A CAUCHY-RIEMANN EQUATION FOR GENERALIZED ANALYTIC FUNCTIONS

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Abstract. We denote by $T^2$ the torus: $z = \exp i\theta, w = \exp i\phi$, and we fix a positive irrational number $\alpha$. $A_\alpha$ denotes the space of continuous functions $f$ on $T^2$ whose Fourier coefficient sequence is supported by the lattice half-plane $n + m\alpha \geq 0$. R. Arens and I. Singer introduced and studied the space $A_\alpha$, and it turned out to be an interesting generalization of the disk algebra. Here we construct a differential operator $X_\Sigma$ on a certain 3-manifold $\Sigma_0$ such that $X_\Sigma$ characterizes $A_\alpha$ in a manner analogous to the characterization of the disk algebra by the Cauchy-Riemann equation in the disk.

1. Introduction

Let $\Gamma$ be the unit circle. The disk algebra $A$ on $\Gamma$ is the space of all continuous functions $f$ on $\Gamma$ such that the Fourier expansion of $f$ is:

$$\sum_{n=0}^{\infty} c_n \exp(in\theta);$$

i.e., the Fourier coefficient sequence of $f$ is supported on the semi-group $n \geq 0$ of $\mathbb{Z}$.

In [1], R. Arens and I.M. Singer studied the following generalization of the disk algebra: we replace $\Gamma$ by the 2-torus $T^2$ and fix a positive irrational number $\alpha$. The dual group of $T^2$ is $\mathbb{Z} \oplus \mathbb{Z}$. We replace the semi-group of nonnegative integers by the semi-group of all pairs of integers $(n,m)$ with $n + m\alpha \geq 0$. We define the algebra $A_\alpha$ as the space of continuous functions on $T^2$ with Fourier expansion on the torus given by

$$\sum_{n+m\alpha \geq 0} c_{nm} \exp(in\theta) \exp(im\phi).$$

$A_\alpha$ is called a space of Generalized Analytic Functions. In [4], H. Helson and D. Lowdenslager made a detailed study of $A_\alpha$ and showed that many basic results of analytic function theory on the unit disk extend from $A$ to $A_\alpha$.

An alternative description to the disk algebra is the following: $A$ consists of those functions $f$ continuous on $\Gamma$ which admit a continuous extension to the closed disk $\Delta$, again denoted by $f$, such that $f$ is smooth on the interior of $\Delta$ and there it
satisfies the equation 

\[ \frac{\delta}{\delta \bar{z}} (f) = 0. \]  

In [1] the disk \( \Delta \) is replaced by the maximal ideal space \( \Pi \) of the Banach algebra \( A_\alpha \), taken in the Gelfand topology. It is shown in [1] that \( \Pi \) has a natural identification with the following compact subset of \( C^2 \):

The set of all points \((z, w)\) in \( C^2 \) such that \(|w| = |z|^\alpha \) and \(|z| \leq 1 \).

We denote this subset of \( C^2 \) by \( \Sigma \). In this identification, \( T^2 \) turns into the set of all points \((z, w)\) in \( C^2 \) such that \(|z| = |w| = 1 \).

Our purpose is to give an equation analogous to (1) on the space \( \Sigma \setminus (T^2 \cup 0) \), which characterizes \( A_\alpha \). To this end we define the differential operator \( X \) on \( C^2 \) by \( X = i z \frac{\partial}{\partial z} + \alpha w \frac{\partial}{\partial w} \). As is shown below, \( X \) restricts to a well-defined differential operator on the smooth manifold: \( \Sigma \setminus T^2 \cup \{0\} \), which we denote by \( \Sigma_0 \).

**Theorem 1.1.** A function \( f \in \mathcal{C}(T^2) \) lies in \( A_\alpha \) if and only if \( f \) admits a continuous extension (denoted \( F \)) to \( \Sigma \) such that \( XF = 0 \), in the sense of distributions, on \( \Sigma \setminus (T^2 \cup 0) \).

**Theorem 1.2.** Given a point \((z_0, w_0) \in \Sigma \setminus (T^2 \cup 0) \), there exists some function \( f \in A_\alpha \) such that the extension \( F \) is not differentiable on \( \Sigma \) at that point.

2. **Proof of Theorem 1.1**

We put \( \Sigma_0 = \Sigma \setminus (T^2 \cup (0,0)) \).

Let \( \phi \) be the function on \( C^2 \setminus z = 0 \) given by \( \phi(z, w) = w\bar{w} - z^\alpha \bar{z}^\alpha \).

\( \Sigma \) has the equation: \( \phi(z, w) = 0 \). We write \( D_\bar{z} \) for the derivative with respect to \( \bar{z} \) and similarly for \( w \). \( X(\phi) = \alpha \phi \delta \) by direct calculation. So \( X(\phi) = 0 \) on \( \Sigma \).

Since \( \Sigma \) is given by the equation \( \phi = 0 \), it follows that the operator \( X \) is well-defined on \( \mathcal{C}_\infty(\Sigma) \). We denote this operator, which acts on functions defined on \( \Sigma \), by \( X_\Sigma \). We wish to express \( X_\Sigma \) in local coordinates on \( \Sigma \). Fix a point \((z_0, w_0) \) on \( \Sigma \). Then \( w_0 = z_0^\alpha \exp(i \theta_0) \) for some \( z_0, \theta_0 \) with \(|z_0| < 1, 0 \leq \theta_0 \leq 2\pi \). We define a neighborhood \( U \) of \((z_0, w_0) \) on \( \Sigma \) by:

\[ U = \langle t, \exp(i \theta) t^\alpha, |t - z_0| < \delta, |\theta - \theta_0| < \delta \rangle. \]

We fix a single-valued branch of \( t^\alpha \) near \( t = z_0 \).

We use \( t, \bar{t}, \theta \) as local coordinates in \( U \). Further, we denote the operator \( \frac{\delta}{\delta \bar{z}} \) by \( D_{\bar{t}} \).

**Claim 1.** \( \bar{t} D_{\bar{t}} = X_\Sigma \) as a differential operator on \( U \). Hence for a continuous function \( f \) on \( U \), \( t D_{\bar{t}} f = X_\Sigma f \), as a distribution on \( U \).

**Proof of Claim 1.** We apply both sides to the functions \( t, \bar{t}, \exp(i \theta), \exp(-i \theta) \). We note that \( t \) is the restriction of \( \bar{z} \) to \( \Sigma \). Since \( X = z D_{\bar{z}} + \alpha \bar{w} D_{\bar{w}} \), \( X_\Sigma(t) = \bar{t} \) on \( U \).

Next, \( \bar{t} \) is the restriction of \( \bar{z} \) to \( \Sigma \). So \( X_\Sigma(\bar{t}) = 0 \). Next, \( X_\Sigma(\exp(i \theta)) = X(\frac{i \theta}{\alpha}) = 0 \). Similarly, \( X_\Sigma(\exp(-i \theta)) = 0 \). On the other hand, \( \bar{t} D_{\bar{t}}(\exp(i \theta)) = 0, \bar{t} D_{\bar{t}}(\exp(-i \theta)) = 0 \).

So \( X_\Sigma \) and \( \bar{t} D_{\bar{t}} \) agree on each of the functions \( t, \bar{t}, \exp i \theta \) on \( U \). Also, \( 0 = X_\Sigma(\exp i \theta) = i \exp i \theta X_\Sigma(\theta) \), so \( X_\Sigma(\theta) = 0 \). Similarly, \( \bar{t} D_{\bar{t}}(\theta) = 0 \). It follows that for all \( G \) in \( \mathcal{C}_\infty(U) \) we have \( \bar{t} D_{\bar{t}}(G) = X_\Sigma(G) \). This proves our claim. \( \square \)
We next follow the Arens-Singer paper in introducing a foliation of the 3-manifold $\Sigma_0$ by a one-parameter family of Riemann surfaces $\Lambda_\theta$. We shall prove

**Theorem 2.1.** Fix $f$ in $C^\infty(\Sigma)$. Then $Xf = 0$ on $\Sigma$ if and only if the restriction of $f$ to $\Lambda_\theta$ is holomorphic on $\Lambda_\theta$ for each $\theta$.

We denote by $H^+$ the right half-plane: $\Re \zeta > 0$. For each $\theta \in [0, 2\pi]$ we put $\chi_\theta(\zeta) = (\exp(\zeta), \exp i\theta \exp(-\alpha \zeta))$, where $\zeta$ is in the closed right half-plane.

**Definition.** Fix $\theta$. $\Lambda_\theta$ is the image in $C^2$ of $H^+$ under the map $\chi_\theta$.

Since $|\exp i\theta \exp(-\alpha \zeta)| = |\exp(-\zeta)|^\alpha$, $\Lambda_\theta$ is a subset of $\Sigma$. The map $\chi_\theta$ is one-one from $H^+$ to $\Lambda_\theta$. We use this map to give $\Lambda_\theta$ the structure of a Riemann surface. We verify that for $\theta$ and $\theta'$ distinct points in $[0, 2\pi]$, the sets $\Lambda_\theta$ and $\Lambda_{\theta'}$ are disjoint.

For a function $g$ defined on $\Lambda_\theta$, we say that “$g$ is holomorphic on $\Lambda_\theta$” if the composition $\zeta \to g(\chi_\theta(\zeta))$ is holomorphic on $H^+$.

Now fix $f$ in $C(\Sigma)$ with $Xf = 0$ on $\Sigma$. Fix $\theta_0$. We must show that $f$, restricted to $\Lambda_{\theta_0}$, is holomorphic on $\Lambda_{\theta_0}$.

Let $(z_0, w_0)$ be a point on $\Lambda_{\theta_0}$. We fix a single-valued branch of the function $z^\alpha$ in a neighborhood $|z - z_0| < \delta$ and fix $\epsilon > 0$. Put $U_\epsilon = \{(z, z^\alpha \exp i\theta)| |z - z_0| < \delta, |\theta - \theta_0| < \epsilon\}$. Let $D$ be the disk $\{(z, z^\alpha \exp i\theta_0), |z - z_0| < \delta\}$, so $D \subset \Lambda_\theta$.

Choose a test function $\phi$ in $C^\infty(D)$, and extend $\phi$ to a smooth function $\tilde{\phi}_\epsilon$ in $C^\infty_0(U_\epsilon)$.

Since $Xf = 0$, by hypothesis, $Xf = 0$ on $U$ (where we suppress the subscript $\epsilon$). So by Claim 1, $\langle D_i (f) \phi \rangle = 0$ as a distribution on $U$. Therefore, $D_i (f) = 0$ as a distribution on $U$. So $\langle D_i (f), \tilde{\phi}_\epsilon \rangle = - \int_D f D_i \tilde{\phi}_\epsilon = - \int_D dt \wedge d\bar{\omega} \int_{\theta_0 - \epsilon}^{\theta_0 + \epsilon} f(t, \exp i\theta \alpha) D_i \tilde{\phi}_\epsilon d\theta$.

Since $D_i (f) = 0$ on $U$, we get for each $\epsilon > 0$: $0 = \int_D dt d\bar{\omega} \int_{\theta_0 - \epsilon}^{\theta_0 + \epsilon} f(t, \exp i\theta \alpha) D_i \tilde{\phi}_\epsilon d\theta$, where the integrand of the inner integral is evaluated at $(t, \exp i\theta \alpha)$. As $\epsilon$ approaches zero, we get in the limit

$$0 = \int_D f(t, \exp i\theta_0 \alpha) D_i \phi(t, \exp i\theta_0 \alpha) dt \wedge d\bar{\omega} = \langle f, D_i \phi \rangle.$$

So $D_i (f) = 0$, since this holds for every test function $\phi$ on $D$. Since $D$ is an arbitrary small disk on $\Lambda_{\theta_0}$, $D_i (f) = 0$ as a distribution on $\Lambda_0$. By Weyl’s Lemma, then, $f$, restricted to $\Lambda_{\theta_0}$, is holomorphic on $\Lambda_{\theta_0}$.

Conversely, fix $f \in C(\Sigma_0)$ such that $f$ restricted to $\Lambda_\theta$ is holomorphic on $\Lambda_\theta$ for each $\theta$. We must show that $Xf = 0$ on $U$, where we write $X$ for $X_\Sigma$.

Fix $(z_0, w_0)$ in $\Sigma_0$. Thus $w_0 = z_0^\alpha \exp i\theta_0$, for some $\theta_0$. We choose a branch of the function $z^\alpha$ and also fix $b > 0$, and form the set

$$U_b = \{(z, z^\alpha \exp i\theta)| |z - z_0| < b, |\theta - \theta_0| < b\}.$$

We claim that $D_i f = 0$ on $U_b$. Choose a test function $\phi$ on $U_b$. We define

$$I = \int_{U_b} f D_i \phi d\bar{\omega} d\theta = \langle D_i f, \phi \rangle.$$

We choose a sequence of smooth functions $\{f_n\}$ on $U_b$ such that for each $n$ the restriction of $f_n$ to $\Lambda_\theta$ is holomorphic on $\Lambda_\theta$ for each $\theta$ in $[\theta_0 - b, \theta_0 + b]$ and $f_n$ converges to $f$ uniformly on $U_b$ as $n \to \infty$ We fix $n$. Put

$$I_n = \int_{U_b} f_n D_i \phi d\bar{\omega} d\theta = - \int_{U_b} D_i f_n \phi d\bar{\omega} d\theta.$$
Since \( f_n \) is holomorphic on \( \Lambda_\theta \) for each \( \theta \), \( I_n \) vanishes. Letting \( n \to \infty \), we have \( I_n \to I \). So \( I = 0 \).

This holds for all test functions \( \phi \) on \( U_b \). So \( D_\ell f = 0 \) as a distribution on \( U_b \). Since \( Xf = ID_\ell f \) on \( U_b \), then \( Xf = 0 \) on \( U_b \). Since \( (z_0, w_0) \) is an arbitrary point on \( \Sigma_0 \), \( Xf = 0 \) on \( \Sigma_0 \).

Theorem 2.1 is proved. We now proceed to the proof of Theorem 1.1.

Proof. For \( f \) in \( C(T^2) \), we put \( ||f|| = \max |f| \), taken over \( T^2 \). We define \( \mathcal{A} = \{ f \in C(T^2) \} \) such that \( f \) has a continuous extension to \( \Sigma \), denoted \( F \), with \( XF = 0 \) on \( \Sigma_0 \), in the sense of distributions. Fix \( f \) in \( \mathcal{A} \). By Theorem 2.1, then, \( F \), restricted to \( \Lambda_\theta \), is holomorphic on \( \Lambda_\theta \) for each \( \theta \); i.e., \( F(\chi_\theta) \) is holomorphic on \( H^+ \), where \( \chi_\theta \) was defined above. Also, since \( F \) is continuous on the compact set \( \Sigma \), \( F(\chi_\theta) \) is bounded on \( H^+ \). Finally, for \( \zeta = it \), \( t \) real, where \( \chi_\theta(\zeta) = (\exp it, \exp i\theta \exp -i\alpha t) \in T^2 \), \( |F(\chi_\theta(\zeta))| \leq ||f|| \).

By the Phragmén-Lindelöf theorem, then, \( |F(\chi_\theta(\zeta))| \leq ||f|| \) for all \( \zeta \in H^+ \), so \( F \) can be holomorphic on \( \mathcal{A} \). Since \( \chi_\theta \) is bounded on \( \Sigma \), the functions in \( \mathcal{A} \), viewed on \( \Sigma \), satisfy the maximum principle relative to \( T^2 \). We note that \( \mathcal{A} \) is a linear space of functions.

Claim 2. \( \mathcal{A} \) is closed under uniform convergence on \( T^2 \).

Proof of Claim 2. Let \( \{f_n\} \) be a sequence of functions in \( \mathcal{A} \) which converges uniformly on \( T^2 \) to a function \( f \). Fix (\( z_0, w_0 \)) \( \in \Sigma \). For each of the indices \( n, m \), we have
\[
|F_n(z_0, w_0) - F_m(z_0, w_0)| \leq ||f_n - f_m||,
\]
since \( f_n - f_m \in \mathcal{A} \). Hence as \( n, m \) tend to \( \infty \), \( F_n \) converges uniformly on \( \Sigma \), to some continuous function \( F \), and \( F = f \) on \( T^2 \). Furthermore, for each of the Riemann surfaces \( \Lambda_\theta \), each \( F_n \) is holomorphic. Hence \( F \) is holomorphic on \( \Lambda_\theta \). By Theorem 2.1, then, \( F \) satisfies \( XF = 0 \) on \( \Sigma_0 \). So \( f \) again belongs to \( \mathcal{A} \). This was the claim. \( \square \)

Claim 3. \( \mathcal{A} \) is an algebra of functions on \( T^2 \).

Proof of Claim 3. Let \( f, g \in \mathcal{A} \), and let \( F, G \) be their corresponding extensions to \( \Sigma \). Since \( F \) and \( G \) are continuous on \( \Sigma \), so is \( FG \), and since \( F \) and \( G \) are each holomorphic on \( \Lambda_\theta \) for each \( \theta \), so is \( FG \). Hence by Theorem 2.1, \( X(FG) = 0 \) on \( \Sigma_0 \). Also \( FG \) is a continuous extension of \( fg \) from \( T^2 \) to \( \Sigma \). So \( fg \) lies in \( \mathcal{A} \). Claim 3 is proved. \( \square \)

Claim 4. \( \mathcal{A} \) contains \( A_\alpha \).

Proof of Claim 4. By Fejér’s theorem, \( A_\alpha \) is the closed span in \( C(T^2) \) of the set of functions \( \phi_{n,m} = \exp in\theta \exp im\phi, n + m\alpha > 0 \). Fix \( n, m \) with \( n + m\alpha > 0 \). We claim that \( \phi_{n,m} \) lies in \( \mathcal{A} \). With \( z, w \) the complex coordinates in \( C^2 \), we consider the extension \( z^n w^m \) of \( \phi_{n,m} \) to \( \Sigma \). The continuity is clear except at the origin. For \( z, w \in \Sigma_0 \),
\[
|z^n w^m| = |z|^n |w|^m = |z|^{n+m\alpha}.
\]
As \( (z, w) \to (0, 0) \), this tends to 0. So \( z^n w^m \) provides a continuous extension of \( \phi_{n,m} \) to \( \Sigma \). Further, \( X(z^n w^m) = 0 \) on \( \Sigma_0 \), since \( z^n w^m \) extends to be holomorphic in a neighborhood of \( \Sigma_0 \) in \( C^2 \). So \( z^n w^m \) provides the desired extension of \( \phi_{n,m} \), and so \( \phi_{n,m} \in \mathcal{A} \). Since \( A_\alpha \) is the closed span of the \( \phi_{n,m} \) in \( C(T^2) \), \( A_\alpha \) is contained in \( \mathcal{A} \). Claim 4 is proved. \( \square \)
By Claims 1 and 2, we know that $\mathcal{A}$ is closed under uniform convergence on $T^2$ and is an algebra of functions on $T^2$. By Claim 3, $\mathcal{A}$ contains $A_\alpha$. Theorem 2.3 in Chapter 7 of T.W. Gamelin’s book [2] gives that $A_\alpha$ is a maximal subalgebra of $C(T^2)$; i.e., no closed subalgebra of $C(T^2)$ lies properly between $A_\alpha$ and $C(T^2)$. So $A_\alpha = \mathcal{A}$. Theorem 1.1 is proved.

We proceed to the proof of Theorem 1.2.

**Proof.** We use the earlier notation.

**Claim.** There exist integers $p_j, q_j \in \mathbb{Z}^+$, $j = 1, 2, \ldots$, such that

1. $-p_j + \alpha q_j > 0$ for all $j$, and
2. $-p_j + \alpha q_j \to 0$, as $j \to \infty$.

**Proof of Claim.** A classical fact from the theory of continued fractions (see Hardy and Wright [3], Chapter X) gives the existence of a sequence of rational numbers $\frac{p_j}{q_j}$ such that

3. $|\alpha - \frac{p_j}{q_j}| < \frac{1}{q_j^2}$, $j = 1, 2, \ldots$ such that $p_j$ and $q_j$ are positive integers tending to $\infty$ as $j \to \infty$, and $\frac{p_j}{q_j} < \alpha$ for each $j$.

Thus for each $j$, we have $\alpha = \frac{p_j}{q_j} + \delta_j$, with $0 < \delta_j < \frac{1}{q_j}$. It follows that we have $-p_j + \alpha q_j = q_j \delta_j$. In view of the bound on $\delta_j$, then, we have (1) and (2). So the Claim is proved.

Let $\{\epsilon_n\}$ be a sequence of real numbers tending to 0. Fix a point $(z_0, \theta_0)$ in $\Sigma$. We now define a sequence of bounded linear functionals $L_n$ on $A_\alpha$, as follows:

For $f$ in $A_\alpha$, and $F$ denoting the extension of $f$ to $\Sigma$, we put

$$L_n f = (\epsilon_n)^{-1}(F(z_0, \exp i\epsilon_n z_0^\alpha) - F(z_0, \theta_0)).$$

Let $p_j, q_j$ be as in the Claim. Define $f_j = \exp -ip_j \theta \exp iq_j \theta$ on $T^2$. Since $-p_j + \alpha q_j > 0$, by (1), $f_j \in A_\alpha$. Further, for $(z, w) \in \Sigma, F_j(z, w) = z^{-p_j} w^{q_j}$. So

$$L_n f_j = (\epsilon_n^{-1})(z_0^{-p_j} \exp i\epsilon_n z_0^\alpha q_j - z_0^{-p_j} \exp i\epsilon_n \theta_0 q_j) = (\epsilon_n^{-1})(z_0^{-p_j} z_0^{q_j}(z_0^\alpha q_j) - \exp i\epsilon_n \theta_0 q_j - 1) = (z_0^{-p_j} z_0^{q_j})(\exp i\epsilon_n \theta_0 q_j - 1).$$

We now take $j = n$ and take absolute values. We get

$$|L_n(f_n)| = |(z_0^{-p_n + \alpha q_n})(\epsilon_n^{-1})(\exp i\epsilon_n \theta_0 q_n - 1)|.$$ 

We next take $\epsilon_n = \frac{z_0^\alpha}{q_n}$. This gives $|L_n(f_n)| = |z_0|^{-p_n + \alpha q_n}(\frac{z_0^\alpha}{q_n}) q_n$. Since $-p_n + \alpha q_n \to 0$ and $q_n \to \infty$ as $n \to \infty$, $|L_n(f_n)| \to \infty$ as $n$ approaches $\infty$. Also, $||f_n|| = 1$ for each $n$. So the norm $||L_n||$, as a functional on $A_\alpha$, becomes unbounded as $n$ grows.

Next, we fix $f \in A_\alpha$ and the point $x_0 = (z_0, \theta_0)$. $F$ denotes the extension of $f$ to $\Sigma$. We put $\Psi(\theta) = F(z_0, \exp i\theta z_0^\alpha), -\pi \leq \theta \leq \pi$. Then

$$L_n f = (\epsilon_n^{-1})(F((z_0, \exp i\epsilon_n z_0^\alpha)) - F(z_0, \theta_0)) = (\epsilon_n^{-1})(\Psi(\epsilon_n) - \Psi(0)).$$

Suppose now that $F$ is differentiable on $\Sigma$ at $x_0$. Then $\Psi$ is differentiable at $\theta = 0$. Hence by the preceding equality, the sequence $\{L_n(f)\}$ converges as $n$ approaches $\infty$. Since $L_n$ converges pointwise on the Banach space $A_\alpha$, the uniform boundedness theorem yields that the sequence $\{L_n\}$ is bounded. This contradicts our earlier result. So for some $f \in A_\alpha, F$ fails to be differentiable at the given point. Theorem 1.2 is proved. \[\square\]
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