A NUCLEAR FRÉCHET SPACE CONSISTING OF $C^\infty$-FUNCTIONS AND FAILING THE BOUNDED APPROXIMATION PROPERTY

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(Communicated by Nigel J. Kalton)

Abstract. An easy and transparent example is given of a nuclear Fréchet space failing the bounded approximation property and consisting of $C^\infty$-functions on a subset of $\mathbb{R}^3$.

The problem of Grothendieck whether every nuclear Fréchet space has the bounded approximation property was open for quite a while. The first counterexample is due to Dubinsky [1, 2]. Another much simpler example was given by the author in [7]. Other examples are due to Floret [4] and Moscatelli [6]. In Dubinsky-Vogt [3] it was shown that every infinite dimensional nuclear Fréchet space not isomorphic to $K^N$, where $K = \mathbb{R}$ or $\mathbb{C}$ is the scalar field, has a quotient failing the bounded approximation property.

For a nuclear Fréchet space $E$ the bounded approximation property means the existence of a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of continuous finite rank operators in $E$ so that $\varphi_n x \to x$ for all $x \in E$.

In [7] the following criterion was established for the bounded approximation property. In [3] it was shown that it is equivalent to $E$ being countably normed. The observation that a non-countably normed Fréchet space admitting a continuous norm will fail the bounded approximation property is due to Pelczyński.

Criterion 1. If $E$ has the bounded approximation property and a continuous norm, then the following holds.

\begin{itemize}
  \item There exists $p_0$ such that for every $p \geq p_0$ we have a $q \geq p$ with the following property:
    \begin{itemize}
      \item Every sequence in $E$ which is Cauchy with respect to $\| \|_q$ and convergent to $0$ with respect to $\| \|_{p_0}$ even converges to $0$ with respect to $\| \|_p$.
    \end{itemize}

  \end{itemize}

Here $\| \|_0 \leq \| \|_1 \leq \cdots$ denotes a fundamental system of continuous seminorms on $E$.

We use Criterion 1 to develop a scheme, how to construct nuclear Fréchet spaces without the bounded approximation property.

Let $F$ and $G$ be nuclear Fréchet spaces. Let again $\| \|_0 \leq \| \|_1 \leq \cdots$ denote fundamental systems of continuous seminorms on $F$ and $G$ respectively. We assume that $G$ admits a continuous norm, which we may assume to be $\| \|_0$. By $F_k$, $k \in \mathbb{N}_0$ we denote the local Banach space with respect to $\| \|_k$, i.e. the completion of

Received by the editors January 11, 2009, and, in revised form, August 13, 2009.

2010 Mathematics Subject Classification. Primary 46A04; Secondary 46A11.

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Let $A : F \supset D(A) \to G$ be a injective closed linear map; i.e. the graph $\Gamma(A)$ is closed in $F \times G$. We assume that for any $p_0$ there is $p > p_0$ such that for any $q \geq p$ there is a sequence $(x_n)_n$ in $F$ with the following properties:

1. $x_n \to 0$ with respect to $\| \cdot \|_{p_0}$;
2. $Ax_n \to 0$ with respect to $\| \cdot \|_q$;
3. there is $x \in F_q$ with $j_q^n x \not= 0$, so that $j_q^n x_n \to x$ in $F_q$.

**Lemma 2.** Then $E := D(A)$ with graph topology given by the norms $\| x \|_k + \| Ax \|_k$, $k \in \mathbb{N}_0$, is a nuclear Fréchet space without the bounded approximation property.

**Proof.** $E$ is isomorphic to a closed subspace of $F \times G$, hence a nuclear Fréchet space. Then, because of (2) and (3) the sequence $(x_n, Ax_n)$, $n \in \mathbb{N}$, is a $\| \cdot \|_q$-Cauchy sequence. Because of (1) and (2) it is a $\| \cdot \|_{p_0}$-null sequence. And, because of (2) and (3), it does not converge to 0 with respect to $\| \cdot \|_p$. Since $A$ is injective, $E$ is a normed space. □

We now proceed to the construction of a concrete example. We identify $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ with the variables $(z, t)$. We apply the lemma to

1. $\mathcal{M} = \{(z, t) : |z| \leq 1, \; |t| < 1\}$,
2. $F = \{f \in C^\infty(\mathcal{M}) : f^{(\alpha)}(0, t) = 0, \; \text{for all } t \; \text{and } \alpha\}$,
3. $G = C^\infty(\mathcal{M})$,
4. $A = \frac{\partial f}{\partial z}$ with $D(A) = \{f \in F : \frac{\partial f}{\partial z} \in G\}$.

Here we identified $C^\infty(\mathcal{M})$ with a linear subspace of $C^\infty(\mathcal{M})$. For the nuclearity of $F$ and $G$ we refer to e.g. [5, 28.6 and 28.9] and remark that, by Whitney’s extension theorem, $C^\infty(\mathcal{M})$ is a quotient of $C^\infty(\Omega)$, where $\Omega = \{(z, t) \in \mathbb{R}^3 : |t| < 1\}$ is an open subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. $A$ is injective since $\ker A$ consists of functions, which are holomorphic in $z$ for $|z| < 1$ and flat in $(0, t)$ for all $t$.

**Theorem 3.** The space $E = \{f \in F : \frac{\partial f}{\partial z} \in G\}$ with its natural Fréchet topology is a nuclear Fréchet space without the bounded approximation property.

**Proof.** We may choose the fundamental system of norms

$\| f \|_k = \sup \{|f^{(\alpha)}(z, t)| : |\alpha| \leq k, \; |z| \leq 1, \; |t| \leq t_k\}$

on $F$, where $t_k = \frac{k}{k+1}$, and

$\| f \|_k = \sup \{|f^{(\alpha)}(z, t)| : |\alpha| \leq k, \; (z, t) \in \mathcal{M}\}$

on $G$. Both spaces are nuclear and clearly $A$ is closed and injective.

Let $p_0$ be given. We put $p = p_0 + 1$ and assume $q \geq p$.

We choose $\varphi \in \mathscr{D}(\mathbb{D})$, $\mathbb{D}$ the unit disc, such that $\varphi \equiv 1$ in a neighborhood of $0$, and $\psi \in \mathscr{D}[t_{p_0}, 1]$ with $\psi(t_p) \not= 0$. We define

$f_n(z, t) = \psi(t)(1 - \varphi(nz))z^{q+2}$.

1. $\| f_n \|_{p_0} = 0$ for all $n$.
2. We have with a suitable constant $C$:

$|f_{n, z}|_q = |\psi(t)\varphi(nz)z^{q+2}|_q \leq C n^{-1}$.

That means $\| Af_n \|_q \to 0$. 
3. With a suitable constant $C$, we have
\[ \|f_n(z,t) - \psi(t) z^{q+2}\|_q = \|\psi(t) \varphi(nz) z^{q+2}\|_q \leq C n^{-2}. \]
That means $f_n \to \psi z^{q+2} \in F_q$ with respect to $\| \cdot \|_q$ where
\[ \|\psi(t) z^{q+2}\|_p \neq 0. \]
The result follows from Lemma 2.

Let us finally remark that in [8] there was given an easy example of a nuclear Fréchet space consisting of $C^\infty$-functions which has no basis, namely the space
\[ E = \{ f \in C^\infty(\mathbb{R}^2) : f|_M \in \mathcal{S}(M) \} \]
with its natural Fréchet topology, where
\[ M = \{(x,y) \in \mathbb{R}^2 : x \geq 0, |\sin y| \leq e^{1-\frac{1}{x}} \}. \]
Here $\mathcal{S}(M)$ denotes the space of all $C^\infty$-functions on $M$ which are rapidly decreasing for $|x| \to \infty$ with all their derivatives and we set $\exp(-\frac{1}{x}) = 0$. It might be useful to remark that the existence of a basis implies the bounded approximation property. So the present example, being likewise easy, provides much sharper properties.

REFERENCES

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