Then \( f(a) \in \bigcup \{C_1, C_2, ..., C_{n_3}\} \), for otherwise, every subcontinuum of \( M \) that contains \( a \) and \( f(a) \) intersects \( \Sigma_j \), which is impossible since \( a \in \Omega_j \). Therefore \( a \in K \). Hence \( K \) is not empty.

See Figure 1. Larger black/white and colored pictures of Figure 1 are posted on http://webpages.csus.edu/~hagopian.

Let \( C_\beta \) be the first link of \( F \) that lies in \( Z \cap B_{i_2} \). Let \( b \) be a point of \( C_\beta \). It follows from the argument above that \( f(b) \in \bigcup \{C_\beta, C_{\beta+1}, ..., C_{n_3}\} \). Therefore \( b \in L \). Hence \( K \) and \( L \) are nonempty disjoint closed sets whose union is \( J \), and this contradicts the connectivity of \( J \). Thus \( i_1 > j_1 \).

Since \( B_{j_2} \) is the last link of \( B \) in \( A_{i_1} \), it follows from the same argument that \( f(B_{j_2}) \) is in an interior link \( A_{k_1} \) of \( A \) and \( i_1 < k_1 \).
Figure 1

The collection \( T_1 \setminus A \) consists of two coherent subcollections \( G \) and \( H \) such that 
\[ \bigcup G \cap \bigcup H = \emptyset, \quad A_1 \cap \bigcup G \neq \emptyset, \quad \text{and} \quad A_{n_1} \cap \bigcup H \neq \emptyset. \]
Let \( A_0 = \bigcup G \) and \( A_{n_1 + 1} = \bigcup H \). Let \( K \) denote the chain \( \{ A_0, A_1, ..., A_{n_1}, A_{n_1 + 1} \} \).
Let \( L \) denote the chain \( \{ B_{i_2}, B_{i_2 + 1}, ..., B_{j_2} \} \). Note \( L = \{ B_{i_2}, B_{j_2} \} \) in Figure 1. However, \( B \) may be folded in \( A \), and each interior link of \( A \) may contain a link of \( L \).
Let \( P \) be a subcontinuum of \( M \) in \( \bigcup L \) that intersects both \( B_{i_2} \) and \( B_{j_2} \). Let \( Q = \{ x \in P : \text{a link of } K \text{ containing } x \text{ follows a link of } K \text{ containing } f(x) \} \).
Let \( R = \{ x \in P : \text{a link of } K \text{ containing } x \text{ precedes a link of } K \text{ containing } f(x) \} \).
Note \( B_{i_2} \cup B_{j_2} \subset A_{i_1} \), \( f(B_{i_2}) \subset A_{j_1} \), \( f(B_{j_2}) \subset A_{k_1} \), and \( j_1 < i_1 < k_1 \). Thus neither \( Q \) nor \( R \) is empty. Since \( \epsilon > \epsilon_1 \), \( P \subset \bigcup A \), and \( f \) is continuous, it follows that \( Q \) and \( R \) are disjoint closed subsets of \( P \) and \( P = Q \cup R \). This contradiction of the connectivity of \( P \) completes the proof. \( \square \)

**Theorem 2.2.** If \( M \) is an indecomposable \( k \)-junctioned tree-like continuum, then \( M \) has the chain-section property.

**Proof.** Let \( \epsilon \) be a given positive number. Let \( U \) be an \( \epsilon \)-tree-chain covering \( M \). Since the composants of \( M \) are dense and disjoint, there exists a collection \( V \) consisting of \( k+1 \) disjoint subcontinua of \( M \) such that each element of \( V \) intersects each element of \( U \). Let \( \delta \) be a positive number less than \( \epsilon \) and the distance between each two elements of \( V \). Let \( T \) be a \( \delta \)-tree-chain covering \( M \) that has only \( k \) junction links. Note that no junction link of \( T \) intersects more than one element of \( V \). Since each element of \( V \) is connected, there is a chain section \( S \) of \( T \) that covers an element...
Theorem 2.3. Suppose that $M$ is an indecomposable $k$-junctioned tree-like continuum. Then every composant-preserving map of $M$ has a fixed point.

Proof. Theorem 2.3 follows directly from Theorems 2.1 and 2.2.

3. Questions

Theorem 2.3 provides partial answers to the following open questions.

Question 1. Must every composant-preserving map of an indecomposable tree-like continuum have a fixed point?

Question 2. Does every $k$-junctioned tree-like continuum have the fixed-point property?

Does every plane continuum that does not separate the plane have the fixed-point property? This is the unsolved classic plane fixed-point problem. It is not known if every tree-like continuum in the plane has the fixed-point property. Questions 1 and 2 are also open for tree-like plane continua. For more information and unsolved fixed-point problems, see [2], [7], and [9].

References


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