NOTE ON BETA ELEMENTS IN HOMOTOPY, 
AND AN APPLICATION TO THE PRIME THREE CASE

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(Communicated by Brooke Shipley)

ABSTRACT. Let $S^0_{(p)}$ denote the sphere spectrum localized at an odd prime $p$. Then we have the first beta element $\beta_1 \in \pi_{2p-2-2}(S^0_{(p)})$, whose cofiber we denote by $W$. We also consider a generalized Smith-Toda spectrum $V_r$ characterized by $BP_*(V_r) = BP_*/(p, v_1^r)$. In this note, we show that an element of $\pi_*(V_r \wedge W)$ gives rise to a beta element of homotopy groups of spheres. As an application, we show the existence of $\beta_{9t+3}$ at the prime three to complete a conjecture of Ravenel’s: $\beta_s \in \pi_{16s-6}(S^0_{(3)})$ exists if and only if $s$ is not congruent to 4, 7 or 8 mod 9.

1. Introduction and Statements of Results

Let $BP$ denote the Brown-Peterson spectrum at an odd prime number $p$. Then, we have the Hopf algebra $BP \wedge BP$ over $BP_*$, where

$$BP_* = \pi_*(BP) = \mathbb{Z}_p[v_1, v_2, \ldots] \quad \text{and} \quad BP \wedge BP = BP_*(BP) = BP_*[t_1, t_2, \ldots]$$

for $v_i \in BP_{2p-2}$ and $t_i \in BP_{2p-3}$. This gives rise to the Adams-Novikov spectral sequence converging to homotopy groups $\pi_*(X)$ of a connective spectrum $X$ with $E_2$-term

$$E_2^{s,t}(X) = \text{Ext}^{s,t}_{BP \wedge BP}(BP_*, BP_*(X)).$$

We consider the sphere spectrum $S^0$, the modulo $p$ Moore spectrum $M$ and a cofiber $V_r$ of the map $\alpha^r : \Sigma^qM \to M$ for a positive integer $r$ and the Adams map $\alpha : \Sigma^qM \to M$, so that

$$S^0 \xrightarrow{p} S^0 \xrightarrow{i} M \xrightarrow{j} S^1 \quad \text{and} \quad \Sigma^qM \xrightarrow{\alpha^r} M \xrightarrow{r} V_r \xrightarrow{j_r} \Sigma^q+1M$$

are cofiber sequences. Here $q = 2p-2$ as usual. Suppose that $v_1^r v_2^s \in E_2^{s,t}(V_r)$ for integers $r > 0$, $s > 0$ and $t \geq 0$. Then, we define the beta element $\beta_{s/r-t}$ in the $E_2$-term by

$$\beta_{s/r-t} = \delta \delta_r(v_1^r v_2^s) \in E_2^{2(s+p+t-r),q}(S^0)$$

for the connecting homomorphisms $\delta_r : E_2^{s,t}(V_r) \to E_2^{s+1,t-r-q}(M)$ and $\delta : E_2^{s,t}(M) \to E_2^{s+1,t}(S^0)$ associated to the cofiber sequences (1.1). We note that if $r-t = r'-t'$, then $\beta_{s/r-t} = \beta_{s/r'-t'}$, and abbreviate $\beta_{s/1}$ to $\beta_s$. In this paper, we study which
of these elements survives to the homotopy groups $\pi_*(S^0)$ of spheres. For a prime greater than three, the following beta elements survive:

$$\beta_s \quad \text{for } s > 0 \text{ by L. Smith [2],}$$

$$\beta_{tp/r} \quad \text{for } t > 0 \text{ and } r \leq p \text{ with } (t, r) \neq (1, p) \text{ by S. Oka [4, 5],}$$

$$\beta_{tp^2/r} \quad \text{for } t > 0 \text{ and } r \leq 2p \text{ by S. Oka [6].}$$

At the prime three, $\beta_s$ survives if $s < 4$ by Toda [10], and does not if $s = 4$, and does if $s = 5$ by Oka [2]. Ravenel then conjectured that $\beta_s$ survives in the spectral sequence if and only if $s \equiv 0, 1, 2, 3, 5, 6 \mod 9$. In [8], we proved the ‘only if’ part. For the survivors, we have

$$\beta_s \quad \text{for } s > 0 \text{ with } s \equiv 0, 1, 2, 5, 6 \mod 9 \text{ by M. Behrens and S. Pemmaraju [1].}$$

We note that the element $\beta_1 \in \pi_{pq-2}(S^0)$ is given by

$$\beta_1 = j j_1[\beta_1] i$$

for the maps $i$, $j$, $i_1$ and $j_1$ are the maps given in (1.1). Here, $[\beta_1]$ denotes $\beta_1$ for the self-map $\beta \in [V(1), V(1)]_{(p+1)q}$ ($V(1) = V_1$) constructed by Smith [9] at a prime greater than three, and the generator of the homotopy group $[M, V(1)]_{16}$ given in [11] at the prime three. We define $W$ by the cofiber sequence

$$(1.3) \quad S^{pq-2} \xrightarrow{\beta_1} S^0 \xrightarrow{t} W \xrightarrow{k} S^{pq-1}.$$ 

Then we have our main theorem:

**Theorem 1.4.** Suppose that there is an element $B_s \in \pi_{s(p+1)q}(V_r \wedge W)$ detected by $v_2^s \in E_2^{0,s(p+1)q}(V_r \wedge W)$. Then, the beta element $\beta_{s/l}$ for $0 < l \leq r - 2$ survives to $\pi_{(sp+s-l)q-2}(S^0)$.

As an example, we have

**Lemma 1.5.** At an odd prime, there exists $B_{tp} \in \pi_{tp(p+1)q}(V_p \wedge W)$ for $t > 0$ detected by $v_2^{tp} \in E_2^{0,tp(p+1)q}(V_p \wedge W)$.

**Corollary 1.6.** The beta elements $\beta_{tp/l}$ for $t > 0$ and $0 < l \leq p - 2$ survive.

This corollary shows that $\beta_{31}$ survives at the prime three and completes a proof of the conjecture.

2. **Proofs of results**

Applying the $BP_*$-homology to the first cofiber sequence of (1.1), we obtain

$$BP_*(M) = BP_*/(p).$$

We observe the $E_2$-term $E_2^t(X)$ of the Adams-Novikov spectral sequence as the cohomology of the reduced cobar complex $\Omega_0^{BP_*/BP}BP_*(X)$. Then, we have a vanishing line for a $(-1)$-connected spectrum $X$:

$$(2.1) \quad E_2^t(X) = 0 \quad \text{if} \quad t < sq.$$ 

The structure maps of the Hopf algebroid $BP_*/BP$ act on generators by

$$\eta_2(v_1) = v_1 + pt_1,$$

$$(2.2) \quad \eta_2(v_2) \equiv v_2 + v_1 t_p^2 - v_1 t_1 \mod (p), \quad \text{and} \quad \Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1$$
Lemma 2.5. Suppose that \( t - s < (p^2 + p + 1)q \). In this range, the Adams-Novikov \( E_2 \)-term \( E_2^{s,t}(M) \) is a subquotient of \( \mathbb{Z}/p[v_1, v_2] \otimes \{ h_0, h_1, h_2, g_0, k_0, k_0 h_0, h_0 h_2 \} \otimes P(b_0, b_1, b_2) \). Here the bi-degrees of the generators are:

\[
|b_0| = (2, p^2q), \quad |b_1| = (2, p^2q), \quad |b_2| = 0.
\]

Proof. We have short exact sequences

\[
0 \rightarrow E^*_p/(p) \xrightarrow{v_1} E^*_p/(p) \rightarrow E^*_p/(p, v_1) \rightarrow 0\]

and

\[
0 \rightarrow E^*_p/(p, v_1) \xrightarrow{v_2} E^*_p/(p, v_1, v_2) \rightarrow 0,
\]

which give rise to Bockstein spectral sequences converging to the Adams-Novikov \( E_2 \)-terms \( E_2^*(V_1) \) with \( E_1 \)-terms \( E_1^*(V_1) \) and \( Ext^*_{BP*BP}(BP^*, BP_*/(p, v_1, v_2)) \), respectively. In our range, we have

\[
Ext^*_{BP*BP}(BP^*, BP_*/(p, v_1, v_2)) = Ext^*_p(\mathbb{Z}/p, \mathbb{Z}/p)
\]

for the subalgebra \( P \) of the Steenrod algebra generated by the reduced power operations. Thus, \( E_2^*(M) \) is a subquotient of \( \mathbb{Z}/p[v_1, v_2] \otimes Ext^*_p(\mathbb{Z}/p, \mathbb{Z}/p) \). We now get the structure of \( Ext^*_p(\mathbb{Z}/p, \mathbb{Z}/p) \) from [12].

Corollary 2.6. In our range, we have a vanishing line: \( E_2^{2s+r,tq}(V) = 0 \) for \( V = M, V_r \), if \( t < ps + \varepsilon \). Here, \( \varepsilon = 0, 1 \).

Lemma 2.7. Let \( \delta: E_2^*(M) \rightarrow E_2^{s+1}(S^0) \) be the connecting homomorphism associated with the first cofiber sequence in (1.1). Then, it is a derivation and

\[
\delta(v_1) = h_0, \quad \delta(h_2) = -b_1 \quad \text{and} \quad \delta(b_0) = 0.
\]

Proof. Note that \( h_i \) and \( b_i \) are represented by cocycles \( t_i^t \) and \( \sum_{k=1}^{p-1} \frac{1}{p}(p^k - 1) t_{i}^p - t_i^k \) of the cobar complex. By (2.2), we see that the differential \( d \) acts on \( v_1 \) and \( t_i^p \) as \( d(v_1) = pt_1 \) and \( d(t_i^p) = -pb_{i-1} \) for \( i > 0 \). The lemma now follows from the definition of the connecting homomorphism.

The cofiber sequence (1.3) induces a split short exact sequence

\[
0 \rightarrow E_2^{s,t}(V) \xrightarrow{\kappa} E_2^{s,t}(V \wedge W) \xrightarrow{\kappa} E_2^{s-t, pq+1}(V) \rightarrow 0
\]

of \( E_2 \)-terms for \( V = M \) and \( V_r \), and so

\[
E_2^s(V \wedge W) = E_2^s(V) \oplus gE_2^s(V),
\]

where \( g \) denotes a generator of degree \( pq - 1 \) such that \( \kappa(xg) = x \). Since the \( E_3 \)-term is a homology of \( E_2 \)-terms and \( d_2(g) = \beta \) for the element \( \beta \) in (1.2), we have the long exact sequence

\[
E_3^{s,t}(M) \xrightarrow{\partial} E_3^{s+2, t+pq}(M) \xrightarrow{\iota_*} E_3^{s+2, t+pq}(M \wedge W) \xrightarrow{\kappa_*} E_3^{s+2, t+1}(M)
\]

(2.8)
with the connecting homomorphism \( \partial \) given by \( \partial(x) = x\beta_1 \).

**Lemma 2.9.** The element \( v_2^p \in E_2^0(V_p \wedge W) \) in (2.3) survives to an element \( B_p \in \pi_{p(p+1)q}(V_p \wedge W) \).

**Proof.** Consider the cofiber sequence (1.1) with \( r = p \). In the Adams-Novikov spectral sequence for computing \( \pi_*(S^0) \), we have the Toda differential \( d_{p+1}(b_1) = h_0b_0^p \in E_2^{q+3,(p^2+1)q}(S^0) \) up to a nonzero scalar. By Lemma 2.5, \( E_2^{q+2,(p^2+1)q}(M) \) is a subquotient of \( \{v_1b_0^p\} \). Since \( d_{q+1}(h_2) = v_1b_0^p \in E_2^{q+2,(p^2+1)q}(M) \) up to a nonzero scalar by Lemma 2.7. Note that \( \beta_1 = b_0 \). In the exact sequence (2.8), \( v_1b_0^p = \partial(v_1b_0^{p-1}) \), and so \( d_{q+1}(h_2) = 0 \) in \( E_3^{q+2,(p^2+1)q}(M \wedge W) \). Besides, Corollary 2.6 shows that \( E_2^{q+2,(p^2+s)q}(M) = 0 \) for \( s > 1 \), and we see that \( \iota_4(h_2) \in E_2^{1,p^2q}(M \wedge W) \) is a permanent cycle, which detects an element \( \beta_p/p \in \pi_{p^2q-1}(M \wedge W) \). Send it by \( \alpha^p \) in (1.1). The element \( \alpha^p\beta_p/p \in \pi_{p(p^2+1)q-1}(M \wedge W) \) is detected by an element of \( E_2^{q+1,(p^2+p+1)q}(M \wedge W) \), since the \( E_2 \)-term \( E_2^{q+1,(p^2+p+1)q}(M) = E_2^{q+1,(p^2+p+1)q}(M \wedge W) \) for \( s > 1 \) is zero by Corollary 2.6. The \( E_2 \)-term \( E_2^{q+1,(p^2+p+1)q}(M) \) for \( s = 1 \) is a subquotient of \( h_0b_0^p \) by Lemma 2.8, and so the \( E_3 \)-term \( E_3^{q+1,(p^2+p+1)q}(M \wedge W) \) for \( s = 0 \) is pulled back to an element \( B_p \) under the map \( j_p \).

We call a spectrum \( R \) a ring spectrum if there exist a multiplicity \( \mu : R \wedge R \rightarrow R \) and an unit \( \iota : S^0 \rightarrow R \) such that \( \mu(\iota \wedge R) = 1_R = \mu(R \wedge \iota) : R \rightarrow R \). By [3 Ex. 2.9] and [3 Ex. 5.7], we have

(2.10) \( W \) and \( V_r \) for \( r > 1 \) are ring spectra.

In particular, the spectrum \( R_r = V_r \wedge W \) for \( r > 1 \) is a ring spectrum with multiplication \( m_r = (\mu_r \wedge \mu_w)(V_r \wedge T \wedge W) : R_r \wedge R_r = V_r \wedge W \wedge V_r \neq W \rightarrow V_r \wedge V_r \wedge W \wedge W \rightarrow V_r \wedge W = R_r \), where \( T \) denotes the switching map and \( \mu_r \) and \( \mu_w \) are the multiplications of \( V_r \) and \( W \), respectively.

**Proof of Lemma 1.1.** Since \( R_p = V_p \wedge W \) is a ring spectrum, we obtain a self-map \( [\beta_p] : R_p \rightarrow R_p \wedge R_p \) inducing \( v_2^p \) on \( BP_* \)-homology. Now put \( B_{tp} = [\beta_p]^{t-1}B_p \) to see the lemma.

We consider the element \( i_2\alpha^2i \in \pi_{2q}(V_r) \cong \pi_{2q}(M) = \mathbb{Z}/p\{\alpha^2\} \) for \( r > 2 \) and for the maps in (1.1), which is detected by the element \( v_1^2 \in E_2^0(V_r) \) by (2.4).

**Lemma 2.11.** Let \( r > 2 \). There exists an element \( \eta_r \in \pi_{(p+2)q-1}(V_r \wedge W) \) such that \( \kappa_r(\eta_r) = i_2\alpha^2i \in \pi_{2q}(V_r) \). Besides, it is detected by \( v_1^2g \in E_2^0(V_r \wedge W) = E_2^0(V_r) \oplus gE_2^0(V_r) \).

**Proof.** Putting \( \delta = ij \) for the maps \( i, j \) in (1.1), we have Yamamoto’s relation \( \alpha^2\delta = 2\alpha^2\alpha = \delta \alpha^2 \in [M, M]_{2q-1} \) (cf. [11]). We compute

\[
\alpha^2\beta_1 = \alpha^2\delta j_1[\beta_1]i = (\delta \alpha^2 + \alpha \delta \alpha)j_1[\beta_1]i = 0,
\]

since \( \alpha j_1 = 0 \) by (1.1). It follows that \( i_2\alpha^2i \in \pi_{2q}(V_r) \) is pulled back to an element \( \eta_r \in \pi_{(p+2)q-1}(V_r \wedge W) \) as desired. Since \( i_2\alpha^2i \) is detected by \( v_1^2 \in E_2^0(V_r) \), \( \eta_r \) is detected by the element \( v_1^2g \in E_2^0(V_r \wedge W) = E_2^0(V_r) \oplus gE_2^0(V_r) \).
Proof of Theorem 1.4 Consider the product $B_s \eta_r \in \pi_*(V_r \land W)$ for the element $\eta_r$ in Lemma 2.11. Then, it is detected by $v_1^2 v_2^2 g$, since $\eta_r$ induces a $BP_*BP$-comodule map $(\eta_r)_*: BP_*(V_r) \to BP_*(V_r \land W)$ such that $(\eta_r)_*(x) = v_1^2 xg$ and $B_s$ is detected by $v_s^2$. The map $\kappa_s: E_0^r(V_r \land W) \to E_1^r(V_r)$ assigns $v_1^2 v_2^2 g$ to $v_s^2 v_2^2$, which is a permanent cycle detected by $\kappa_s \eta_r$. Now putting $\beta_{s/l} = \alpha_{r,l}^{r-2} - \alpha_{r}^{-2-l} \kappa_s(\eta_r) = j \alpha_{r,l}^{r-2} - \alpha_{r}^{-2-l} \kappa_s(\eta_r)$ for $l < r - 2$, we see the theorem by the Geometric Boundary Theorem (cf. [7]). Here, $a^{r,k}$ denotes a map in the cofiber sequence $V_{r-k} \to V_r \xrightarrow{a^{r,k}} V_k$ obtained from applying the $3 \times 3$ Lemma to the cofiber sequences of $V_r$ for $r - k$, $r$ and $k$. We note that it satisfies $j \alpha_{r,l}^{r-2} = \alpha_{r,k}^{r-2} j_r$. □

References

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