THE PINCHING CONSTANT OF MINIMAL HYPERSURFACES
IN THE UNIT SPHERES

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ABSTRACT. In this paper, we prove that if $M^n (n \leq 8)$ is a closed minimal hypersurface in a unit sphere $S^{n+1}(1)$, then there exists a positive constant $\alpha(n)$ depending only on $n$ such that if $n \leq S \leq n + \alpha(n)$, then $M$ is isometric to a Clifford torus, where $S$ is the squared norm of the second fundamental form of $M$.

1. Introduction

Let $M^n$ be an $n$-dimensional closed minimal hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n + 1$. Denote by $S$ the squared norm of the second fundamental form of $M^n$. Lawson [2], Simons [1] and Chern, do Carmo, Kobayashi [3] obtained independently the famous rigidity theorem, which says that if $S \leq n$, then $S \equiv 0$ or $S \equiv n$; i.e., $M^n$ is the great sphere $S^n(1)$ or the Clifford torus. Further discussions in this direction have been carried out by many other authors [6, 8, 9, 10, 11, 12, 13]. In [4], Peng and Terng proved that if the scalar curvature of $M^n$ is constant, then there exists a positive constant $\alpha(n)$ depending only on $n$ such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$. Later, Cheng and Yang [14] improved the pinching constant $\alpha(n)$ to $n/3$. More generally, Peng and Terng [5] proved that if $M^n (n \leq 5)$ is a closed minimal hypersurface in $S^{n+1}$, then there exists a positive constant $\alpha(n)$ depending only on $n$ such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$. So they proposed the following problem.

Let $M^n (n \geq 6)$ be a closed minimal hypersurface in $S^{n+1}$. Does there exist a positive constant $\alpha(n)$ depending only on $n$ such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$?

In [15], Cheng gives a positive answer under the additional condition that $M$ has only two distinct principal curvatures. Later, Cheng and Ishikawa [6] improved the result of Peng and Terng [5] when $n \leq 5$.

In this paper, we solve the problem proposed by Peng and Terng [5] for $n \leq 8$.

Theorem 1.1. Let $M^n (n \leq 8)$ be a closed minimal hypersurface in $S^{n+1}(1)$. If $n \leq S \leq n + \alpha(n)$, then $S \equiv n$ and $M^n$ is isometric to a Clifford torus $S^m (\sqrt{\frac{2}{n}}) \times S^{n-m} (\sqrt{\frac{n-m}{n}})$, where $\alpha(n) = \frac{2(n+4)(3-n\delta)}{9n+30}$, $\delta(3) = 0$, $\delta(4) = 0.16$, $\delta(5) = 0.23$, $\delta(6) = 0.28$, $\delta(7) = 0.32$ and $\delta(8) = 0.34$.
For $n \leq 5$, Cheng and Ishikawa [6] proved the following: Let $M$ be an $n$-dimensional ($n \leq 5$) closed minimal hypersurface of a unit sphere $S^{n+1}(1)$. If $n \leq S \leq n + \alpha(n)$, then $S = n$, where $\alpha(3) = \frac{42}{85}$, $\alpha(4) = \frac{8}{31}$ and $\alpha(5) = \frac{3(21 - 5\sqrt{17})}{28 + 3\sqrt{17}}$. It is obvious that our pinching constant is larger than theirs. Up to now, the open problem for $n \geq 9$ is still open and it is a very hard problem.

2. Fundamental formulas

Let $M^n$ be an $n$-dimensional hypersurface in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame field $e_1, \cdots, e_n$ in $S^{n+1}(1)$, restricted to $M^n$, so that $e_1, \cdots, e_n$ are tangent to $M^n$. Let $\omega_1, \cdots, \omega_n$ denote the dual coframe field in $S^{n+1}(1)$. Then in $M^n$, $\omega_{n+1} = 0$. It follows from Cartan’s Lemma that

$$\omega_{n+1} = \sum_j h_{ij} \omega_j,$$

The second fundamental form $\alpha$ and the mean curvature $H$ of $M^n$ are defined by

$$\alpha = \sum_{i,j} h_{ij} \omega_i \omega_j e_{n+1}, \quad nH = \sum_i h_{ii},$$

respectively. If $M^n$ is a minimal hypersurface, then $\sum_i h_{ii} = 0$. The connection form $\omega_{ij}$ is characterized by the structure equations

$$d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij},$$

$$\Omega_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

where $\Omega_{ij}$ (resp. $R_{ijkl}$) denotes the curvature form (resp. the components of the curvature tensor) of $M^n$. The Gauss equation is given by

$$R_{ijkl} = (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + (h_{ik} h_{jl} - h_{il} h_{jk}).$$

Denote by $h_{ijk}, h_{ijkl}, h_{ijklm}$ the components of the first, second and third covariant derivatives of the second fundamental form, respectively. Then

$$h_{ij} = h_{kij} = h_{jik},$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im} R_{mjkl} + \sum_m h_{mj} R_{mikl},$$

$$h_{ijklm} - h_{ijkml} = \sum_r h_{rjk} R_{rilm} + \sum_r h_{irk} R_{rjlm} + \sum_r h_{ijr} R_{rklm}.$$

For any fixed point $p$ in $M^n$, we take a local orthonormal frame field $e_1, \cdots, e_n$ such that

$$h_{ij} = \left\{ \begin{array}{ll} \lambda_i, & i = j, \\ 0, & i \neq j. \end{array} \right.$$
Let $S := \sum_{i,j} h^2_{ij} = \sum \lambda_i^2$. The following formulas can be obtained by a direct computation (cf. [7]):

\[(2.11) \quad \frac{1}{2} \Delta S = \sum_{i,j,k} h^2_{ijk} - S(S - n),\]

\[(2.12) \quad \frac{1}{2} \sum_{i,j,k} h^2_{ijk} = \sum_{i,j,k,l} h^2_{ijkl} + (2n + 3 - S) \sum_{i,j,k} h^2_{ijk} + 3(2B - A) - \frac{3}{2} |\nabla S|^2,\]

where $A = \sum_{i,j,k} \lambda_i^2 h^2_{ijk}$, $B = \sum_{i,j,k} \lambda_i \lambda_j h^2_{ijk}$.


At first we give a proposition and some lemmas which will play a crucial role in the proof of our theorem. For convenience, we let

$$b_i = h_{i\ell}, \quad b = \sum_{i \neq 1} b_i^2 + \frac{1}{3} b_1^2, \quad f = \sum_{i \neq 1} (\lambda_i^2 - 4\lambda_i \lambda_1) b_i^2 - \lambda_1^2 b_1^2.$$

**Proposition 3.1.** Let $M^n$ be a closed minimal hypersurface in $S^{n+1}(1)$. Suppose that

$$3(A - 2B) \leq [2 + \delta(n)] \sum_{i,j,k} h^2_{ijk},$$

where $\delta(n)$ is a number depending only on $n$ such that $0 \leq \delta(n) < \min\{\frac{1}{2}, \frac{3}{n}\}$. Then there exists a positive constant $\alpha(n)$ depending only on $n$ such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$; i.e., $M^n$ is isometric to a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$. Here, $\alpha(n) = \frac{2(n + 4)(3 - n\delta)}{9n + 30}$.

**Proof.** Since $M^n$ is a minimal hypersurface in $S^{n+1}(1)$, from (2.11) and (2.12) we have

\[(3.1) \quad \int_M \sum_{i,j,k} h^2_{ijk} dM = \int_M S(S - n) dM,\]

\[(3.2) \quad -\frac{1}{2} \int_M |\nabla S|^2 dM = \int_M \left[ S \sum_{i,j,k} h^2_{ijk} - S^2(S - n) \right] dM,\]

\[(3.3) \quad \int_M \sum_{i,j,k,l} h^2_{ijkl} dM = \int_M \left[ (S - 2n - 3) \sum_{i,j,k} h^2_{ijk} + 3(A - 2B) + \frac{3}{2} |\nabla S|^2 \right] dM.\]

Letting $f_3 = \sum \lambda_i^3$ and $f_4 = \sum \lambda_i^4$, we have (cf. [6])

\[(3.4) \quad \sum_{i,j,k,l} h^2_{ijkl} \geq \frac{3}{2} (S f_4 - f_3^2 - 2S^2 + nS) + \frac{3S(S - n)^2}{2(n + 4)},\]

\[(3.5) \quad \int_M (A - 2B) dM = \int_M \left[ S f_4 - f_3^2 - S^2 - \frac{1}{4} |\nabla S|^2 \right] dM.\]

From (3.3), (3.4) and (3.5), we have

\[(3.6) \quad \int_M \left[ (S - 2n - 3) \sum_{i,j,k} h^2_{ijk} + \frac{3}{2} (A - 2B) + \frac{3}{2} S(S - n) + \frac{9}{8} |\nabla S|^2 - \frac{3S(S - n)^2}{2(n + 4)} \right] dM \geq 0.\]
Noticing that $S^2 = S(S - n) + nS$, from (3.1), (3.2) and (3.6), we have
\begin{equation}
\int_M \left[ \frac{3}{2}(A - 2B) + \frac{9n + 30}{4(n + 4)} S(S - n)^2 - \left( \frac{5}{4} S - \frac{n}{4} + \frac{3}{2} \right) \sum_{i,j,k} h_{ijk}^2 \right] dM \geq 0.
\end{equation}

Suppose $3(A - 2B) \leq (2 + \delta(n)) \sum_{i,j,k} h_{ijk}^2$ and $n \leq S \leq n + \alpha(n)$. From the above inequality, we have
\begin{equation}
\int_M \left\{ \frac{9n + 30}{4(n + 4)} \alpha(n) - \frac{1 - 2\delta(n)}{4} (S - n) - \frac{3 - n\delta(n)}{2} \right\} \sum_{i,j,k} h_{ijk}^2 dM \geq 0.
\end{equation}
Since $\alpha(n) = \frac{2(n + 4)(3 - n\delta)}{9n + 30}$ and $\delta(n) < \min \{ \frac{1}{2}, \frac{n}{4} \}$, we have
\begin{equation}
- \int_M (S - n) \sum_{i,j,k} h_{ijk}^2 dM \geq 0.
\end{equation}
Hence, $S \equiv n$. This completes the proof of Proposition 3.1.

**Lemma 3.2.** Let $M^n$ be a closed minimal hypersurface in $S^{n+1}(1)$. If $\lambda_1^2 - 4\lambda_1\lambda_2 \geq tS$ ($t \geq 2$), then $\lambda_1^2 - 4\lambda_1\lambda_2 - (\lambda_2^2 - 4\lambda_1\lambda_i) \geq rS$ ($i \neq 1, 2$). Here, $r = \frac{16t - 8 - 12\sqrt{-2t^2 + 2t + 8}}{17}$.

**Proof.** Let $\lambda_1^2 = x^2S$, $\lambda_2^2 = y^2S$ ($x, y > 0$). Since $\lambda_1^2 - 4\lambda_1\lambda_2 \geq tS$ ($t \geq 2$), we have $x^2 + 4xy \geq t$, that is, $y \geq \frac{t - x^2}{4x}$. Hence, we have
\begin{align*}
\lambda_1^2 - 4\lambda_1\lambda_i & \leq \left( x^2 + 4x\sqrt{1 - x^2 - y^2} \right) S \\
& \leq \left\{ x^2 + \sqrt{16x^2 - 16x^4 - (t - x^2)^2} \right\} S \\
& = \left\{ x^2 + \sqrt{17x^4 + (16 + 2t)x^2 - t^2} \right\} S.
\end{align*}
Let $g(z) = z + \sqrt{17z^2 + (16 + 2t)z - t^2}$ ($0 < z < 1$). Then $g'(z) = 1 - \frac{17z - (8 + t)}{\sqrt{-17z^2 + (16 + 2t)z - t^2}}$.

Letting $g'(z_0) = 0$, we have
\begin{equation}
z_0 = \frac{3(8 + t) + 2\sqrt{-2t^2 + 2t + 8}}{51}.
\end{equation}
Hence we have
\begin{equation}
g(z) \leq g(z_0) = \frac{t + 8 + 12\sqrt{-2t^2 + 2t + 8}}{17},
\end{equation}
which implies that
\begin{equation}
\lambda_1^2 - 4\lambda_1\lambda_i \leq \frac{t + 8 + 12\sqrt{-2t^2 + 2t + 8}}{17}.
\end{equation}
Since $\lambda_1^2 - 4\lambda_1\lambda_2 \geq tS$, we have
\begin{equation}
(\lambda_2^2 - 4\lambda_1\lambda_2) - (\lambda_1^2 - 4\lambda_1\lambda_i) \geq \frac{16t - 8 - 12\sqrt{-2t^2 + 2t + 8}}{17}.
\end{equation}
This completes the proof of Lemma 3.2. \qedsymbol
Lemma 3.3. Let \( f_n(t) = 17[t - 2 - \delta(n)] [3(n - 2)t + (n + 2)\delta(n) + 10 - 4n] \) and \( g_n(t) = [8 + 16\delta(n)] (4t - 2 - 3\sqrt{-2t^2 + 2t + 8}) \). Then
\[
h_n(t) = f_n(t) - g_n(t) \leq 0 \quad (t \geq 2, 4 \leq n \leq 8).
\]
Here, \( \delta(4) = 0.16, \delta(5) = 0.23, \delta(6) = 0.28, \delta(7) = 0.32 \) and \( \delta(8) = 0.34 \).

Proof. By a direct computation, we obtain
\[
h_n''(t) = 51 \left\{ 2(n - 2) - [8 + 16\delta(n)] (-2t^2 + 2t + 8)^{-3/2} \right\}
\]
and
\[
h_n''(t) = -153 [8 + 16\delta(n)] (2t - 1)(-2t^2 + 2t + 8)^{-3/2} < 0 \quad (t \geq 2).
\]
On the other hand, \( h_n''(2) > 0 \) and \( h_n(2) < 0 \). Hence, if there exist real numbers \( t_i > 2 \) \((i = 1, 2)\) such that \( h_n'(t_1) > 0, h_n'(t_2) < 0 \) and \( h_n(t) \leq 0 \) \((t_1 \leq t \leq t_2)\), then \( h_n(t) \leq 0 \) \((\forall t \geq 2)\).

In the case \( n = 4 \), since \( \delta(4) = 0.16 \), we have
\[
f_4(t) = 17(6t^2 - 18t + 10.8864),
g_4(t) = 10.56(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}),
\]
\[
h_4'(t) = 204t - 348.24 - 31.68(2t - 1)(-2t^2 + 2t + 8)^{-3/2}.
\]
By a direct computation, we obtain
\[
h_4'(2.48) > 0, \ h_4'(2.5) < 0.
\]
On the other hand, when \( 2.48 \leq t \leq 2.5 \), we have
\[
f_4(t) \leq f_4(2.5) \leq 57.6, \ g_4(t) \geq g_4(2.48) \geq 57.8.
\]
This implies that
\[
h_4(t) \leq 0 \quad (2.48 \leq t \leq 2.5).
\]
From (3.10) and (3.11), we know that Lemma 3.3 is true in the case \( n = 4 \).

In the case \( n = 5 \), since \( \delta(5) = 0.23 \), we have
\[
f_5(t) = 17(9t^2 - 28.46t + 18.7097),
g_5(t) = 11.68(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}),
\]
\[
h_5'(t) = 306t - 530.54 - 35.04(2t - 1)(-2t^2 + 2t + 8)^{-3/2}.
\]
By a direct computation, we obtain
\[
h_5'(2.51) > 0, \ h_5'(2.52) < 0.
\]
On the other hand, when \( 2.51 \leq t \leq 2.52 \), we have
\[
f_5(t) \leq f_5(2.52) \leq 70.5, \ g_5(t) \geq g_5(2.51) \geq 71.
\]
This implies that
\[
h_5(t) \leq 0 \quad (2.51 \leq t \leq 2.52).
\]
From (3.12) and (3.13), we know that Lemma 3.3 is true in the case \( n = 5 \).

In the case \( n = 6 \), since \( \delta(6) = 0.28 \), we have
\[
f_6(t) = 17(12t^2 - 39.12t + 26.8128),
g_6(t) = 12.48(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}),
\]
\[
h_6'(t) = 408t - 714.96 - 37.44(2t - 1)(-2t^2 + 2t + 8)^{-3/2}.
\]
By a direct computation, we obtain
\begin{equation}
(3.14)\quad h^\prime_6(2.53) > 0, \quad h^\prime_6(2.535) < 0.
\end{equation}
On the other hand, when $2.53 \leq t \leq 2.535$, we have
$$f_6(t) \leq f_6(2.535) \leq 81, \quad g_6(t) \geq g_6(2.51) \geq 82.$$  
This implies that
\begin{equation}
(3.15)\quad h_6(t) \leq 0 (2.53 \leq t \leq 2.535).
\end{equation}
From \((3.14)\) and \((3.15)\), we know that Lemma 3.3 is true in the case $n = 6$.

In the case $n = 7$, since $\delta(7) = 0.32$, we have
$$f_7(t) = 17(15t^2 - 49.92t + 35.0784),$$
$$g_7(t) = 13.12(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}),$$
$$h^\prime_7(t) = 510t - 901.12 - 39.36(2t - 1)(-2t^2 + 2t + 8)^{-\frac{3}{2}}.$$  
By a direct computation, we obtain
\begin{equation}
(3.16)\quad h^\prime_7(2.54) > 0, \quad h^\prime_7(2.544) < 0.
\end{equation}
On the other hand, when $2.54 \leq t \leq 2.544$, we have
$$f_7(t) \leq f_7(2.544) \leq 88, \quad g_7(t) \geq g_7(2.54) \geq 90.$$  
This implies that
\begin{equation}
(3.17)\quad h_7(t) \leq 0 (2.54 \leq t \leq 2.544).
\end{equation}
From \((3.16)\) and \((3.17)\), we know that Lemma 3.3 is true in the case $n = 7$.

In the case $n = 8$, since $\delta(8) = 0.34$, we have
$$f_8(t) = 17(18t^2 - 60.72t + 43.524),$$
$$g_8(t) = 13.44(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}),$$
$$h^\prime_8(t) = 612t - 1086 - 40.32(2t - 1)(-2t^2 + 2t + 8)^{-\frac{3}{2}}.$$  
By a direct computation, we obtain
\begin{equation}
(3.18)\quad h^\prime_8(2.5465) > 0, \quad h^\prime_8(2.5468) < 0.
\end{equation}
On the other hand, when $2.5465 \leq t \leq 2.5468$, we have
$$f_8(t) \leq f_8(2.5468) \leq 95.78, \quad g_8(t) \geq g_8(2.5465) \geq 95.8.$$  
This implies that
\begin{equation}
(3.19)\quad h_8(t) \leq 0 (2.5465 \leq t \leq 2.5468).
\end{equation}
From \((3.18)\) and \((3.19)\), we know that Lemma 3.3 is true in the case $n = 8$. This completes the proof of Lemma 3.3. \hfill $\Box$

**Lemma 3.4.** Let $M^n$ be a closed minimal hypersurface in $S^{n+1}(1)$. Then
$$f \leq \left[2 + \delta(n)\right]Sb, \quad 3 \leq n \leq 8.$$  
Here, $\delta(3) = 0, \delta(4) = 0.16, \delta(5) = 0.23, \delta(6) = 0.28, \delta(7) = 0.32$ and $\delta(8) = 0.34$.  

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Proof. In the case \( n = 3 \), if \( b_1 = 0 \), then \( b_2^2 = b_3^2 = \frac{1}{3}b \). Hence

\[
f = \frac{1}{2}(\lambda_1^2 - 4\lambda_1\lambda_2 + \lambda_2^2 - 4\lambda_1\lambda_3)b
= \left\{ \lambda_1^2 - 2 \cdot \frac{\lambda_1}{\sqrt{2}} \cdot \sqrt{2}\lambda_2 - 2 \cdot \frac{\lambda_1}{\sqrt{2}} \cdot \sqrt{2}\lambda_3 \right\}b
\leq 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)b \leq 2Sb.
\]

When \( b_1 \neq 0 \), we let \( b_2 = (x - \frac{1}{2})b_1 \) and \( \lambda_2 = (y - \frac{1}{2})\lambda_1 \). Then

\[
f = \frac{36x^2 + 48xy + 3}{48x^2y^2 + 36x^2 + 20y^2 + 15} \cdot 2Sb
= \frac{36x^2 + 48xy + 3}{36x^2 + 48xy + 3 + 48(xy - 1/2)^2 + 20y^2} \cdot 2Sb
\leq 2Sb.
\]

From the above discussion, we know that Lemma 3.4 is true in the case \( n = 3 \).

In the case \( 4 \leq n \leq 8 \), if \( \lambda_1^2 - 4\lambda_1\lambda_i \leq [2 + \delta(n)]S (\forall i \neq 1) \), then Lemma 3.3 is true. Otherwise, without loss of generality, we suppose that \( \lambda_1^2 - 4\lambda_1\lambda_2 = tS (t \geq 2) \) and \( b_1 = xb_2 \). Then

\[
\sum_{i \neq 1,2} b_i^2 \geq \frac{(1+x)^2}{n-2}b_2^2, \quad \lambda_1^2 \geq (t-2)S.
\]

From the above inequalities and Lemma 3.2 we have

\[
f - [2 + \delta(n)]Sb \leq [t - 2 - \delta(n)]Sb^2 + [t - r - 2 - \delta(n)]S \sum_{i \neq 1,2} b_i^2
- \left\{ t - 2 + \frac{2 + \delta(n)}{3} \right\} Sb^2
\leq [t - 2 - \delta(n)]Sb^2 + \frac{t - r - 2 - \delta(n)}{n - 2} (1+x)^2Sb^2
- \left\{ t - 2 + \frac{2 + \delta(n)}{3} \right\} x^2Sb^2.
\]

Here, \( r = \frac{16t - 8 - 2\sqrt{17t^2 + 2t + 8}}{17} \).

Let \( F(n, t, x) = t - 2 - \delta(n) + \frac{t - r - 2 - \delta(n)}{n - 2} (1+x)^2 - \left\{ t - 2 + \frac{2 + \delta(n)}{3} \right\} x^2 \).

The above inequality becomes

\[
(3.20) \quad f - [2 + \delta(n)]Sb \leq F(n, t, x)Sb^2.
\]

Since

\[
\frac{1}{2} \frac{\partial F(n, t, x)}{\partial x} = \frac{t - r - 2 - \delta(n)}{n - 2} (1+x) - \left\{ t - 2 + \frac{2 + \delta(n)}{3} \right\} x,
\]

we have

\[
(3.21) \quad F(n, t, x) \leq F(n, t, x_0)
= \frac{t - 2 - \delta(n)}{G} \left\{ (n - 2)t + \frac{(n+2)\delta(n) + 10 - 4n}{3} \right\} - \frac{2 + 4\delta(n)}{3G}r.
\]
Here, \( x_0 = \frac{3[r + 2 + \delta(n) - t]}{3r + 3(n - t) + 14 + (n + 1)\delta(n) - 4n} \), \( \partial F(n, t, x_0) = 0 \), and \( G = r + 2 + \delta(n) - t + (n - 2)\left(t - 2 + \frac{2 + \delta(n)}{3}\right) \).

Notice that \( h_n(t) = 51F(n, t, x_0)G \), where \( h_n(t) \) is defined as in Lemma 3.3. From (3.20), (3.21) and Lemma 3.3, we have
\[
f \leq [2 + \delta(n)]S_b.
\]
This completes the proof of Lemma 3.3. \( \square \)

Now we are in a position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** From Lemma 3.4, we have obtained
\[
f = \sum_{i \neq 1}(\lambda_i^2 - 4\lambda_i\lambda_j)h_{iij}^2 - \lambda_j^2h_{jjj}^2 \leq (2 + \delta)S\left(\sum_{i \neq 1}h_{iij}^2 + \frac{1}{3}h_{jjj}^2\right),
\]
In general,
\[
f_j = \sum_{i \neq j}(\lambda_i^2 - 4\lambda_i\lambda_j)h_{iij}^2 - \lambda_j^2h_{jjj}^2 \leq (2 + \delta)S\left(\sum_{i \neq j}h_{iij}^2 + \frac{1}{3}h_{jjj}^2\right), \forall j.
\]
Hence we get
\[
3(A - 2B) = \sum_{i \neq j \neq k \neq i}[2(\lambda_i^2 + \lambda_j^2 + \lambda_k^2) - (\lambda_i + \lambda_j + \lambda_k)^2]h_{iij}^2 - 3\sum_i\lambda_i^2h_{iij}^2 + 3\sum_{i \neq j}(\lambda_i^2 - 4\lambda_i\lambda_j)h_{iij}^2
\]
\[
\leq 2S\sum_{i \neq j}h_{iij}^2 + 3\sum_{i \neq j}(\lambda_i^2 - 4\lambda_i\lambda_j)h_{iij}^2 - \lambda_j^2h_{jjj}^2
\]
\[
\leq (2 + \delta)S\left\{\sum_{i \neq j}h_{iij}^2 + 3\sum_{i \neq j}h_{iij}^2 + \sum_j h_{jjj}^2\right\}
\]
\[
= (2 + \delta)S\sum_{i,j,k}h_{iij}^2.
\]
Notice that \( \delta(3) = 0, \delta(4) = 0.16, \delta(5) = 0.23, \delta(6) = 0.28, \delta(7) = 0.32, \delta(8) = 0.34 \) and \( \alpha(n) = \frac{2(n + 4)(3 - n\delta)}{9n + 30} \). We conclude from Proposition 3.1 that \( S \equiv n \). This completes the proof of Theorem 1.1. \( \square \)

**References**


15. Q. M. Cheng, *The rigidity of Clifford torus* $S^1(\sqrt{\frac{m}{n}}) \times S^{n-1}(\sqrt{\frac{n}{n-m}})$, Comment. Math. Helvetici, **71** (1996), 60–69. MR1371678 (97a:53094)


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