ON GROMOV’S SCALAR CURVATURE CONJECTURE

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Abstract. We prove the Gromov conjecture on the macroscopic dimension of the universal covering of a closed spin manifold with a positive scalar curvature under the following assumptions on the fundamental group.

0.1. Theorem. Suppose that a discrete group $\pi$ has the following properties:
1. The Strong Novikov Conjecture holds for $\pi$.
2. The natural map $\text{per} : K_{0n}(B\pi) \to KO_n(B\pi)$ is injective.
Then the Gromov Macroscopic Dimension Conjecture holds true for spin $n$-manifolds $M$ with the fundamental groups $\pi_1(M)$ that contain $\pi$ as a finite index subgroup.

1. Introduction

In his study of manifolds with positive scalar curvature M. Gromov observed some large scale dimensional deficiency of their universal coverings: For an $n$-dimensional manifold $M$, its universal covering has to be at most $(n-2)$-dimensional from the macroscopic point of view. For example, the product of a closed $(n-2)$-manifold $N$ and the standard 2-sphere is an $n$-manifold $M = N \times S^2$ that admits a metric of positive scalar curvature (by making the 2-sphere small). The universal covering $\widetilde{M} = \widetilde{N} \times S^2$ looks like an $(n-2)$-dimensional space $\widetilde{N}$. Gromov predicted similar behavior for all manifolds with positive scalar curvature. He stated it in [G1] as the following.

1.1. Conjecture (Gromov). For every closed Riemannian $n$-manifold $(M,g)$ with a positive scalar curvature there is the inequality

$$\dim_{mc}(\widetilde{M},\tilde{g}) \leq n-2,$$

where $(\widetilde{M},\tilde{g})$ is the universal cover of $M$ with the pull-back metric.

Here $\dim_{mc}$ stands for the macroscopic dimension [G1]. The first time this conjecture was stated was in the famous “filling” paper [G2] in a different language.

In [GL] the conjecture was proved for 3-manifolds.

1.2. Definition. A map $f : X \to K$ of a metric space is called uniformly cobounded if there is $D > 0$ such that $\text{diam}(f^{-1}(y)) \leq D$.
A metric space $X$ has the macroscopic dimension $\dim_{mc} X \leq n$ if there is a uniformly cobounded proper continuous map $f : X \to K^n$ to an $n$-dimensional polyhedron.

In [G1] Gromov asked the following questions related to his conjecture which were stated in [B1], [B2] in the form of a conjecture:

1.3. Conjecture (C1). Let $(M, g)$ be a closed Riemannian $n$-manifold with torsion free fundamental group, and let $\widetilde{M}$ be the universal covering of $M$ with the pull-back metric. Suppose that $\dim_{mc} \widetilde{M} < n$. Then

(A) If $\dim_{mc} \widetilde{M} < n$, then $\dim_{mc} \widetilde{M} \leq n - 2$.

(B) If a classifying map for the universal covering map $f : M \to B\pi$ can be deformed to a map with $f(M) \subset B\pi^{(n-1)}$, then it can be deformed to a map with $f(M) \subset B\pi^{(n-2)}$.

The Conjecture C1 is proven for $n = 3$ by D. Bolotov in [B1]. In [B2] it was disproved for $n > 3$ by a counterexample. It turns out that Bolotov’s example does not admit a metric of positive scalar curvature [B3], and hence it does not affect the Gromov Conjecture 1.1.

Perhaps the most famous conjecture on manifolds of positive scalar curvature is

**The Gromov-Lawson Conjecture** [GL]. A closed spin $n$-manifold $M$ admits a metric of positive scalar curvature if and only if $f_*([M]_{KO}) = 0$ in $KO_n(B\pi)$, where $f : M \to B\pi$ is a classifying map for the universal covering of $M$ and $[M]_{KO}$ is the $KO$-theory fundamental class of $M$.

J. Rosenberg connected the Gromov-Lawson conjecture with the Novikov conjecture. Namely, he proved [R] that $\alpha f_*([M]_{KO}) = 0$ in $KO_n(C^*(\pi))$ in the presence of positive scalar curvature where $\alpha$ is the assembly map.

1.4. Conjecture (Strong Novikov Conjecture). The analytic assembly map

$$\alpha : KO_*(B\pi) \longrightarrow KO_*(C^*(\pi))$$

is a monomorphism.

Then Rosenberg and Stolz proved the Gromov-Lawson conjecture for manifolds with the fundamental group $\pi$ which satisfies the Strong Novikov conjecture and has the natural transformation map

$$\text{per} : ko_*(B\pi) \to KO_*(B\pi)$$

injective ([RS], Theorem 4.13).

The main goal of this paper is to prove the Gromov Conjecture 1.1 under the Rosenberg-Stolz conditions.

2. Connective spectra and $n$-connected complexes

We refer to the textbook [Ru] on the subject of spectra. We recall that for every spectrum $E$ there is a connective cover $e \to E$, i.e., the spectrum $e$ with the morphism $e \to E$ that induces the isomorphisms for $\pi_i(e) = \pi_i(E)$ for $i \geq 0$ and with $\pi_i(e) = 0$ for $i < 0$. By $KO$ we denote the spectrum for real $K$-theory, by $ko$ its connective cover, and by $\text{per} : ko \to KO$ the corresponding transformation (morphism of spectra). We will use both the old and new notation for an $E$-homology of a space $X$: old-fashioned $E_*(X)$ and modern $H_*(X; E)$. We recall
that $KO_n(pt) = \mathbb{Z}$ if $n = 0$ or $n = 4 \mod 8$, $KO_n(pt) = \mathbb{Z}_2$ if $n = 1$ or $n = 2 \mod 8$, and $KO_n(pt) = 0$ for all other values of $n$. By $S$ we denote the spherical spectrum. Note that for any ring spectrum $E$ there is a natural morphism $S \to E$ which leads to the natural transformation of the stable homotopy to $E$-homology $\pi^*_s(X) \to H_*(X;E)$.

2.1. Proposition. Let $X$ be an $(n-1)$-connected $(n+1)$-dimensional CW complex. Then $X$ is homotopy equivalent to the wedge of spheres of dimensions $n$ and $n+1$ together with the Moore spaces $M(\mathbb{Z}_m,n)$.

Proof. This is a partial case of the Minimal Cell Structure Theorem (see Proposition 4C.1 and Example 4C.2 in [Ha]).

2.2. Proposition. The natural transformation $\pi^*_s(pt) \to ko_*(pt)$ induces an isomorphism $\pi^*_n(K KO^{(n-2)}) \to ko_n(K KO^{(n-2)})$ for any CW complex $K$.

Proof. Since $\pi^*$ and $ko$ are both connective, it suffices to show that $\pi^*_n K KO^{(n+1)}/K KO^{(n-2)} \to ko_n K KO^{(n+1)}/K KO^{(n-2)}$ is an isomorphism. Consider the diagram generated by exact sequences of the pair $(K KO^{(n+1)}/K KO^{(n-2)}, K(KO)^{n-2})$

$$
\begin{array}{ccc}
\oplus \mathbb{Z} & \longrightarrow & \pi^*_n K KO^{(n)}/K KO^{(n-2)} \\
\downarrow & & \downarrow \\
\oplus \mathbb{Z} & \longrightarrow & ko_n K KO^{(n)}/K KO^{(n-2)}
\end{array}
$$

Since the left vertical arrow is an isomorphism and the right vertical arrow is an isomorphism of zero groups, it suffices to show that $\pi^*_n K KO^{(n)}/K KO^{(n-2)} \to ko_n K KO^{(n)}/K KO^{(n-2)}$ is an isomorphism.

Note that $\pi_n(S^k) \to ko_n(S^k)$ is an isomorphism for $k = n, n - 1$. In view of Proposition 2.1 it suffices to show that $\pi^*_n(N(\mathbb{Z}_m,n-1)) \to ko_n(N(\mathbb{Z}_m,n-1))$ is an isomorphism for any $m$ and $n$. This follows from the Five Lemma applied to the cofibration $S^{n-1} \to S^n \to M(\mathbb{Z}_m,n-1)$.

3. Inessential Manifolds

We recall the following definition, which is due to Gromov.

3.1. Definition. An $n$-manifold $M$ is called inessential if it admits a map $f : M \to K^{n-1}$ to an $(n-1)$-dimensional complex that induces an isomorphism on fundamental groups. Note that one can always take $K^{n-1}$ to be the $(n-1)$-skeleton $B\pi_1^{(n-1)}$ of the classifying space $B\pi$ of the fundamental group $\pi = \pi_1(M)$.

The following is well-known to experts.

3.2. Proposition. An orientable $n$-manifold $M$ is inessential if and only if $f_*(\lbrack M \rbrack) \in H_n(B\pi)$ is zero for a map $f : M \to B\pi_1(M)$ classifying the universal covering of $M$.

Proof. If $M$ admits a classifying map $f : M \to B\pi^{(n-1)}$, then clearly $f_*(\lbrack M \rbrack) = 0$.

Let $f_*(\lbrack M \rbrack) = 0$ for some map $f : M \to B\pi_1(M)$ that induces an isomorphism of the fundamental groups. Let $o_n(f) \in H^n(M;\pi_{n-1}(F))$ be the primary obstruction to deform $f$ to the $(n-1)$-dimensional skeleton $B\pi^{(n-1)}$, and let $o_n(1_{B\pi}) \in H^n(B\pi;\pi_{n-1}(F))$ be the primary obstruction to retraction of $B\pi$ to the
(n − 1)-skeleton. Here F denotes the homotopy fiber of the inclusion $Bπ^{(n−1)} → Bπ$ and $π_{n−1}(F)$ is considered as a π-module. Since $f_*$ induces an isomorphism of the fundamental groups, $f_* : H_0(M; π_{n−1}(F)) = π_{n−1}(F)_π → H_0(Bπ; π_{n−1}(F)) = π_{n−1}(F)_π$ is an isomorphism. Then $f_*([M] ∩ α_n(f)) = f_*([M]) ∩ α_n(1_{Bπ}) = 0$. By the Poincaré duality, $α_n(f) = 0$. □

3.3. Proposition. An orientable spin n-manifold M is inessential if $f_*([M]_{ko}) ∈ ko_n(Bπ)$ is zero for a map $f : M → Bπ$ classifying the universal covering.

Proof. By the definition, a ko-fundamental class $[M]_{ko}$ of an n-manifold goes to the generator of $ko_n(S^n) = Z$ under any degree-one map $g : M → S^n$.

There is a natural transformation $T$ of ko to the Eilenberg-MacLane spectrum $H(Z)$ which is induced by the 0-dimensional cohomology of the spectrum ko (see [Ru]). Since on the n-sphere this transformation induces an isomorphism of n-homology groups $ko_n(S^n) → H_n(S^n)$, it sends the ko-fundamental class $[M]_{ko}$ to the fundamental class $[M]$. From the commutativity of the diagram

$$
\begin{array}{ccc}
k_0(M) & \xrightarrow{f_*} & k_0(Bπ) \\
\downarrow & & \downarrow \\
H_n(M) & \xrightarrow{f_*} & H_n(Bπ),
\end{array}
$$

it follows that $f_*([M]) = 0$. Proposition 3.2 completes the proof. □

There are many ways to detect essentiality of manifolds. One of them deals with the Lusternik-Schnirelmann category of X, catLS X, which is the minimal m such that X admits an open cover $U_0, \ldots, U_m$ contractible in X.

3.4. Theorem. A closed n-manifold is essential if and only if its Lusternik-Schnirelmann category equals n.

We refer to [CLOT] for the proof and more facts about the Lusternik-Schnirelmann category. Note that catLS X is estimated from below by the cup-length of X possibly with twisted coefficient and it’s estimated from above by the dimension of X. The definition of the Lusternik-Schnirelmann category can be reformulated in terms of existence of a section of some universal fibration (called Ganea’s fibration). The characteristic class arising from the universal Ganea fibration over the classifying space $Bπ$ is called the Berstein-ˇSvarc class $β_π ∊ H^1(π; I(π))$ of π, where $I(π)$ is the augmentation ideal of the group ring $Z(π)$ (see [Ber], [SV], [CLOT]). Formally, $β_π$ is the image of the generator under the connecting homomorphism $H^0(π; Z) → H^1(π; I(π))$ in the long exact sequence generated by the short exact sequence of coefficients

$$0 → I(π) → Z(π) → Z → 0.$$

The main property of $β_π$ is universality: Every cohomology class $α ∈ H^k(π; L)$ is the image of $(β_π)^k$ under a suitable coefficients homomorphism $I(π)^k = I(π) ⊗ \cdots ⊗ I(π) → L$. We refer to [DR] (see also [SV]) for more details.

3.5. Lemma. Let M be a closed inessential n-manifold, $n ≥ 4$, supplied with a CW complex structure and let $π = π_1(M)$. Then M admits a classifying map $f : M → Bπ$ of the universal covering such that $f(M) ⊂ Bπ^{(n−1)}$ and $f(M^{(n−1)}) ⊂ Bπ^{(n−2)}$. 

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We may assume that $f(M^{(2)}) \to B\pi^{(2)}$ is the identity map. First we show that $f^{*}(\beta^{n-1}) = 0$, where $\beta = \beta_{n}$ is the Berstein-Švarc class of $\pi$. Assume that $f^{*}(\beta^{n-1}) \neq 0$. Then $a = [M] \cap f^{*}(\beta^{n-1}) \neq 0$ by the Poincaré Duality Theorem [Br].

There is $u \in H^{1}(X; A)$ such that $a \cap u \neq 0$ for some local system $A$ (see Proposition 2.3 in [DKR]). Then $f^{*}(\beta^{n-1}) \cup u \neq 0$. Thus the twisted cup-length of $M$ is at least $n$, and hence $\text{cat}_{G} M = n$. This contradicts Theorem 3.3.

4. The Main Theorem

4.1. **Lemma.** Suppose that a classifying map $f : M \to B\pi$ of a closed spin $n$-manifold, $n > 3$, takes the $ko$ fundamental class to 0, $f_{*}(\langle M \rangle_{ko}) = 0$. Then $f$ is homotopic to a map $g : M \to B\pi^{(n-2)}$.

**Proof.** In view of Proposition 3.3 we may assume that $f(M) \subset B\pi^{(n-1)}$. In view of Lemma 3.3 we may additionally assume that $f(M^{(n-1)}) \subset B\pi^{(n-2)}$. Also we assume that $M$ has one $n$-dimensional cell. As in the proof of Proposition 3.3 we can say that the primary obstruction for moving $f$ into the $(n-2)$-skeleton is defined by the cocycle $c_{f} : \pi_{n}(M, M^{(n-1)}) \to \pi_{n}(B\pi, B\pi^{(n-2)})$ which defines the cohomology class $\sigma_{f} = [c_{f}]$ that lives in the group of coinvariants $\pi_{n}(B\pi, B\pi^{(n-2)})_{\pi} = \pi_{n}(B\pi/B\pi^{(n-2)})$ and is represented by $f_{*}(1)$ for the homomorphism $\tilde{f}_{*} : \pi_{n}(M/M^{(n-1)}) \to \pi_{n}(B\pi/B\pi^{(n-2)})$.

We assume that the obstruction $[c_{f}]$ is nonzero. We show that $\tilde{f}_{*} : ko_{n}(S^{n}) \to ko_{n}(B\pi/B\pi^{(n-2)})$ is nontrivial to obtain a contradiction as in the proof of Proposition 3.3. Thus, $\tilde{f}_{*}(1)$ defines a nontrivial element of $\pi_{n}(B\pi/B\pi^{(n-2)})$. The restriction $n > 3$ implies that $\tilde{f}_{*}(1)$ survives in the stable homotopy group. In view of Proposition 2.2 the element $\tilde{f}_{*}(1)$ survives in the composition $\pi_{n}(B\pi/B\pi^{(n-2)}) \to \pi_{n}(B\pi^{(n-2)}) \to ko_{n}(B\pi/B\pi^{(n-2)})$. 

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The commutative diagram

\[
\begin{array}{ccc}
\pi_n(S^n) & \xrightarrow{f_*} & \pi_n(B\pi/B\pi^{(n-2)}) \\
\cong & & \cong \\
\pi_n^\ast(S^n) & \xrightarrow{f_*} & \pi_n^\ast(B\pi/B\pi^{(n-2)}) \\
\cong & & \cong \\
k_{0n}(S^n) & \xrightarrow{f_*} & k_{0n}(B\pi/B\pi^{(n-2)})
\end{array}
\]

implies that \( f_*(1) \neq 0 \) for \( k_{0n} \), a contradiction. \( \square \)

The Strong Novikov Conjecture is connected to the Gromov Conjecture by means of the following theorem, which is due to J. Rosenberg.

4.2. Theorem \([R]\). Suppose \( M^n \) is a spin manifold with a fundamental group \( \pi \). Let \( f \) be the classifying map \( f : M^n \to B\pi \). If \( M^n \) is a positive scalar curvature manifold, then \( \alpha f_*([M^n]_{KO}) = 0 \), where \( \alpha : KO_\ast(B\pi) \to KO_\ast(C_\ast'(\pi)) \) is the analytic assembly map.

4.3. Theorem. Suppose that a discrete group \( \pi \) contains a finite index subgroup \( \pi' \) that has the following properties:

1. The Strong Novikov Conjecture holds for \( \pi' \).
2. The natural map \( \text{per} : k_{0n}(B\pi') \to KO_{n}(B\pi') \) is injective.

Then the Gromov Conjecture holds for spin \( n \)-manifolds \( M \) with fundamental group \( \pi_1(M) = \pi \).

Proof. Let \( M \) be a closed spin \( n \)-manifold that admits a metric with positive scalar curvature. Then its finite cover \( \tilde{M} \) corresponding to the subgroup \( \pi' \) is a spin manifold with positive scalar curvature. By Theorem \([R]\), \( \alpha \circ \text{per} \circ f_*([\tilde{M}]_{KO}) = 0 \). The conditions on \( \pi' \) imply that \( f_*([M^n]_{ko}) = 0 \) for the classifying map \( f : M' \to B\pi' \). Then by Lemma \([R]\), \( f \) is homotopic to \( g : M' \to B\pi'^{(n-2)} \). In view of the fact that \( \tilde{M}' = \tilde{M} \), the induced map of the universal covering spaces \( \tilde{M}' \to E\pi'^{(n-2)} \) produces the inequality \( \dim_{\text{mc}} \tilde{M} \leq n-2 \). \( \square \)

We recall that the virtual cohomological dimension \( vcd(\pi) \) of a discrete group \( \pi \) is the cohomological dimension of a torsion free finite index subgroup. It is known that \( vcd(\pi) \) is well-defined if it is finite; i.e., it does not depend on the choice of a torsion free finite index subgroup.

4.4. Corollary. The Gromov Conjecture holds for spin \( n \)-manifolds \( M \) with the fundamental group \( \pi_1(M) = \pi \) having \( vcd(\pi) \leq n+4 \) and satisfying the Strong Novikov Conjecture.

Proof. We consider a finite index subgroup \( \pi' \) with \( cd(\pi') \leq n+4 \). We show that \( \text{per} \) is an isomorphism in dimension \( n \) for \( \pi' \). Let \( F \to ko \to KO \) be the fibration of spectra induced by the morphism \( ko \to KO \). Since \( \pi_k(ko) \to \pi_k(KO) \) is an isomorphism for \( k \geq 0 \) and \( \pi_{-1}(F) = 0 \), we have \( \pi_k(F) = 0 \) for \( k \geq -1 \). Also \( \pi_k(F) = \pi_{k+1}(KO) = KO_{k+1}(pt) = 0 \) if \( k = -2, -3, -4 \) mod 8. The Atiyah-Hirzebruch \( F \)-homology spectral sequence for \( B\pi' \) implies that \( H_n(B\pi'; F) = 0 \)
since all entries on the $n$-diagonal in the $E^2$-term are 0. Then the coefficient exact sequence for homology
\[ H_n(B\pi'; F) \to ko_n(B\pi') \to KO_n(B\pi') \to \ldots \]
implies that $\text{per} : ko_n(B\pi') \to KO_n(B\pi')$ is a monomorphism. \qed

We note that this corollary for $cd(\pi) \leq n - 1$ was first proven in \[B3\].

4.5. Corollary. The Gromov Conjecture holds for spin $n$-manifolds $M$ with the fundamental group $\pi_1(M) = \pi$ having a finite index subgroup $\pi'$ with finite complex $B\pi'$ and with asymptotic dimension $\text{asdim} \pi \leq n + 4$.

Proof. This is a combination of the fact that the Strong Novikov conjecture holds true for such groups $\pi'$ (\[Ba\], \[DFW\]), the above corollary, and the inequality $\text{vcd}(\pi) \leq \text{asdim} \pi$ proven in \[Dr\]. \qed

4.6. Corollary. The Gromov conjecture holds for spin $n$-manifolds $M$ with the fundamental group $\pi_1(M)$ equal to the product of free groups $F_1 \times \cdots \times F_n$. In particular, it holds for free abelian groups.

Proof. For reduced homology with coefficients in a spectrum $E$, the formula
\[ H_i(X \times S^1; E) \cong H_i(X; E) \oplus H_{i-1}(X; E) \oplus H_i(S^1) \]
implies that if $ko_*(X) \to KO_*(X)$ is a monomorphism, then $ko_*(X \times S^1) \to KO_*(X \times S^1)$ is a monomorphism. By induction on $m$ using the Mayer-Vietoris sequence this formula can be generalized to the following:
\[ H_i(X \times (\bigvee_m S^1); E) \cong H_i(X; E) \oplus \bigoplus_m (H_{i-1}(X; E) \oplus H_i(S^1)) \]
Therefore,
\[ ko_*(X \times (\bigvee_m S^1)) \to KO_*(X \times (\bigvee_m S^1)) \]
is a monomorphism. \qed

As we noted in the proof of Theorem \[B3\] the Gromov Conjecture for spin manifolds with fundamental group $\pi'$ implies the Gromov Conjecture for spin manifolds with fundamental groups $\pi$ containing $\pi'$ as a finite index subgroup. Since every finitely generated abelian group has the torsion subgroup finite, in view of the above corollary, we obtain the following.

4.7. Corollary. The Gromov Conjecture holds true for spin $n$-manifolds $M$ with virtually abelian fundamental groups $\pi_1(M)$.

References


\[B3\] D. Bolotov, About the macroscopic dimension of certain PSC-manifolds, Algebr. Geom. Topol. 9 (2009), 21-27. MR2471130

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