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THE PINCHING CONSTANT OF MINIMAL HYPERSURFACES IN THE UNIT SPHERES

QIN ZHANG

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Abstract. In this paper, we prove that if M^n $(n \leq 8)$ is a closed minimal hypersurface in a unit sphere $S^{n+1}(1)$, then there exists a positive constant $\alpha(n)$ depending only on n such that if $n \leq S \leq n + \alpha(n)$, then M is isometric to a Clifford torus, where S is the squared norm of the second fundamental form of M.

1. Introduction

Let M^n be an n-dimensional closed minimal hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. Denote by S the squared norm of the second fundamental form of M^n. Lawson [2], Simons [1] and Chern, do Carmo, Kobayashi [3] obtained independently the famous rigidity theorem, which says that if $S \leq n$, then $S \equiv 0$ or $S \equiv n$; i.e., M^n is the great sphere $S^n(1)$ or the Clifford torus. Further discussions in this direction have been carried out by many other authors [6, 8, 9, 10, 11, 12, 13]. In [4], Peng and Terng proved that if the scalar curvature of M is constant, then there exists a positive constant $\alpha(n)$ depending only on n such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$. Later, Cheng and Yang [14] improved the pinching constant $\alpha(n)$ to $n/3$. More generally, Peng and Terng [5] proved that if $M^n (n \leq 5)$ is a closed minimal hypersurface in S^{n+1}, then there exists a positive constant $\alpha(n)$ depending only on n such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$. So they proposed the following problem.

Let M^n $(n \geq 6)$ be a closed minimal hypersurface in S^{n+1}. Does there exist a positive constant $\alpha(n)$ depending only on n such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$?

In [15], Cheng gives a positive answer under the additional condition that M has only two distinct principal curvatures. Later, Cheng and Ishikawa [6] improved the result of Peng and Terng [5] when $n \leq 5$.

In this paper, we solve the problem proposed by Peng and Terng [5] for $n \leq 8$.

Theorem 1.1. *Let M^n $(n \leq 8)$ be a closed minimal hypersurface in $S^{n+1}(1)$. If $n \leq S \leq n + \alpha(n)$, then $S \equiv n$ and M^n is isometric to a Clifford torus $S^m\big(\sqrt{\frac{m}{n}}\big) \times S^{n-m}\big(\sqrt{\frac{n-m}{n}}\big)$, where $\alpha(n) = \dfrac{2(n+4)(3-n\delta)}{9n+30}$, $\delta(3) = 0$, $\delta(4) = 0.16$, $\delta(5) = 0.23$, $\delta(6) = 0.28$, $\delta(7) = 0.32$ and $\delta(8) = 0.34$.*

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For $n \leq 5$, Cheng and Ishikawa [6] proved the following: Let M be an n-dimensional ($n \leq 5$) closed minimal hypersurface of a unit sphere $S^{n+1}(1)$. If $n \leq S \leq n+\alpha(n)$, then $S = n$, where $\alpha(3) = \dfrac{42}{85}$, $\alpha(4) = \dfrac{8}{31}$ and $\alpha(5) = \dfrac{3(21-5\sqrt{17})}{28+3\sqrt{17}}$. It is obvious that our pinching constant is larger than theirs. Up to now, the open problem for $n \geq 9$ is still open and it is a very hard problem.

2. Fundamental formulas

Let M^n be an n-dimensional hypersurface in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame field $e_1, \cdots, e_{n+1}$ in $S^{n+1}(1)$, restricted to M^n, so that $e_1, \cdots, e_n$ are tangent to M^n. Let $\omega_1, \cdots, \omega_{n+1}$ denote the dual coframe field in $S^{n+1}(1)$. Then in M^n, $\omega_{n+1} = 0$. It follows from Cartan's Lemma that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j. \tag{2.1}$$

The second fundamental form α and the mean curvature H of M^n are defined by

$$\alpha = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}, \quad nH = \sum_i h_{ii}, \tag{2.2}$$

respectively. If M^n is a minimal hypersurface, then $\sum_i h_{ii} = 0$. The connection form ω_{ij} is characterized by the structure equations

$$d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.3}$$

$$d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \tag{2.4}$$

$$\Omega_{ij} = \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.5}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the curvature form (resp. the components of the curvature tensor) of M^n. The Gauss equation is given by

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.6}$$

Denote by h_{ijk}, h_{ijkl}, h_{ijklm} the components of the first, second and third covariant derivatives of the second fundamental form, respectively. Then

$$h_{ijk} = h_{ikj} = h_{jik}, \tag{2.7}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}, \tag{2.8}$$

$$h_{ijklm} - h_{ijkml} = \sum_r h_{rjk}R_{rilm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{ijr}R_{rklm}. \tag{2.9}$$

For any fixed point p in M^n, we take a local orthonormal frame field $e_1, \cdots, e_n$ such that

$$h_{ij} = \begin{cases} \lambda_i, & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.10}$$

Let $S := \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2$. The following formulas can be obtained by a direct computation (cf. [7]):

$$\frac{1}{2}\triangle S = \sum_{i,j,k} h_{ijk}^2 - S(S-n), \tag{2.11}$$

$$\frac{1}{2}\sum_{i,j,k} h_{ijk}^2 = \sum_{i,j,k,l} h_{ijkl}^2 + (2n+3-S)\sum_{i,j,k} h_{ijk}^2 + 3(2B-A) - \frac{3}{2}|\nabla S|^2, \tag{2.12}$$

where $A = \sum_{i,j,k} \lambda_i^2 h_{ijk}^2$, $B = \sum_{i,j,k} \lambda_i \lambda_j h_{ijk}^2$.

3. Proof of Theorem 1.1

At first we give a proposition and some lemmas which will play a crucial role in the proof of our theorem. For convenience, we let

$$b_i = h_{ii1}, b = \sum_{i\neq 1} b_i^2 + \frac{1}{3}b_1^2, f = \sum_{i\neq 1}(\lambda_1^2 - 4\lambda_1\lambda_i)b_i^2 - \lambda_1^2 b_1^2.$$

Proposition 3.1. *Let M^n be a closed minimal hypersurface in $S^{n+1}(1)$. Suppose that*

$$3(A-2B) \leq \big[2+\delta(n)\big]\sum_{i,j,k} h_{ijk}^2,$$

where $\delta(n)$ is a number depending only on n such that $0 \leq \delta(n) < \min\{\frac{1}{2}, \frac{3}{n}\}$. Then there exists a positive constant $\alpha(n)$ depending only on n such that if $n \leq S \leq n+\alpha(n)$, then $S \equiv n$; i.e., M^n is isometric to a Clifford torus $S^m\big(\sqrt{\frac{m}{n}}\big) \times S^{n-m}\big(\sqrt{\frac{n-m}{n}}\big)$. Here, $\alpha(n) = \dfrac{2(n+4)(3-n\delta)}{9n+30}$.

Proof. Since M^n is a minimal hypersurface in $S^{n+1}(1)$, from (2.11) and (2.12) we have

$$\int_M \sum_{i,j,k} h_{ijk}^2 dM = \int_M S(S-n)dM, \tag{3.1}$$

$$-\frac{1}{2}\int_M |\nabla S|^2 dM = \int_M \Big[S\sum_{i,j,k} h_{ijk}^2 - S^2(S-n)\Big]dM, \tag{3.2}$$

$$\int_M \sum_{i,j,k,l} h_{ijkl}^2 dM = \int_M \Big[(S-2n-3)\sum_{i,j,k} h_{ijk}^2 + 3(A-2B) + \frac{3}{2}|\nabla S|^2)\Big]dM. \tag{3.3}$$

Letting $f_3 = \sum_i \lambda_i^3$ and $f_4 = \sum_i \lambda_i^4$, we have (cf. [6])

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq \frac{3}{2}(Sf_4 - f_3^2 - 2S^2 + nS) + \frac{3S(S-n)^2}{2(n+4)}, \tag{3.4}$$

$$\int_M (A-2B)dM = \int_M \Big[Sf_4 - f_3^2 - S^2 - \frac{1}{4}|\nabla S|^2\Big]dM. \tag{3.5}$$

From (3.3), (3.4) and (3.5), we have

$$\int_M \Big[(S-2n-3)\sum_{i,j,k} h_{ijk}^2 + \frac{3}{2}(A-2B) + \frac{3}{2}S(S-n) + \frac{9}{8}|\nabla S|^2 - \frac{3S(S-n)^2}{2(n+4)}\Big]dM \geq 0. \tag{3.6}$$

Noticing that $S^2 = S(S-n) + nS$, from (3.1), (3.2) and (3.6), we have

$$\int_M \Big[\frac{3}{2}(A-2B) + \frac{9n+30}{4(n+4)}S(S-n)^2 - \Big(\frac{5}{4}S - \frac{n}{4} + \frac{3}{2}\Big)\sum_{i,j,k} h_{ijk}^2\Big]dM \geq 0. \tag{3.7}$$

Suppose $3(A-2B) \leq \big[2+\delta(n)\big]S\sum_{i,j,k} h_{ijk}^2$ and $n \leq S \leq n+\alpha(n)$. From the above inequality, we have

$$\int_M \Big\{\frac{9n+30}{4(n+4)}\alpha(n) - \frac{1-2\delta(n)}{4}(S-n) - \frac{3-n\delta(n)}{2}\Big\}\sum_{i,j,k} h_{ijk}^2 dM \geq 0. \tag{3.8}$$

Since $\alpha(n) = \dfrac{2(n+4)(3-n\delta)}{9n+30}$ and $\delta(n) < \min\big\{\frac{1}{2}, \frac{3}{n}\big\}$, we have

$$-\int_M (S-n)\sum_{i,j,k} h_{ijk}^2 dM \geq 0. \tag{3.9}$$

Hence, $S \equiv n$. This completes the proof of Proposition 3.1. □

Lemma 3.2. *Let M^n be a closed minimal hypersurface in $S^{n+1}(1)$. If $\lambda_1^2 - 4\lambda_1\lambda_2 \geq tS$ $(t \geq 2)$, then $(\lambda_1^2 - 4\lambda_1\lambda_2) - (\lambda_1^2 - 4\lambda_1\lambda_i) \geq rS$ $(i \neq 1, 2)$. Here,* $r = \dfrac{16t - 8 - 12\sqrt{-2t^2+2t+8}}{17}$.

Proof. Let $\lambda_1^2 = x^2 S, \lambda_2^2 = y^2 S$ $(x, y > 0)$. Since $\lambda_1^2 - 4\lambda_1\lambda_2 \geq tS$ $(t \geq 2)$, we have $x^2 + 4xy \geq t$, that is, $y \geq \dfrac{t-x^2}{4x}$. Hence, we have

$$\begin{aligned}
\lambda_1^2 - 4\lambda_1\lambda_i &\leq \Big(x^2 + 4x\sqrt{1-x^2-y^2}\Big)S \\
&\leq \Big\{x^2 + \sqrt{16x^2 - 16x^4 - (t-x^2)^2}\Big\}S \\
&= \Big\{x^2 + \sqrt{-17x^4 + (16+2t)x^2 - t^2}\Big\}S.
\end{aligned}$$

Let $g(z) = z + \sqrt{-17z^2 + (16+2t)z - t^2}$ $(0 < z < 1)$. Then

$$g'(z) = 1 - \frac{17z - (8+t)}{\sqrt{-17z^2 + (16+2t)z - t^2}}.$$

Letting $g'(z_0) = 0$, we have

$$z_0 = \frac{3(8+t) + 2\sqrt{-2t^2+2t+8}}{51}.$$

Hence we have

$$g(z) \leq g(z_0) = \frac{t + 8 + 12\sqrt{-2t^2+2t+8}}{17},$$

which implies that

$$\lambda_1^2 - 4\lambda_1\lambda_i \leq \frac{t + 8 + 12\sqrt{-2t^2+2t+8}}{17}.$$

Since $\lambda_1^2 - 4\lambda_1\lambda_2 \geq tS$, we have

$$(\lambda_1^2 - 4\lambda_1\lambda_2) - (\lambda_1^2 - 4\lambda_1\lambda_i) \geq \frac{16t - 8 - 12\sqrt{-2t^2+2t+8}}{17}.$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Let* $f_n(t) = 17\big[t-2-\delta(n)\big]\big[3(n-2)t+(n+2)\delta(n)+10-4n\big]$ *and* $g_n(t) = \big[8+16\delta(n)\big]\big(4t-2-3\sqrt{-2t^2+2t+8}\big)$. *Then*

$$h_n(t) = f_n(t) - g_n(t) \le 0 \quad (t \ge 2,\ 4 \le n \le 8).$$

Here, $\delta(4) = 0.16$, $\delta(5) = 0.23$, $\delta(6) = 0.28$, $\delta(7) = 0.32$ *and* $\delta(8) = 0.34$.

Proof. By a direct computation, we obtain

$$h_n''(t) = 51\Big\{2(n-2) - \big[8+16\delta(n)\big](-2t^2+2t+8)^{-\frac{3}{2}}\Big\}$$

and

$$h_n'''(t) = -153\big[8+16\delta(n)\big](2t-1)(-2t^2+2t+8)^{-\frac{5}{2}} < 0 \quad (t \ge 2).$$

On the other hand, $h_n''(2) > 0$ and $h_n(2) < 0$. Hence, if there exist real numbers $t_i > 2\,(i = 1, 2)$ such that $h_n'(t_1) > 0$, $h_n'(t_2) < 0$ and $h_n(t) \le 0$ $(t_1 \le t \le t_2)$, then $h_n(t) \le 0$ $(\forall t \ge 2)$.

In the case $n = 4$, since $\delta(4) = 0.16$, we have

$$\begin{aligned} f_4(t) &= 17(6t^2 - 18t + 10.8864),\\ g_4(t) &= 10.56(4t - 2 - 3\sqrt{-2t^2+2t+8}),\\ h_4'(t) &= 204t - 348.24 - 31.68(2t-1)(-2t^2+2t+8)^{-\frac{1}{2}}. \end{aligned}$$

By a direct computation, we obtain

$$h_4'(2.48) > 0,\ h_4'(2.5) < 0. \tag{3.10}$$

On the other hand, when $2.48 \le t \le 2.5$, we have

$$f_4(t) \le f_4(2.5) \le 57.6,\ g_4(t) \ge g_4(2.48) \ge 57.8.$$

This implies that

$$h_4(t) \le 0\ (2.48 \le t \le 2.5). \tag{3.11}$$

From (3.10) and (3.11), we know that Lemma 3.3 is true in the case $n = 4$.

In the case $n = 5$, since $\delta(5) = 0.23$, we have

$$\begin{aligned} f_5(t) &= 17(9t^2 - 28.46t + 18.7097),\\ g_5(t) &= 11.68(4t - 2 - 3\sqrt{-2t^2+2t+8}),\\ h_5'(t) &= 306t - 530.54 - 35.04(2t-1)(-2t^2+2t+8)^{-\frac{1}{2}}. \end{aligned}$$

By a direct computation, we obtain

$$h_5'(2.51) > 0,\ h_5'(2.52) < 0. \tag{3.12}$$

On the other hand, when $2.51 \le t \le 2.52$, we have

$$f_5(t) \le f_5(2.52) \le 70.5,\ g_5(t) \ge g_5(2.51) \ge 71.$$

This implies that

$$h_5(t) \le 0\ (2.51 \le t \le 2.52). \tag{3.13}$$

From (3.12) and (3.13), we know that Lemma 3.3 is true in the case $n = 5$.

In the case $n = 6$, since $\delta(6) = 0.28$, we have

$$\begin{aligned} f_6(t) &= 17(12t^2 - 39.12t + 26.8128),\\ g_6(t) &= 12.48(4t - 2 - 3\sqrt{-2t^2+2t+8}),\\ h_6'(t) &= 408t - 714.96 - 37.44(2t-1)(-2t^2+2t+8)^{-\frac{1}{2}}. \end{aligned}$$

By a direct computation, we obtain

$$h_6'(2.53) > 0,\ h_6'(2.535) < 0. \tag{3.14}$$

On the other hand, when $2.53 \le t \le 2.535$, we have

$$f_6(t) \le f_6(2.535) \le 81,\ g_6(t) \ge g_6(2.51) \ge 82.$$

This implies that

$$h_6(t) \le 0\ (2.53 \le t \le 2.535). \tag{3.15}$$

From (3.14) and (3.15), we know that Lemma 3.3 is true in the case $n = 6$.

In the case $n = 7$, since $\delta(7) = 0.32$, we have

$$\begin{aligned} f_7(t) &= 17(15t^2 - 49.92t + 35.0784), \\ g_7(t) &= 13.12(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}), \\ h_7'(t) &= 510t - 901.12 - 39.36(2t - 1)(-2t^2 + 2t + 8)^{-\frac{1}{2}}. \end{aligned}$$

By a direct computation, we obtain

$$h_7'(2.54) > 0,\ h_7'(2.544) < 0. \tag{3.16}$$

On the other hand, when $2.54 \le t \le 2.544$, we have

$$f_7(t) \le f_7(2.544) \le 88,\ g_7(t) \ge g_7(2.54) \ge 90.$$

This implies that

$$h_7(t) \le 0\ (2.54 \le t \le 2.544). \tag{3.17}$$

From (3.16) and (3.17), we know that Lemma 3.3 is true in the case $n = 7$.

In the case $n = 8$, since $\delta(8) = 0.34$, we have

$$\begin{aligned} f_8(t) &= 17(18t^2 - 60.72t + 43.524), \\ g_8(t) &= 13.44(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}), \\ h_8'(t) &= 612t - 1086 - 40.32(2t - 1)(-2t^2 + 2t + 8)^{-\frac{1}{2}}. \end{aligned}$$

By a direct computation, we obtain

$$h_8'(2.5465) > 0,\ h_8'(2.5468) < 0. \tag{3.18}$$

On the other hand, when $2.5465 \le t \le 2.5468$, we have

$$f_8(t) \le f_8(2.5468) \le 95.78,\ g_8(t) \ge g_8(2.5465) \ge 95.8.$$

This implies that

$$h_8(t) \le 0\ (2.5465 \le t \le 2.5468). \tag{3.19}$$

From (3.18) and (3.19), we know that Lemma 3.3 is true in the case $n = 8$. This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let M^n be a closed minimal hypersurface in $S^{n+1}(1)$. Then*

$$f \le \big[2 + \delta(n)\big] Sb,\ 3 \le n \le 8.$$

Here, $\delta(3) = 0$, $\delta(4) = 0.16$, $\delta(5) = 0.23$, $\delta(6) = 0.28$, $\delta(7) = 0.32$ and $\delta(8) = 0.34$.

Proof. In the case $n = 3$, if $b_1 = 0$, then $b_2^2 = b_3^2 = \frac{1}{2}b$. Hence

$$\begin{aligned} f &= \frac{1}{2}(\lambda_1^2 - 4\lambda_1\lambda_2 + \lambda_1^2 - 4\lambda_1\lambda_3)b \\ &= \Big\{\lambda_1^2 - 2\cdot\frac{\lambda_1}{\sqrt{2}}\cdot\sqrt{2}\lambda_2 - 2\cdot\frac{\lambda_1}{\sqrt{2}}\cdot\sqrt{2}\lambda_3\Big\}b \\ &\le 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)b \le 2Sb. \end{aligned}$$

When $b_1 \neq 0$, we let $b_2 = \big(x - \frac{1}{2}\big)b_1$ and $\lambda_2 = \big(y - \frac{1}{2}\big)\lambda_1$. Then

$$\begin{aligned} f &= \frac{36x^2 + 48xy + 3}{48x^2y^2 + 36x^2 + 20y^2 + 15}\cdot 2Sb \\ &= \frac{36x^2 + 48xy + 3}{36x^2 + 48xy + 3 + 48(xy - 1/2)^2 + 20y^2}\cdot 2Sb \\ &\le 2Sb. \end{aligned}$$

From the above discussion, we know that Lemma 3.4 is true in the case $n = 3$.

In the case $4 \le n \le 8$, if $\lambda_1^2 - 4\lambda_1\lambda_i \le \big[2 + \delta(n)\big]S\ (\forall i \neq 1)$, then Lemma 3.4 is true. Otherwise, without loss of generality, we suppose that $\lambda_1^2 - 4\lambda_1\lambda_2 = tS\ (t \ge 2)$ and $b_1 = xb_2$. Then

$$\sum_{i\neq 1,2} b_i^2 \ge \frac{(1+x)^2}{n-2}b_2^2,\ \lambda_1^2 \ge (t-2)S.$$

From the above inequalities and Lemma 3.2, we have

$$\begin{aligned} f - \big[2 + \delta(n)\big]Sb &\le \big[t - 2 - \delta(n)\big]Sb_2^2 + \big[t - r - 2 - \delta(n)\big]S\sum_{i\neq 1,2} b_i^2 \\ &\quad - \Big\{t - 2 + \frac{2 + \delta(n)}{3}\Big\}Sb_1^2 \\ &\le \big[t - 2 - \delta(n)\big]Sb_2^2 + \frac{t - r - 2 - \delta(n)}{n-2}(1+x)^2Sb_2^2 \\ &\quad - \Big\{t - 2 + \frac{2 + \delta(n)}{3}\Big\}x^2Sb_2^2. \end{aligned}$$

Here, $r = \dfrac{16t - 8 - 12\sqrt{-2t^2 + 2t + 8}}{17}$.

Let $\quad F(n,t,x) = t - 2 - \delta(n) + \dfrac{t - r - 2 - \delta(n)}{n-2}(1+x)^2 - \Big\{t - 2 + \dfrac{2 + \delta(n)}{3}\Big\}x^2.$

The above inequality becomes

$$f - \big[2 + \delta(n)\big]Sb \le F(n,t,x)Sb_2^2. \tag{3.20}$$

Since

$$\frac{1}{2}\frac{\partial F(n,t,x)}{\partial x} = \frac{t - r - 2 - \delta(n)}{n-2}(1+x) - \Big\{t - 2 + \frac{2 + \delta(n)}{3}\Big\}x,$$

we have

(3.21)

$$\begin{aligned} F(n,t,x) &\le F(n,t,x_0) \\ &= \frac{t - 2 - \delta(n)}{G}\Big\{(n-2)t + \frac{(n+2)\delta(n) + 10 - 4n}{3}\Big\} - \frac{2 + 4\delta(n)}{3G}r. \end{aligned}$$

Here, $-x_0 = \dfrac{3[r+2+\delta(n)-t]}{3r+3(n-3)t+14+(n+1)\delta(n)-4n}$, $\dfrac{\partial F(n,t,x_0)}{\partial x} = 0$, and $G = r+2+\delta(n)-t+(n-2)\Big(t-2+\dfrac{2+\delta(n)}{3}\Big)$.
Notice that $h_n(t) = 51F(n,t,x_0)G$, where $h_n(t)$ is defined as in Lemma 3.3. From (3.20), (3.21) and Lemma 3.3, we have

$$f \le \big[2+\delta(n)\big]Sb.$$

This completes the proof of Lemma 3.4. □

Now we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. From Lemma 3.4, we have obtained

$$f = \sum_{i\neq 1}(\lambda_1^2 - 4\lambda_1\lambda_i)h_{ii1}^2 - \lambda_1^2 h_{111}^2 \le (2+\delta)S\Big(\sum_{i\neq 1} h_{ii1}^2 + \frac{1}{3}h_{111}^2\Big).$$

In general,

$$f_j = \sum_{i\neq j}(\lambda_j^2 - 4\lambda_i\lambda_j)h_{iij}^2 - \lambda_j^2 h_{jjj}^2 \le (2+\delta)S\Big(\sum_{i\neq j} h_{iij}^2 + \frac{1}{3}h_{jjj}^2\Big),\ \forall j.$$

Hence we get

$$\begin{aligned}
3(A-2B) &= \sum_{i\neq j\neq k\neq i}[2(\lambda_i^2+\lambda_j^2+\lambda_k^2)-(\lambda_i+\lambda_j+\lambda_k)^2]h_{ijk}^2 \\
&\quad - 3\sum_i \lambda_i^2 h_{iii}^2 + 3\sum_{i\neq j}(\lambda_j^2 - 4\lambda_i\lambda_j)h_{iij}^2 \\
&\le 2S\sum_{i\neq j\neq k\neq i} h_{ijk}^2 + 3\sum_{i\neq j}[(\lambda_j^2-4\lambda_i\lambda_j)h_{iij}^2 - \lambda_j^2 h_{jjj}^2] \\
&\le (2+\delta)S\Big\{\sum_{i\neq j\neq k\neq i} h_{ijk}^2 + 3\sum_{i\neq j} h_{iij}^2 + \sum_j h_{jjj}^2\Big\} \\
&= (2+\delta)S\sum_{i,j,k} h_{ijk}^2.
\end{aligned}$$

Notice that $\delta(3)=0$, $\delta(4)=0.16$, $\delta(5)=0.23$, $\delta(6)=0.28$, $\delta(7)=0.32$, $\delta(8)=0.34$ and $\alpha(n) = \dfrac{2(n+4)(3-n\delta)}{9n+30}$. We conclude from Proposition 3.1 that $S \equiv n$. This completes the proof of Theorem 1.1. □

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Institute of Mathematics and Software Science, Sichuan Normal University, Chengdu 610066, People's Republic of China

E-mail address: zhangdiligence@126.com