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A NUCLEAR FRÉCHET SPACE CONSISTING OF C^∞-FUNCTIONS AND FAILING THE BOUNDED APPROXIMATION PROPERTY

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Abstract. An easy and transparent example is given of a nuclear Fréchet space failing the bounded approximation property and consisting of C^∞-functions on a subset of $\mathbb{R}^3$.

The problem of Grothendieck whether every nuclear Fréchet space has the bounded approximation property was open for quite a while. The first counterexample is due to Dubinsky [1, 2]. Another much simpler example was given by the author in [7]. Other examples are due to Floret [4] and Moscatelli [6]. In Dubinsky-Vogt [3] it was shown that every infinite dimensional nuclear Fréchet space not isomorphic to $\mathbb{K}^{\mathbb{N}}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the scalar field, has a quotient failing the bounded approximation property.

For a nuclear Fréchet space E the bounded approximation property means the existence of a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of continuous finite rank operators in E so that $\varphi_n x \longrightarrow x$ for all $x \in E$.

In [7] the following criterion was established for the bounded approximation property. In [3] it was shown that it is equivalent to E being countably normed. The observation that a non-countably normed Fréchet space admitting a continuous norm will fail the bounded approximation property is due to Pełczyński.

Criterion 1. If E has the bounded approximation property and a continuous norm, then the following holds.

($*$) There exists p_0 such that for every $p \geq p_0$ we have a $q \geq p$ with the following property:

Every sequence in E which is Cauchy with respect to $\| \|_q$ and convergent to 0 with respect to $\| \|_{p_0}$ even converges to 0 with respect to $\| \|_p$.

Here $\| \|_0 \leq \| \|_1 \leq \cdots$ denotes a fundamental system of continuous seminorms on E.

We use Criterion 1 to develop a scheme, how to construct nuclear Fréchet spaces without the bounded approximation property.

Let F and G be nuclear Fréchet spaces. Let again $\| \|_0 \leq \| \|_1 \leq \cdots$ denote fundamental systems of continuous seminorms on F and G respectively. We assume that G admits a continuous norm, which we may assume to be $\| \|_0$. By F_k, $k \in \mathbb{N}_0$ we denote the local Banach space with respect to $\| \|_k$, i.e. the completion of

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$(F/\ker\|\,\|_k, \|\,\|_k)$ and by $j^k_m : F_m \longrightarrow F_k$ for $m \geq k$, $j^k : F \longrightarrow F_k$ the natural linking maps (see [5, p. 294]).

Let $A : F \supset D(A) \longrightarrow G$ be a injective closed linear map; i.e. the graph $\Gamma(A)$ is closed in $F \times G$. We assume that for any p_0 there is $p > p_0$ such that for any $q \geq p$ there is a sequence $(x_n)_n$ in F with the following properties:

(1) $x_n \longrightarrow 0$ with respect to $\|\,\|_{p_0}$;
(2) $Ax_n \longrightarrow 0$ with respect to $\|\,\|_q$;
(3) there is $x \in F_q$ with $j^p_q x \neq 0$, so that $j^q x_n \longrightarrow x$ in F_q.

Lemma 2. *Then $E := D(A)$ with graph topology given by the norms $\|x\|_k + \|Ax\|_k$, $k \in \mathbb{N}_0$, is a nuclear Fréchet space without the bounded approximation property.*

Proof. E is isomorphic to a closed subspace of $F \times G$, hence a nuclear Fréchet space. Then, because of (2) and (3) the sequence (x_n, Ax_n), $n \in \mathbb{N}$, is a $\|\cdot\|_q$-Cauchy sequence. Because of (1) and (2) it is a $\|\cdot\|_{p_0}$-null sequence. And, because of (2) and (3), it does not converge to 0 with respect to $\|\cdot\|_p$. Since A is injective, E is a normed space. □

We now proceed to the construction of a concrete example. We identify $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ with the variables (z, t). We apply the lemma to

(1) $\mathbb{M} = \{(z,t) : |z| \leq 1,\ |t| < 1\}$,
(2) $F = \{f \in C^\infty(\mathbb{M}) : f^{(\alpha)}(0,t) = 0,\ \text{for all } t \text{ and } \alpha\}$,
(3) $G = C^\infty(\overline{\mathbb{M}})$,
(4) $A = \frac{\partial}{\partial \bar z}$ with $D(A) = \{f \in F : \frac{\partial f}{\partial \bar z} \in G\}$.

Here we identified $C^\infty(\overline{\mathbb{M}})$ with a linear subspace of $C^\infty(\mathbb{M})$. For the nuclearity of F and G we refer to e.g. [5, 28.6 and 28.9] and remark that, by Whitney's extension theorem, $C^\infty(\mathbb{M})$ is a quotient of $C^\infty(\Omega)$, where $\Omega = \{(z,t) \in \mathbb{R}^3 : |t| < 1\}$ is an open subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. A is injective since $\ker A$ consists of functions, which are holomorphic in z for $|z| < 1$ and flat in $(0,t)$ for all t.

Theorem 3. *The space $E = \{f \in F : \frac{\partial f}{\partial \bar z} \in G\}$ with its natural Fréchet topology is a nuclear Fréchet space without the bounded approximation property.*

Proof. We may choose the fundamental system of norms

$$\|f\|_k = \sup\{|f^{(\alpha)}(z,t)| : |\alpha| \leq k,\ |z| \leq 1,\ |t| \leq t_k\}$$

on F, where $t_k = \frac{k}{k+1}$, and

$$|f|_k = \sup\{|f^{(\alpha)}(z,t)| : |\alpha| \leq k,\ (z,t) \in \mathbb{M}\}$$

on G. Both spaces are nuclear and clearly A is closed and injective.

Let p_0 be given. We put $p = p_0 + 1$ and assume $q \geq p$.

We choose $\varphi \in \mathscr{D}(\mathbb{D})$, $\mathbb{D}$ the unit disc, such that $\varphi \equiv 1$ in a neighborhood of 0, and $\psi \in \mathscr{D}[t_{p_0}, 1[$ with $\psi(t_p) \neq 0$. We define

$$f_n(z,t) = \psi(t)(1 - \varphi(nz))\, z^{q+2}.$$

1. $\|f_n\|_{p_0} = 0$ for all n.
2. We have with a suitable constant C:

$$|f_{n,\bar z}|_q = |\psi(t)\varphi(nz)_{\bar z} z^{q+2}|_q \leq C\, n^{-1}.$$

That means $|Af_n|_q \longrightarrow 0$.

3. With a suitable constant C, we have

$$\|f_n(z,t) - \psi(t)\, z^{q+2}\|_q = \|\psi(t)\varphi(nz)z^{q+2}\|_q \le C\, n^{-2}.$$

That means $f_n \longrightarrow \psi\, z^{q+2} \in F_q$ with respect to $\|\cdot\|_q$ where

$$\|\psi(t)\, z^{q+2}\|_p \neq 0.$$

The result follows from Lemma 2. □

Let us finally remark that in [8] there was given an easy example of a nuclear Fréchet space consisting of C^∞-functions which has no basis, namely the space

$$E = \{f \in C^\infty(\mathbb{R}^2) \,:\, f|_M \in \mathscr{S}(M)\}$$

with its natural Fréchet topology, where

$$M = \{(x,y) \in \mathbb{R}^2 \,:\, x \ge 0,\, |\sin y| \le e^{1-\frac{1}{x}}\}.$$

Here $\mathscr{S}(M)$ denotes the space of all C^∞-functions on M which are rapidly decreasing for $|x| \longrightarrow \infty$ with all their derivatives and we set $\exp(-\frac{1}{0}) = 0$. It might be useful to remark that the existence of a basis implies the bounded approximation property. So the present example, being likewise easy, provides much sharper properties.

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