INTEGRATION ON \( p \)-ADIC GROUPS AND CRYSTAL BASES

DANIEL BUMP AND MAKI NAKASUJI

(Communicated by Wen-Ching Winnie Li)

Abstract. Let \( G = \text{GL}_{r+1} \) over a nonarchimedean local field \( F \). The Kashiwara crystal \( B(\infty) \) is the quantized enveloping algebra of the lower triangular maximal unipotent subgroup \( N_- \). Examples are given where an integral over \( N_-(F) \) may be replaced by a sum over \( B(\infty) \). Thus the Gindikin-Karpelevich formula evaluates the integral of the standard spherical vector in the induced model of a principal series representation as a product
\[
\prod (1 - q^{-1}z^\alpha)/(1 - z^\alpha)
\]
where \( z \) is the Langlands parameter and the product is over positive roots. This may also be expressed as a sum over \( B(\infty) \). The corresponding equivalence over a metaplectic cover of \( \text{GL}_{r+1} \) is deduced by using Kashiwara’s similarity of crystals.

1. Introduction

Kashiwara defined the notion of a \( \text{crystal} \) and gave examples of crystal structures associated with bases of representations of quantum groups. We recommend the expository article [7], written a few years after the original papers, and the book of Hong and Kang [5].

One particular crystal defined by Kashiwara is denoted by \( B(\infty) \). It is a basis of the quantized universal enveloping algebra \( U_q(n_-) \) where \( n_- \) is the Lie algebra of the maximal unipotent subgroup \( N_- \) of a reductive algebraic group \( G \) or more generally its \( n \)-fold metaplectic cover. Our basic philosophy is that an integral over \( N_-(F) \) where \( F \) is a nonarchimedean local field can sometimes be replaced by a sum over \( B(\infty) \). We will demonstrate this for \( G = \text{GL}_{r+1} \), and later for the \( n \)-fold metaplectic cover. In this introduction we will consider the “nonmetaplectic case” where \( n = 1 \).

Let \( L \rightarrow G = \text{GL}_{r+1}(\mathbb{C}) \) be the (connected) Langlands dual group. Then the diagonal group \( T(\mathbb{C}) \) in \( L \) has character group \( \Lambda = X^*(T) \cong \mathbb{Z}^{r+1} \), and we may identify this with the full weight lattice.

If \( z = \text{diag}(z_1, \ldots, z_{r+1}) \in T(\mathbb{C}) \) where \( z_i \in \mathbb{C}^\times \), then in this identification \( \mu \in \mathbb{Z}^{r+1} \) is the character \( \mu \mapsto z^\mu = \prod z_i^{\mu_i} \). The simple positive roots are \( \alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \) where the 1 is in the \( i \)-th place. The dominant weights are \( \lambda = (\lambda_1, \ldots, \lambda_{r+1}) \) such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+1} \). If all \( \lambda_i \geq 0 \), then we call a weight \( \lambda \) effective. Thus an effective dominant weight is a partition. We will write \( \rho = (r, r-1, \ldots, 2, 1, 0) \). It differs from half of the positive roots by a vector orthogonal to the roots, so it may substitute for \( \frac{1}{2} \sum \alpha \) in many formulas, such as the Weyl character formula.

Received by the editors August 12, 2009.

2010 Mathematics Subject Classification. Primary 17B37; Secondary 22E35, 11F85.

©2009 American Mathematical Society
Reverts to public domain 28 years from publication

1595
The conjugacy class in $L^G$ parametrizes a spherical representation of $G(F)$. The induced model of this representation acts on the space of smooth functions $f$ on $G$ that satisfy $f(bg) = \delta^{1/2} \chi(b)f(g)$, where $b$ lies in the Borel subgroup $B(F)$ of upper triangular matrices, $\delta$ is the modular quasicharacter on $B(F)$ and $\chi$ is the quasicharacter of $B(F)$ defined by

$$
\chi = \begin{pmatrix}
y_1 & * & \cdots & * 
y_2 & & \cdots & * 
& \ddots & & \vdots 
y_{r+1} & & & 1
\end{pmatrix} = \prod_{i=1}^{r+1} z_i^{\alpha_i(y_i)}.
$$

Various integrals that we write down will be convergent if $|z_i/z_{i+1}| < 1$, and we will assume this. Let $\mathfrak{o}$ be the ring of integers in $F$ and let $q$ be the cardinality of the residue field. Let $p$ be the prime ideal of $\mathfrak{o}$ and $\varpi$ be a prime element.

The standard spherical vector $f^\circ$ in this representation is the function such that $f^\circ(bk) = \delta^{1/2} \chi(b)$ when $b \in B(F)$ and $k \in K = GL_{r+1}(\mathfrak{o})$. We mention two important integrals that illustrate the principle we stated above. The first is the formula of Gindikin and Karpelevich, which asserts that

$$
\int_{N_-(F)} f^\circ(\mathfrak{n}) \, d\mathfrak{n} = \prod_{\alpha \in \Phi^+} \frac{1-q^{-1}z^\alpha}{1-z^\alpha}.
$$

The second is the formula of Casselman and Shalika.

The formula (1.1) was first proved by Langlands [10]. Another proof may be found in Casselman [2]. (The original paper of Gindikin and Karpelevich [4] is concerned with the archimedean case.) MacNamara [12] also gives a proof of a generalization of this formula, as well as the Casselman-Shalika formula, to metaplectic covers.

We will show that (1.1) may also be expressed as a sum over $B(\infty)$. This is striking since $B(\infty)$ is obtained from $N_-$ by quantization. The work of MacNamara [12] may clarify this phenomenon by showing how to decompose $N_-(F)$ into cells parametrized by elements of $B(\infty)$.

The integral $\int_{N_-(F)} f(\mathfrak{n}) \psi(\mathfrak{n}) \, d\mathfrak{n}$, where $\psi$ is a nondegenerate additive character of $N_-(F)$, is evaluated in the formula of Casselman and Shalika [3]. Making use of a formula of Tokuyama [14] this evaluation may be rewritten in terms of crystals. This was done by Brubaker, Bump and Friedberg [1]. We will describe a variant of their formula. The difference is that we will use the Kashiwara operators $e_i$ where they use the $f_i$.

Let $\lambda \in \mathbb{Z}^{r+1}$. Define

$$
\psi_\lambda = \begin{pmatrix}
1 
x_{2,1} & 1 
\vdots & \ddots 
x_{r+1,1} & \cdots & x_{r+1,r} & 1
\end{pmatrix} = \psi_0(\varpi^{\lambda_1-\lambda_2 x_{r+1,r} + \cdots + \varpi^{\lambda_r-\lambda_{r+1}} x_{2,1}})
$$

where $\psi_0$ is a fixed additive character on $F$ that is trivial on $\mathfrak{o}$ but not on $p^{-1}$. The integral $\int_{N_-(F)} f(\mathfrak{n}) \psi_\lambda(\mathfrak{n}) \, d\mathfrak{n}$ is zero unless the weight $\lambda$ is dominant, which we now assume.

In order to give the relevant definitions, we recall some facts and definitions about crystals. Let $\Phi$ be a root system, which in this paper will be mainly $A_r$. \[\text{...}\]
Let $\alpha_i$ ($i = 1, \cdots, r$) be the simple roots and $\alpha_i^\vee$ their associated coroots. Let $\Lambda$ be the associated weight lattice. By a crystal for $\Phi$ we mean a set $\mathcal{B}$ together with a map $\text{wt} : \mathcal{B} \to \Lambda$ and, for $1 \leq i \leq r$, maps $\phi_i, \varepsilon_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$ and $f_i, e_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$, where $0$ is an auxiliary element. It is assumed that $\phi_i(v) = \langle \text{wt}(v), \alpha_i^\vee \rangle + \varepsilon_i(v)$. If $e_i(v) \neq 0$, then it is assumed that $f_i e_i(v) = v$ and that $\text{wt}(e_i(v)) = \text{wt}(v) + \alpha_i$, and if $f_i(v) \neq 0$, then it is assumed that $e_i f_i(v) = v$ and that $\text{wt}(f_i(v)) = \text{wt}(v) - \alpha_i$.

In Kashiwara’s papers the maps we have denoted by $e_i$ and $f_i$ are denoted by $\tilde{e}_i$ and $\tilde{f}_i$, because the letters $e_i$ and $f_i$ were already in use with a different meaning.

One may impose on $\mathcal{B}$ the structure of a directed graph with labeled edges, called the crystal graph, in which elements are vertices and there is an edge $x \xrightarrow{i} y$ if $f_i(x) = y$. Examples of crystal graphs may be seen in Figure 1 in the next section.

If $\mathcal{C}$ and $\mathcal{D}$ are crystals, a morphism $m : \mathcal{C} \to \mathcal{D}$ is a map $\mathcal{C} \to \mathcal{D} \cup \{0\}$ such that if $x \in \mathcal{C}$ and $m(x) \neq 0$, then $\text{wt}(m(x)) = \text{wt}(x)$, $\varepsilon_i(m(x)) = \varepsilon_i(x)$ and $\phi_i(m(x)) = \phi_i(x)$, and if $x, y \in \mathcal{C}$ and $m(x), m(y) \neq 0$, then $e_i(x) = y$ if and only if $e_i(m(x)) = m(y)$ and $f_i(y) = x$ if and only if $f_i(m(y)) = m(x)$. Crystals form a category.

Let $G$ be a complex analytic group and $T$ a maximal torus such that $\Phi$ is the root system of $G$ with respect to $T$. Assuming that the derived group of $G$ is simply connected, we may identify $\Lambda$ with the group $X^*(T)$ of rational characters of $T$. A crystal $\mathcal{B}_\lambda$ is defined with the property that

$$\sum_{v \in \mathcal{B}_\lambda} z^{\text{wt}(v)}$$

$(z \in T)$ is the character of the highest weight module $V_\lambda$ for $\lambda$.

By a long word $\Omega$ we mean a reduced expression of the long element $w_0$ of $W$ as a product of simple reflections. Thus

$$\Omega = (\omega_1, \omega_2, \cdots, \omega_N)$$

where $N$ is the number of positive roots ($N = \frac{1}{2}r(r + 1)$ for $\Phi = A_r$) and $\omega_j \in \{1, 2, \cdots, r\}$ are such that $w_0 = s_{\omega_1} \cdots s_{\omega_N}$. Let $v \in \mathcal{B}_\lambda$. Let $b_1$ (depending on $v$ and $\Omega$) be the largest integer such that $e_{\omega_1}^{b_1} v \neq 0$. Let $b_2$ be the largest integer such that $e_{\omega_2}^{b_2} e_{\omega_1}^{b_1} v \neq 0$, and so forth. It is known (see Littelmann [1]) that $e_{\omega_N}^{b_N} \cdots e_{\omega_2}^{b_2} e_{\omega_1}^{b_1} v$ is the unique element $v_{\text{high}}$ of $\mathcal{B}_\lambda$ with $\text{wt}(v_{\text{high}}) = \lambda$, the highest weight.

We decorate the pattern

$$(1.2) \quad \text{BZL}(v) = (b_1, \cdots, b_N)$$

by “circling” or “boxing” certain entries. We will describe the boxing rule for all $\Omega$, but we will describe the circling rule only for $\Omega = \Omega_\Gamma$ or $\Omega = \Omega_\Delta$ where

$$\Omega_\Gamma = (1, 2, 1, 3, 2, 1, \cdots, r, r - 1, \cdots, 3, 2, 1),$$

$$\Omega_\Delta = (r, r - 1, r, r - 2, r - 1, r, \cdots, 1, 2, 3, \cdots, r).$$
If $f_\omega e_{\omega_i-1} \cdots e_{\omega_1} v = 0$, then we decorate $b_1$ by boxing it. In the case where $\Omega = \Omega_{T}$ or $\Omega_{\Delta}$ it was proved by Littelmann \[1\] that

\[
\begin{align*}
b_1 & \geq 0, \\
b_2 & \geq b_3 \geq 0, \\
b_4 & \geq b_5 \geq b_6 \geq 0,
\end{align*}
\]

(1.3)

If $b_1 = 0$, then we decorate $b_1$ by circling it. If $b_2 = b_3$, then we decorate $b_2$ by circling it. If $b_3 = 0$, then we decorate $b_1$ by circling it, and so forth.

Now let us recall from \[1\] the definition

\[
G_\Omega(v) = G_\Omega^{(e)}(v) = \prod_{i=1}^{N} \begin{cases} 
h(b_i) & \text{if } b_i \text{ is neither circled nor boxed,} \\
g(b_i) & \text{if } b_i \text{ is boxed but not circled,} \\
q^{b_i} & \text{if } b_i \text{ is circled but not boxed,} \\
0 & \text{if } b_i \text{ is both circled and boxed.} \end{cases}
\]

(1.4)

In \[1\] (and in the final section below), $h$ and $g$ are $n$-th order Gauss sums, where $n$ is an integer prime to the residue characteristic such that the ground field contains the $n$-th roots of unity. In the case at hand, $n = 1$ and $g$ and $h$ can be made explicit:

\[
g(a) = -q^{a-1}, \quad h(a) = (q-1)q^{a-1}.
\]

(1.5)

We may also dualize these definitions by interchanging the roles of the $e_i$ and $f_i$. Thus we would alternatively let $b_1$ be the largest integer such that $f_{\omega_1}^{b_1} v \neq 0$. Let $b_2$ then be the largest integer such that $f_{\omega_2}^{b_2} f_{\omega_1}^{b_1} v \neq 0$, and so forth. It is known (see Littelmann \[1\]) that $f_{\omega_N}^{b_N} \cdots f_{\omega_2}^{b_2} f_{\omega_1}^{b_1} v$ is the unique element $v_{\min}$ of $B_\lambda$ with $
olimits \wt(v_{\min}) = w_0 \lambda$, the lowest weight. In this scheme, we box $b_i$ if $e_{\omega_i} f_{\omega_i-1}^{b_i-1} \cdots f_{\omega_1}^{b_1} v = 0$. The inequalities (1.3) are again satisfied, and as before, if $b_1 = 0$, then we decorate $b_1$ by circling it, and so forth. Then we may define

\[
G_\Omega^{(f)}(v) = \prod_{i=1}^{N} \begin{cases} 
h(b_i) & \text{if } b_i \text{ is neither circled nor boxed,} \\
g(b_i) & \text{if } b_i \text{ is boxed but not circled,} \\
q^{b_i} & \text{if } b_i \text{ is circled but not boxed,} \\
0 & \text{if } b_i \text{ is both circled and boxed.} \end{cases}
\]

We can make exactly the same definitions for $v \in B(\infty)$. However, only the definition of $G_\Omega^{(e)}(v)$ makes sense, since there is no largest integer such that $f_{\omega_1}^{b_1} v \neq 0$. Indeed, if $v \in B(\infty)$, then $f_{\omega_i}^{k} v \neq 0$ for all $k$. Therefore we may define $G_\Omega^{(e)}(v)$ but not $G_\Omega^{(f)}(v)$. Also circling can occur but not boxing; indeed $f_{\omega_i} e_{\omega_i-1} \cdots e_{\omega_1} v \neq 0$ for the same reason.

If $\lambda$ is any weight, there is a crystal $T_\lambda$ having one element $t_\lambda$ with weight $\lambda$. It has the properties that $e_i(t_\lambda) = f_i(t_\lambda) = 0$ and $\phi_i(t_\lambda) = \varepsilon_i(t_\lambda) = -\infty$. We have $T_\lambda \otimes T_\mu \cong T_{\lambda + \mu}$. Tensoring any crystal $B$ with $T_\lambda$ produces a crystal that is isomorphic to $B$ as a directed graph but in which the weights are shifted: $\wt(x \otimes t_\lambda) = \wt(x) + \lambda$ for $x \in B$.

If $\lambda$ is a dominant weight, let $\chi_\lambda$ be the irreducible character of $L G = GL_{r+1}(\mathbb{C})$ with highest weight $\lambda$. 
Theorem 1.1. If $\lambda$ is a dominant weight and $\Omega = \Omega_\epsilon$ or $\Omega_\Delta$, then
\[
\int_{N_-(F)} f^\sigma(n) \psi_\lambda(n) \, dn = \prod_{\alpha \in \Phi^+} (1 - q^{-1}z^\alpha) \chi_\lambda(z)
\]
\[= \sum_{B_{\lambda+\rho} \otimes T_{-\lambda+\rho}} G_\Omega(v) q^{-(w_0(\wt(v)), \rho)} z^{w_0(\wt(v))}.
\]

The first equality is the Casselman-Shalika formula. We will also rewrite the formula of Gindikin and Karpelevich in the following similar way.

Theorem 1.2. We have
\[
\int_{N_-(F)} f^\sigma(n) \, dn = \prod_{\alpha \in \Phi^+} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha} = \sum_{\mathcal{B}(\infty)} G_\Omega(v) q^{(\wt(v), \rho)} z^{-\wt(v)}.
\]

In fact in both these theorems, the final sum may be written as a sum over $B(\infty)$. Indeed, there is a morphism $M_{\lambda+\rho} : B(\infty) \rightarrow B_{\lambda+\rho} \otimes T_{-\lambda+\rho}$ due to Kashiwara (see [7], Theorem 8.1), which we will make use of in the next section, and the sum over $B_{\lambda+\rho} \otimes T_{-\lambda+\rho}$ may therefore be interpreted as a sum over $B(\infty)$, with only finitely many nonzero terms (those that do not map to zero in the morphism).

Thus both theorems illustrate the philosophy that we can sometimes replace integrals over $N_-(F)$ by sums over $B(\infty)$, which is a basis of the quantized enveloping algebra of $N_-(F)$.

We would like to thank Ben Brubaker and Solomon Friedberg for helpful conversations. This work was supported in part by a JSPS Research Fellowship for Young Scientists and by NSF grant DMS-0652817.

2. Proofs of the theorems

The paper of Hong and Lee [6] describes $B(\infty)$ in explicit terms by means of tableaux. We will not review their work here, but it was useful in the preparation of this paper.

We have already mentioned the crystal $T_\lambda$ having just one element $t_\lambda$ of weight $\lambda$, such that $\epsilon_i(t_\lambda) = f_i(t_\lambda) = 0$ and $\phi_i(t_\lambda) = \epsilon_i(t_\lambda) = -\infty$. There is a morphism $M_\lambda : B(\infty) \rightarrow B_\lambda \otimes T_{-\lambda}$ that was introduced by Kashiwara (see [7], Theorem 8.1), which we will make use of. Let $u_0$ and $b_\lambda$ be the highest weight vectors in $B(\infty)$ and $B_\lambda$, so $\wt(u_0) = 0$ and $\wt(b_\lambda) = \lambda$. The morphism maps $u_0$ to $b_\lambda \otimes t_{-\lambda}$. It maps all but a finite number of elements to 0. Those elements $u$ of $B(\infty)$ that do not map to zero form a directed subgraph of the crystal graph of $B(\infty)$ that is a copy of $B_\lambda$ as a colored directed graph. To illustrate this morphism, Figure 1 shows $B_\lambda$ (using Kashiwara’s notation for the crystal elements as tableaux) in the case where $\lambda = (2, 1, 0)$; tensoring this with $T_{-\lambda}$ so that the highest weight vector has weight 0, this is embedded in $B(\infty)$, where the labeling is a modification of the notation in Hong and Lee [6]. (From the partial tableaux in Figure 1 one obtains representatives of the crystal $T_\infty$ in [6] by adding sufficiently many 1’s at the beginning of the first row, 2’s at the beginning of the second row, etc.)

We will prove Theorem 1.1. If $\psi_\lambda$ is an additive character of $N_-(F)$ as defined in the introduction, the Casselman-Shalika formula for $GL_{r+1}$ is written as follows:
\[
\int_{N_-(F)} f^\sigma(n) \psi_\lambda(n) \, dn = z^{-w_0(\lambda)} \left[ \prod_{\alpha \in \Phi^+} (1 - q^{-1}z^\alpha) \right] s_\lambda(z_1, \ldots, z_{r+1}),
\]
where the integral is absolutely convergent if $|z^\alpha| < 1$, and $s_\lambda(z_1, \ldots, z_r)$ is the standard Schur polynomial.

On the other hand, Brubaker, Bump and Friedberg show the following Tokuyama deformation of the Weyl character formula for crystals.

**Theorem 2.1** ([1], Theorem 5). If $\lambda$ is a dominant weight and if $z_1, \ldots, z_{r+1}$ are the eigenvalues of $g \in \text{GL}_{r+1}(\mathbb{C})$, then

$$\prod_{\alpha \in \Phi^+} (1 - q^{-1}z^\alpha) \chi_\lambda(g) = \sum_{v \in \mathcal{B}_{\rho+\lambda}} G^{(f)}_{\text{th}}(v) q^{-(\text{wt}(v) - w_0(\lambda + \rho), \rho)} z^{\text{wt}(v) - w_0(\rho)} ,$$

where $\chi_\lambda$ is the character of the irreducible representation with highest weight $\lambda$. When $z_i$ are the eigenvalues of $g \in \text{GL}_{r+1}(\mathbb{C})$, we have $s_\lambda(z_1, \ldots, z_{r+1}) = \chi_\lambda(g)$. Therefore, by this theorem, the integral $\int_{N_-(F)} f^\sigma(n) \psi_\lambda(n) dn$ in the formula of Casselman and Shalika is evaluated in terms of crystal graphs ([1], (3.7)):

$$(2.1) \quad \int_{N_-(F)} f^\sigma(n) \psi_\lambda(n) dn = \sum_{v \in \mathcal{B}_{\rho+\lambda}} G^{(f)}_{\text{th}}(v) q^{-(\text{wt}(v) - w_0(\lambda + \rho), \rho)} z^{\text{wt}(v) - w_0(\rho + \lambda)} .$$

Now we will replace the right hand side by an expression involving $G^{(c)}_{\text{th}}$. The following equivalence of two descriptions is obtained in [1].

**Theorem 2.2** ([1], Statement A').

$$\sum_{v \in \mathcal{B}_{\rho+\lambda}} G^{(f)}_{\Omega_\lambda}(v) = \sum_{v \in \mathcal{B}_{\rho+\lambda}} G^{(f)}_{\Omega_\Delta}(v) .$$

By this theorem, the right hand side of (2.1) is written as

$$\sum_{v \in \mathcal{B}_{\rho+\lambda}} G^{(f)}_{\Omega_\Delta}(v) q^{-(\text{wt}(v) - w_0(\lambda + \rho), \rho)} z^{\text{wt}(v) - w_0(\rho + \lambda)} .$$

There is a map $\text{Sch} : \mathcal{B}_{\rho+\lambda} \to \mathcal{B}_{\rho+\lambda}$ called the Schützenberger involution such that $\text{Sch} \circ e_i = f_{r+1-i} \circ \text{Sch}$ and $\text{Sch} \circ f_i = e_{r+1-i} \circ \text{Sch}$. Let $v' = \text{Sch}(v)$ for $v \in \mathcal{B}_{\rho+\lambda}$. 

Figure 1. The crystal $\mathcal{B}_{\lambda} \otimes \mathcal{T}_-\lambda$, with $\lambda = (2, 1, 0)$, and its image in $\mathcal{B}(\infty)$.
Since \( wt(v') = w_0 wt(v) \) and \( G^{(f)}_{\Omega^\varepsilon}(v) = G^{(c)}_{\Omega^\varepsilon}(\text{Sch}(v)) = G^{(c)}_{\Omega^\varepsilon}(v') \), the right hand side of (2.1) becomes
\[
\sum_{v' \in \mathcal{B}_{\rho}^{\lambda,\mu}} G^{(c)}_{\Omega^\varepsilon}(v') q^{-\langle w_0(wt(v') - \rho - \lambda), \rho \rangle} z^{w_0(wt(v') - \rho - \lambda)}.
\]
Let \( v'' := v' \circ t_{-\lambda - \rho} \) with \( v' \in \mathcal{B}_{\lambda + \rho} \) and \( t_{-\lambda - \rho} \in T_{-\lambda - \rho} \). Since \( wt(v'') = wt(v') - \lambda - \rho \) and \( G^{(c)}_{\Omega^\varepsilon}(v'') = G^{(c)}_{\Omega^\varepsilon}(v') \), with the morphism \( M_{\lambda + \rho} : \mathcal{B}(\infty) \to \mathcal{B}_{\lambda + \rho} \otimes T_{-\lambda - \rho} \), we obtain
\[
\sum_{v'' \in \mathcal{B}_{\lambda + \rho} \otimes T_{-\lambda - \rho}} G^{(c)}_{\Omega^\varepsilon}(v'') q^{-\langle w_0(wt(v''), \rho) \rangle} z^{w_0(wt(v''))}.
\]
This proves Theorem 1.1.

In order to prove Theorem 1.2, we need to discuss the limiting argument first.

Given \( n \in N_-(F) \) we may write \( n = t n_+ k \) where \( t \in T, n_+ \in N \) and \( k \in \text{GL}_{r+1}(\mathfrak{o}) \). The element \( t \) is not uniquely determined but its image \( \bar{t} \) in \( T/T(\mathfrak{o}) \) is uniquely determined. The group \( T/T(\mathfrak{o}) \) is discrete, and \( v : T/T(\mathfrak{o}) \to \mathbb{Z}^{r+1} \) defined by
\[
v \left( \begin{array}{c}
t_1 \\
\vdots \\
t_{r+1}
\end{array} \right) = (\text{ord}(t_1), \ldots, \text{ord}(t_{r+1}))
\]
is an isomorphism. Define a map \( \beta : N_-(F) \to \mathbb{Z}^{r+1} \) by \( \beta(n) = v(\bar{t}) \).

**Proposition 2.3.** The map \( \beta \) is proper.

We recall that if \( X \) and \( Y \) are Hausdorff topological spaces, then a map \( f : X \to Y \) is proper if the inverse image of a compact set is compact. Since \( \mathbb{Z}^{r+1} \) is discrete, this means that the inverse image of a finite set is compact in \( N_-(F) \).

**Proof.** Write \( n = t n_+ k \) with \( t \in T, n_+ \in N \) and \( k \in K \). Let \( S \) be a subset of \( \{1, \ldots, r+1\} \) with \( k = |S| \). If \( A = (a_{ij}) \) is an \( (r+1) \times (r+1) \) matrix, denote by \( M_S(A) \) the minor
\[
det(a_{i,j} \mid i \in \{r+2-k, r+3-k, \ldots, r+1\}, j \in S)
\]
formed with the bottom \( k \) rows of \( A \) and columns in \( j \). We call \( M_S(A) \) a bottom minor. Since \( n_+ \) is upper triangular and unipotent, \( M_S(n_+ k) = M_S(k) \), and since \( t \) is diagonal,
\[
M_S(n) = \left[ \prod_{j=r+2-k}^{r+1} t_j \right] M_S(k).
\]
Since the entries in \( M_S(k) \) are in \( \mathfrak{o} \), this means that
\[
|M_S(n)| \leq \left| \prod_{j=r+2-k}^{r+1} t_j \right|.
\]
Now since \( n \) is lower triangular and unipotent it is easy to see that each entry \( n_{ij} \) in \( n \) (with \( i > j \)) equals \( M_S(n) \) where \( S = \{j, i+1, i+2, \ldots, r+1\} \). For example if \( r+1 = 4 \) and
\[
n = \left( \begin{array}{ccc}
1 & 1 & 1 \\
n_{21} & n_{31} & n_{32} \\
n_{41} & n_{42} & n_{43}
\end{array} \right),
\]
then \( n_{31} = M_S(n) \) where \( S = \{1,4\} \). It is now clear that if \( t \) is confined to a compact subset of \( T \), then the entries of \( n \) are bounded, and it follows that \( \beta \) is a proper map.

Let \( R = \mathbb{C}[q][[z^{\alpha_1}, \cdots, z^{\alpha_r}]] \) and \( P := \{ \sum k_i \alpha_i \mid 1 \leq i \leq r, k_i \geq 0 \} \). If \( v \in B_{\lambda+\rho} \), then \( \omega(v) - w_0(\lambda + \rho) \in P \). It follows from (2.1) that \( \int_{N_-(F)} f^\alpha(n) \psi(n) \, dn \in R \). Applying Proposition 2.3, we have the following result:

**Proposition 2.4.** \( \int_{N_-(F)} f^\alpha(n) \psi_\lambda(n) \, dn \) converges to \( \int_{N_-(F)} f^\alpha(n) \, dn \) in the topology of the ring \( R \) as \( \lambda \) goes to \( \infty \).

**Proof.** Let \( S \) be a finite subset of \( \Lambda \) contained in \( P \). By Proposition 2.3 there is a compact subset \( C \) of \( N_-(F) \) such that, for \( n \in N_-(F) - C \), \( \beta(n) = \sum k_i \alpha_i \notin S \). Assume \( \lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots > M \) for some integer \( M \). The difference \( \int_{N_-(F)} f^\alpha(n) \, dn - \int_{N_-(F)} f^\alpha(n) \psi_\lambda(n) \, dn \) is written in two parts:

\[
\int_C f^\alpha(n)(1 - \psi_\lambda(n)) \, dn + \int_{N_-(F) - C} f^\alpha(n)(1 - \psi_\lambda(n)) \, dn.
\]

Choose \( M \) so large that \( \psi_\lambda = 1 \) on \( C \). Then the first term vanishes. Let \( E_S \) be the additive subgroup of \( R \) consisting of \( \sum c_{k_1 \cdots k_r} (q) z^{k_1 \alpha_1 + \cdots + k_r \alpha_r} \) such that \( c_{k_1 \cdots k_r} (q) = 0 \) if \( \sum k_i \alpha_i \notin S \). These form a base of neighborhoods of the identity in \( R \). Since \( f^\alpha(n) \in R \), this means that the second term converges in \( R \). \( \square \)

We will prove Theorem 1.2.

When \( \lambda \) goes to \( \infty \), the limiting argument as above and Theorem 1.1 lead to

\[
\int_{N_-(F)} f^\alpha(n) \, dn = \sum_{v \in \mathcal{B}(\infty)} G^\alpha_{\Omega_1}(v) q^{\omega_0(\omega(v), \rho)} z^{\omega_0(\omega(v))}.
\]

There is a map \( \iota : \mathcal{B}_\lambda \to \mathcal{B}_{\omega_0(\lambda + \rho)} \) which satisfies \( \iota_\lambda \circ f_1 = f_{r+1-i} \circ \iota_\lambda \) and \( \iota_{\lambda + \rho} \circ e_{r+1-i} = e_i \circ \iota_{\lambda + \rho} \). There is a corresponding bijection \( \iota : \mathcal{B}(\infty) \to \mathcal{B}(\infty) \) such that

\[
\begin{array}{ccc}
\mathcal{B}(\infty) & \xleftarrow{\iota} & \mathcal{B}(\infty) \\
M_{\lambda+\rho} & \downarrow & M_{\omega_0(\lambda + \rho)} \\
\mathcal{B}_\lambda \otimes \mathcal{T}_{\lambda + \rho} & \xrightarrow{\iota_{\lambda + \rho}} & \mathcal{B}_{\omega_0(\lambda + \rho)} \otimes \mathcal{T}_{\omega_0(\lambda + \rho)}
\end{array}
\]

Let \( \tilde{v} = \iota(v) \) for \( v \in \mathcal{B}(\infty) \). Then since \( G^\alpha_{\Omega_1}(\tilde{v}) = G^\alpha_{\Omega_1}(v) \) and \( \omega(\tilde{v}) = -\omega_0(\omega(v)) \), we have

\[
\int_{N_-(F)} f^\alpha(n) \, dn = \sum_{\tilde{v} \in \mathcal{B}(\infty)} G^\alpha_{\Omega_1}(\tilde{v}) q^{\omega(\tilde{v}), \rho} z^{-\omega(\tilde{v})}.
\]

This concludes the proof of Theorem 1.2.

3. **The Metaplectic Case**

Finally, we give metaplectic analogs of these formulas. We assume that the ground field \( F \) has residue characteristic prime to \( n \) and contains the group \( \mu_n \) of \( n \)-th roots of unity in the algebraic closure of \( F \). We fix an isomorphism of \( \mu_n \) with the group of \( n \)-th roots of unity in \( \mathbb{C}^\times \). To avoid unnecessary minor complications we will take \( G = \text{SL}_{r+1} \) rather than \( \text{GL}_{r+1} \) in this section.
Let $\hat{G}(F)$ be the $n$-fold metaplectic cover of $\text{SL}_{r+1}(F)$, first constructed by Matsumoto [13], that splits over $K = \text{SL}_{r+1}(O)$. Let $K^*$ be the image of $K$ in $\hat{G}(F)$ under the splitting. It is a central extension

$$1 \rightarrow \mu_n \rightarrow \hat{G}(F) \rightarrow \text{SL}_{r+1}(F) \rightarrow 1.$$  

We choose a section $s : \text{SL}_{r+1}(F) \rightarrow \hat{G}(F)$ and a cocycle $\sigma : \text{SL}_{r+1}(F) \times \text{SL}_{r+1}(F) \rightarrow \mu_n$ whose class in $H^2(\hat{G}(F), \mu_n)$ determines the extension so that, upon identifying $\mu_n$ with its image in $\hat{G}(F)$, we have $s(g)s(g') = \sigma(g, g')s(gg')$. We may choose $s$ and $\sigma$ so that

$$\sigma \left( s \left( \begin{array}{ccc} t_1 & \cdots & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdots \\
 \cdot & & t_{r+1} \end{array} \right), s \left( \begin{array}{ccc} u_1 & \cdots & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdots \\
 \cdot & & u_{r+1} \end{array} \right) \right) = \prod_{i<j} (t_i, u_j)^{-1},$$

where $(t, u)$ is the $n$-th order Hilbert symbol, and so that $\sigma(n, g) = \sigma(g, n) = 1$ when $n$ is in the group $N(F)$ of upper triangular unipotent matrices in $\text{SL}_{r+1}(F)$.

Identifying $\mu_n$ both with its image in $\hat{G}(F)$ and with its image in $\mathbb{C}$, we call a function $f : \hat{G}(F) \rightarrow \mathbb{C}$ genuine if $f(\varepsilon g) = \varepsilon f(g)$ for $\varepsilon \in \mu_n$. There exists a unique genuine function $\tilde{f}^\circ$ on $\hat{G}(F)$ that satisfies

$$\tilde{f}^\circ \left( s \left( \begin{array}{ccc} t_1 & \cdots & * \\ \cdots & & \cdots \\ * & \cdots & * \\
 \cdots & & t_{r+1} \end{array} \right) \right) k = \left\{ \begin{array}{ll} \prod z_i^{\text{ord}(t_i)} & \text{if } n|\text{ord}(t_i) \text{ for } 1 \leq i \leq r+1, \\
 0 & \text{otherwise,} \end{array} \right.$$  

when $k \in K^*$. Let $i : N_-(F) \rightarrow \hat{G}(F)$ be the canonical splitting homomorphism, which satisfies $s(w_0)i(n)s(w_0)^{-1} = s(w_0\nu w_0^{-1})$ when $n \in N_-$, where $w_0$ is a representative of the long Weyl group element.

In the Introduction, $G_{11}$ was defined for $n = 1$. In [1], the definition [13.4] is given for general $n$. It is the same, except that [13.5] is generalized. We make use of the $n$-th order Gauss sum defined, with $\psi_0$ as in the Introduction, by

$$g(m, c) = \sum_{d \mod c \atop \text{gcd}(d, c) = 1} (d, c)\psi_0 \left( \frac{md}{c} \right).$$

Then with $\varpi$ a fixed prime element, we have $g(a) = g(\varpi^{a-1}, \varpi^a)$ and $h(a) = g(\varpi^a, \varpi^a)$. Since boxing does not occur for $B(\infty)$, the function $h$ is most relevant here, and it can be made explicit, as

$$h(a) = \left\{ \begin{array}{ll} (q-1)q^{a-1} & \text{if } n|a, \\
 0 & \text{otherwise.} \end{array} \right.$$  

We may now generalize Theorem [1.2] as follows.

**Theorem 3.1.** We have

$$\int_{N_-(F)} \tilde{f}^\circ(n) d\mu = \prod_{\alpha \in \Phi^+} \frac{1 - q^{-1}z^{n\alpha}}{1 - z^{n\alpha}} = \sum_{\mathcal{B}(\infty)} G_{11}(v)q^{\langle \text{wt}(v), \rho \rangle}z^{-\text{wt}(v)}.$$
Proof. The formula of Gindikin and Karpelevich in this context is
\[
\int_{\mathcal{N}_c(F)} \hat{f}^c(n) \, dn = \prod_{\alpha \in \Phi^+} \frac{1 - q^{-1}z_{\alpha}^{n\lambda}}{1 - z_{\alpha}^{n\lambda}},
\]
and it is the same as Proposition I.2.4 of Kazhdan and Patterson \[9\]. Another proof, closely related to our point of view in this paper, can be found in MacNamara \[12\].

We will prove the second equality. With \(v \in \mathcal{B}(\infty)\) and with \(b_i\) as in \[12\] we have \((\text{wt}(v),\rho) = -\sum b_i\). Thus
\[
\sum_{\mathcal{B}(\infty)} G_{\Omega}(v) q^{(\text{wt}(v),\rho)} z^{-\text{wt}(v)} = \sum_{\mathcal{B}(\infty)} G'_{\Omega}(v) z^{-\text{wt}(v)}
\]
where (since boxing does not occur for \(\mathcal{B}(\infty)\)) we have
\[
G'_{\Omega}(v) = \prod_{i=1}^{N} \begin{cases} q^{-b_i}h(b_i) & \text{if } b_i \text{ is not circled,} \\ 1 & \text{if } b_i \text{ is circled.} \end{cases}
\]

Using \[9\], \(G'_{\Omega}(v) = (1 - q^{-1})^{s(v)}\) where \(s(v)\) is the number of \(b_i\) that are not circled, provided that these uncircled \(b_i\) are all multiples of \(n\), while \(G'_{\Omega}(v) = 0\) if any \(b_i\) that is not circled is not a multiple of \(n\). Thus we must show that
\[
\prod_{\alpha \in \Phi^+} \frac{1 - q^{-1}z_{\alpha}^{n\lambda}}{1 - z_{\alpha}^{n\lambda}} = \sum_{\text{BZL}(v) = (b_1, \ldots, b_N)} (1 - q^{-1})^{s(v)} z^{-\text{wt}(v)}.
\]

Now we argue that this may actually be written as
\[
(3.3) \quad \prod_{\alpha \in \Phi^+} \frac{1 - q^{-1}z_{\alpha}^{n\lambda}}{1 - z_{\alpha}^{n\lambda}} = \sum_{\text{BZL}(v) = (b_1, \ldots, b_N)} (1 - q^{-1})^{s(v)} z^{-\text{wt}(v)}.
\]

Thus we claim that if \(n|b_i\) for all uncircled \(b_i\), then \(n\) divides all \(b_i\), whether circled or not. Indeed, if \(b_i\) is circled, then either it is zero (and hence a multiple of \(n\)) or \(b_i = b_{i+1}\). If \(b_{i+1}\) is circled, then \(n|b_{i+1}\), so \(n|b_i\), and the claim is proved; otherwise, we may repeat the argument. We have \(b_i = b_{i+1} = \ldots = b_j\) and the last \(b_j\) is uncircled, so \(n|b_j\) and therefore \(n|b_i\). (This observation also appears as the “Circling Lemma” in \[1\].) Thus we are reduced to proving \[3.3\].

Now Kashiwara \[8\] proved a similarity property of crystals: let \(\lambda\) be a dominant weight. Then there exists a similarity map, which we will denote by \(n : \mathcal{B}_\lambda \rightarrow \mathcal{B}_{n\lambda}\), such that \(\text{wt}(n \cdot v) = n \cdot \text{wt}(v)\) and \(f_{\alpha}^n(n \cdot v) = n \cdot (f_{\alpha} v)\). It follows from the description of \(\mathcal{B}(\infty)\) that there exists a corresponding similarity map \(n : \mathcal{B}(\infty) \rightarrow \mathcal{B}(\infty)\), and we may summarize what we have learned by saying that the right hand side of \[3.2\] is the sum over \(v\) in the image of the similarity map. Pulling the sum back to \(\mathcal{B}(\infty)\) through the similarity map, we may now apply Theorem \[12\] (with \(z^n\) replacing \(z\)), since that theorem proves \[8\] in the \(n = 1\) case. \(\square\)

References


License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use