ON MEMS EQUATION WITH FRINGING FIELD

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Abstract. We consider the MEMS equation with fringing field

\[-\Delta u = \lambda (1 + \delta |\nabla u|^2)(1 - u)^{-2} \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]

where \(\lambda, \delta > 0\) and \(\Omega \subset \mathbb{R}^n\) is a smooth and bounded domain. We show that when the fringing field exists (i.e. \(\delta > 0\)), given any \(\mu > 0\), we have a uniform upper bound of classical solutions \(u\) away from the rupture level 1 for all \(\lambda \geq \mu\).

Moreover, there exists \(\lambda^*_\delta > 0\) such that there are at least two solutions when \(\lambda \in (0, \lambda^*_\delta)\); a unique solution exists when \(\lambda = \lambda^*_\delta\); and there is no solution when \(\lambda > \lambda^*_\delta\). This represents a dramatic change of behavior with respect to the zero fringing field case (i.e., \(\delta = 0\)) and confirms the simulations in a paper by Pelesko and Driscoll as well as a paper by Lindsay and Ward.

1. Introduction

We consider the elliptic equation

\((E_\lambda) \quad -\Delta u = \frac{\lambda (1 + \delta |\nabla u|^2)}{(1 - u)^2} \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\)

where \(\delta, \lambda\) are positive constants, and \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^n\) \((n \geq 2)\).

Problem \((E_\lambda)\) arises in the study of electrostatic Micro-Electromechanical System (MEMS) devices. We refer to [5] and the book [13] for detailed discussions on MEMS devices modeling. The parameter \(\lambda\) is called the voltage and the term \(\delta |\nabla u|^2\) is called a fringing field (cf. [14]). The eventual singular set \(\{x \in \Omega, u(x) = 1\}\) is called a rupture set. When \(\delta = 0\), problem \((E_\lambda)\) becomes

\((S_\lambda) \quad -\Delta u = \frac{\lambda}{(1 - u)^2} \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.\)

Recently there have been many studies on \((S_\lambda)\). We summarize some of the results here:

- There exists a critical number \(\lambda^* > 0\) such that for \(0 < \lambda < \lambda^*\) problem \((S_\lambda)\) has a minimal stable solution \(u^*_\lambda\), while for \(\lambda > \lambda^*\) there are no solutions to \((S_\lambda)\) (see [6]).
• Either the solution branch stops at $\lambda^*$ and $\lim_{\lambda \to \lambda^*} \| \nabla u \|_{\infty} = 1$ (if $\Omega$ is a ball in $\mathbb{R}^n$ with $n \geq 8$ for example) or the solution branch bends back and we could have another critical parameter $0 < \lambda < \lambda^*$ (when $\Omega$ is a ball in $\mathbb{R}^n$ with $2 \leq n \leq 7$ or a convex domain with two axes of symmetry in $\mathbb{R}^2$) such that the solution branch takes infinitely many turns and converges to a rupture solution of $(S_{\lambda^*})$ (see [4], [10]).

• For general strictly convex domains with $n \geq 2$, it can be shown that for $\lambda > 0$ small, the minimal solution is the unique one for $(S_{\lambda})$ (see [3], [14]).

So we must have a family of solutions $(u^k, \lambda^k)$ such that $\lim_{k \to \infty} \lambda^k = \lambda > 0$ and $\lim_{k \to \infty} \| u^k \|_{\infty} = 1$.

In this short paper, we show that the fringing field dramatically changes the structure of solutions of $(E_{\lambda})$ (see Theorem 2 below): we prove that there exists a critical parameter $\lambda^*_{\delta}$ such that for $\lambda > \lambda^*_{\delta}$ there are no solutions to $(E_{\lambda})$, for $0 < \lambda < \lambda^*_{\delta}$ there are at least two solutions, and when $\lambda = \lambda^*_{\delta}$ there exists a unique solution. Furthermore, for any fixed $\mu > 0$, all solutions to $(E_{\lambda})$ with $\lambda \geq \mu$ are below $C_{\mu} < 1$; i.e., no ruptures can occur by using solutions with $\lambda$ tending to some $\lambda > 0$. Our study holds for any dimension and confirms the numerical results obtained in [14], [11]. Here all solutions considered are classical solutions.

The results of this paper are also true for the generalized MEMS equation

$$(E_{\lambda,p}) \quad -\Delta u = \frac{\lambda(1 + \delta|\nabla u|^2)}{(1-u)^p} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

where $p > 1$.

2. A KEY TRANSFORMATION

To study the structure of solutions for $(E_{\lambda})$, we present a suitable transformation, which leads to considering a semilinear equation. More precisely, we have

**Lemma 1.** Let

$$v = \zeta_{\lambda}(u) = \int_0^u e^{\frac{\lambda s}{1-s}} ds, \quad \forall u \in [0,1).$$

Then $u : \Omega \to [0,1)$ is a solution (resp. supersolution, subsolution) of $(E_{\lambda})$ if and only if $v$ is a solution (resp. supersolution, subsolution) for

$$(F_{\lambda}) \quad -\Delta v = \rho_{\lambda}(v), \quad v > 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega,$$

where $\rho_{\lambda}$ is a smooth increasing function from $\mathbb{R}_+$ into $(0, \infty)$, defined by

$$\rho_{\lambda}(v) = \xi_{\lambda} \circ \xi_{\lambda}^{-1} \quad \text{with} \quad \xi_{\lambda}(u) = \frac{\lambda e^{\frac{\lambda u}{1-u}}}{(1-u)^2}.$$ 

**Proof.** As $\xi_{\lambda}$, $\lambda$ are increasing in $[0,1)$ and $\lim_{u \to 1^-} \xi_{\lambda}(u) = \infty$, so $\rho_{\lambda}$ is also increasing in $\mathbb{R}_+$. By direct calculus, $v = \zeta_{\lambda}(u)$ satisfies

$$-\Delta v = -e^{\frac{\lambda u}{1-u}} \Delta u - \frac{\lambda \delta e^{\frac{\lambda u}{1-u}} |\nabla u|^2}{(1-u)^2};$$

all conclusions are straightforward.

Otherwise, it is not difficult to prove the following.
Theorem 1. Fixing $\delta > 0$, there exists $\lambda_\delta \in (0, \infty)$ such that for any $\lambda < \lambda_\delta$, the equation $(E_\lambda)$ has a minimal solution $u_\lambda$, while for any $\lambda > \lambda_\delta$, no solution exists for $(E_\lambda)$. Moreover $\lambda \mapsto u_\lambda$ is increasing for $\lambda \in (0, \lambda_\delta)$.

Here the minimal solution means that for any solution $u$ to $(E_\lambda)$, we have $u_\lambda \leq u$ in $\Omega$.

Proof. The result is a direct consequence of the following claims:

(i) If $(E_\lambda)$ is solvable with $\lambda > 0$, then $(E_{\lambda'})$ is solvable for any $\lambda' \in (0, \lambda)$.
(ii) The equation $(E_\lambda)$ has no solution for $\lambda$ sufficiently large.
(iii) For $\lambda > 0$ small enough, we have a solution to $(E_\lambda)$.
(iv) If $(E_\lambda)$ is solvable, then there exists a minimal solution $u_\lambda$.

If $u$ is a solution to $(E_\lambda)$, it is clearly a supersolution to $(E_{\lambda'})$, so $v = \zeta_\lambda(u)$ is a supersolution to $(E_{\lambda'})$ by Lemma 1. As $\rho_\lambda(0) = \lambda'e^{\lambda\delta} > 0$, 0 is always a subsolution. Moreover $\rho_{\lambda'}$ is locally Lipschitz in $\mathbb{R}_+$, so we have a solution to $(F_{\lambda'})$, which yields the claim (i).

The claim (ii) comes from the fact that any solution of $(E_\lambda)$ is a supersolution for the equation $(S_\lambda)$, which has no solution for large $\lambda$. Let $-\Delta \xi = 1$ in $\Omega$ and $\xi = 0$ on $\partial\Omega$, and fix $c > 0$ such that $c\xi||\xi||_\infty < 1$. We can check that $c\xi$ is a supersolution of $(E_\lambda)$ if $\lambda > 0$ is small enough; this leads to the claim (iii).

The last claim is due to the monotonicity of $\rho_\lambda$ (cf. (1) below), $\zeta_\lambda$ and the monotone iteration for $(F_\lambda)$ as $-\Delta v^{n+1} = \rho_\lambda(v^n)$ with Dirichlet boundary condition and $v^0 \equiv 0$.

Remark 1. Of course, the transformation $v = \zeta_\lambda$ is not really necessary for the above proof. Thanks to the monotonicity of the function $g(u) = (1 - u)^{-2}$, we can consider directly the iteration operator $w = Tu$, the unique solution of

$$-\Delta w = \frac{\lambda(1 + \delta|\nabla u|^2)}{(1 - u)^2} \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

3. Stability of minimal solutions

The minimal solution for $(E_\lambda)$ will ensure some stability properties, even though the equation $(E_\lambda)$ does not have a variational structure. First, for the linearized operator of $(E_\lambda)$,

$$L_\lambda \varphi = -\Delta \varphi - \frac{2\lambda(1 + \delta|\nabla u|^2)}{(1 - u)^3} \varphi - \frac{2\lambda\delta \nabla u \nabla \varphi}{(1 - u)^2},$$

we can define the principal eigenvalue $\mu_1$ of $L_\lambda$, associated to the Dirichlet boundary condition (cf. [12]). Then a solution $u$ of $(E_\lambda)$ is said to be stable if and only if $\mu_1(L_\lambda) \geq 0$. Another idea is to use the transformation $v = \zeta_\lambda(u)$ and the corresponding linearized operator. Following the ideas in [1], we obtain

Theorem 2. Letting $\lambda \in (0, \lambda_\delta)$, the minimal solution $v_\lambda$ of $(F_\lambda)$ satisfies

$$(3) \quad \int_\Omega |\nabla \varphi|^2 \geq \int_\Omega \rho_\lambda(v_\lambda) \varphi^2 dx, \quad \forall \varphi \in H_0^1(\Omega).$$

Furthermore, $v_\lambda$ is the unique solution of $(F_\lambda)$ satisfying (3), and $u_\lambda$ is the unique stable solution of $(E_\lambda)$. 

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Moreover, \( u = \zeta_\lambda^{-1}(v) \) implies that
\[
\rho'_\lambda(v) = (\xi_\lambda \circ \zeta_\lambda^{-1})'(v) = \frac{\xi_\lambda'}{\xi_\lambda} \cdot \zeta_\lambda^{-1}(v) = \frac{\lambda^2 \delta}{(1-u)^2} + \frac{2\lambda}{(1-u)^3} > 0.
\]
As the minimal solution \( u_\lambda \) of \((E_\lambda)\) is just \( \zeta_\lambda^{-1}(v_\lambda) \), we conclude then

**Theorem 3.** For \( \lambda \in (0, \lambda_0^*) \), the minimal solution \( u_\lambda \) is the unique solution of \((E_\lambda)\) satisfying the stability condition
\[
\int_\Omega |\nabla \varphi|^2 \geq \int_\Omega \left[ \frac{\lambda^2 \delta}{(1-u)^2} + \frac{2\lambda}{(1-u)^3} \right] \varphi^2 dx, \quad \forall \varphi \in H^1_0(\Omega).
\]

4. **Bifurcation and uniform estimate**

Using the equation \((F_\lambda)\) and the standard bifurcation theory of Rabinowitz (section 3 of [15]), we can say that a solution curve \((\lambda, v)\) exists in \( \mathbb{R}_+ \times C(\overline{\Omega}) \); it goes from \((0,0)\) to “infinity”. By Theorem [1] the only possibility is that \( \|v\|_\infty \) tends to \( \infty \). For \((F_\lambda)\), when \( \|v\|_\infty \to \infty \), we show that \( \lambda \) must tend to \( 0 \) by the following result.

**Theorem 4.** For any \( \mu > 0 \), there exists a constant \( C_\mu > 0 \) such that any solution of \((F_\lambda)\) with \( \lambda \geq \mu \) satisfies \( \|v\|_\infty < C_\mu \). Consequently, there exists \( c_\mu \in (0, 1) \) such that any solution \( u \) of \((E_\lambda)\) with \( \lambda \geq \mu \) is such that \( u \leq c_\mu < 1 \).

**Proof.** In fact, using integration by parts, we can see that
\[
v = \zeta_\lambda(u) \sim \frac{(1-u)^2}{\lambda \delta} c^{\frac{4}{\lambda}} \text{ as } u \to 1^-.\]
Hence for \( \mu \in (0, \lambda_0^*) \) fixed, there exist positive constants \( C, C' \) such that
\[C v (\ln v)^4 \leq \rho_\lambda(v) \leq C' v (\ln v)^4 \quad \forall (\lambda, v) \in [\mu, \lambda_0^*) \times [2, \infty).
\]
We also have the uniform estimate \( \rho_\lambda(v) \geq C v + \mu \) for \( (\lambda, v) \in [\mu, \lambda_0^*) \times \mathbb{R}_+ \). The proof of Theorem 2.1 in [2] holds and shows that there exists \( \bar{C}_\mu > 0 \) such that \( \|v\|_\infty < C_\mu < \infty \). The conclusion for \( u \) is an immediate consequence.

An important consequence is just the uniqueness of a solution for \((E_{\lambda_0^*})\). We shall use the problem \((F_\lambda)\). Now \( v^* = \lim_{\lambda \to \lambda_0^*} v_\lambda \) is a smooth solution for the limit problem \((F_{\lambda_0^*})\); we claim that \( \mu_1 \left[ -\Delta - \rho_{\lambda_0^*}(v^*) \right] = 0 \). In fact, the stability of \( v^* \) (in the sense of [3]) means that \( \mu_1 \left[ -\Delta - \rho_{\lambda_0^*}(v^*) \right] \geq 0 \), while the definition of \( \lambda_0^* \) prevents having \( \mu_1 \left[ -\Delta - \rho_{\lambda_0^*}(v^*) \right] > 0 \). Hence we get a positive eigenfunction \( \varphi_1 \) satisfying \( -\Delta \varphi_1 - \rho_{\lambda_0^*}(v^*) \varphi_1 = 0 \) in \( \Omega \) and \( \varphi_1 = 0 \) on \( \partial \Omega \).

If we have a solution \( v \) of \((F_{\lambda_0^*})\) such that \( v \neq v^* \), we know that \( v \geq v^* \) as \( v \geq v_\lambda \) for any \( \lambda < \lambda_0^* \). Letting \( \phi = v - v^* \), so that \( -\Delta \phi = \rho_{\lambda_0^*}(v) - \rho_{\lambda_0^*}(v^*) \geq 0 \) by (3), the strong maximum principle implies that \( \phi > 0 \) in \( \Omega \). Remarkably also that \( \rho''_\lambda > 0 \) in \( \mathbb{R}_+ \) for any \( \lambda > 0 \), then \( -\Delta \phi - \rho_{\lambda_0^*}(v^*) \phi > 0 \) in \( \Omega \). By multiplying with \( \varphi_1 \) and integrating by parts, we immediately get a contradiction.

Another consequence is that \( v^* \) is a bifurcation point for the solution curve, which will continue with \( \|v\|_\infty \) tending to \( \infty \) and the associated \( \lambda \) must go to zero. So we get at least two solutions to \((F_\lambda)\) for any \( \lambda \in (0, \lambda_0^*) \). Coming back to \( u \), we obtain the main theorem of this paper.
Theorem 5. If a family of solutions \( \{u^k\} \) of \( (E_{\lambda^k}) \) satisfies \( \lim_{k \to \infty} \|u^k\|_\infty = 1 \), then \( \lim_{k \to \infty} \lambda^k = 0 \). Furthermore, \( u^* = \lim_{\lambda \to \lambda^*_d} u_\lambda \) is the unique solution of the limit equation \( (E_{\lambda^*_d}) \) while for any \( \lambda \in (0, \lambda^*_d) \), the equation \( (E_\lambda) \) has at least two solutions.

5. **Estimate of \( \lambda^*_d \)**

Here we compare \( \lambda^*_d \) with \( \lambda^* \) in the lower dimension situation.

Theorem 6. For \( n < 8 \) and \( \delta > 0 \), we have

\[
\frac{\lambda^*}{1 + \delta \|\nabla u^*\|_\infty^2} \leq \lambda^*_d \leq \lambda^*,
\]

where \( \lambda^* \) is the critical value for the problem \((S_\lambda)\) and \( \pi_* \) is the unique solution of \((S_{\pi^*})\).

**Proof.** As any solution of \( (E_\lambda) \) is a supersolution of \((S_\lambda)\), it is clear that \( \lambda^*_d \leq \lambda^* \). On the other hand, when \( n < 8 \), \( \pi_* \) is a smooth function with \( \|\pi_*\|_\infty < 1 \) (see [3]). Obviously \( \pi_* \) is a supersolution for \((E_\lambda)\) with

\[
\lambda = \frac{\lambda^*}{1 + \delta \|\nabla u^*\|_\infty^2},
\]

so we get the lower bound. \( \square \)

Therefore \( \lambda^*_d = \lambda^* + O(\delta) \) in dimension two. This confirms somehow the formal result in [11] (see also another bound of \( \lambda^*_d \) in section 5 of [14]).

6. **Remarks and open questions**

As we have seen in Theorem 5, the introduction of a fringing field basically destroys the infinite fold point structure of the basic membrane problem \((S_\lambda)\) for any smooth domain.

There are still some interesting questions:

- Do we have some weak solutions with \( \|u\|_\infty = 1 \) for \((E_\lambda)\)? We turn to conjecture that no weak solution exists for the fringing field model. In fact, using Sobolev embedding and a boot-strap argument, any weak solution of \((F_\lambda)\) satisfying \( \rho_\lambda(v) \in L^1(\Omega) \) is indeed smooth. However, if \( u \) is just a weak solution for \((E_\lambda)\), it is not clear that \( v = \rho_\lambda(u) \) is then a weak solution for \((F_\lambda)\).
- In [11], Lindsay and Ward derived the following asymptotic behavior of \( \lambda^*_d \):

\[
\lambda^*_d = \lambda^* - C\delta + O(\delta^2)
\]

in the case of a unit disk or a slab in \( \mathbb{R}^2 \), where \( C > 0 \) is a constant depending on \( \pi_* \) of the unit disk or slab without the fringing field. Can we prove rigorously this first-order expansion \( (7) \)? A key point seems to prove a uniform upper bound for \( v^* \) as \( \delta \) tends to zero.
- In nice domains (disks, convex domains with two axes of symmetry in \( \mathbb{R}^2 \)), it has been shown that for the problem \((S_\lambda)\), there exists a \( \lambda_* > 0 \) such that
the solution branch has infinitely many turns as \( \lambda \) crosses \( \lambda_* \) (see [9, 10]). On the other hand, in the presence of a fringing field, there are at most finitely many turns. What is the asymptotic behavior of the solutions near \( \lambda_* \) as \( \delta \to 0^+ \)?

- It seems that there are no studies on the corresponding parabolic equation

\[
(8) \quad u_t - \Delta u = \frac{\lambda(1 + \delta |\nabla u|^2)}{(1 - u)^2}.
\]

What is the effect of the fringing field on \( (8) \)? Can we establish results similar to [1, 7, 8]?

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