PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 5, May 2010, Pages 1569–1574
S 0002-9939(10)10050-1
Article electronically published on January 8, 2010

REDUCTIONS OF IDEALS IN LOCAL RINGS WITH FINITE RESIDUE FIELDS

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(Communicated by Bernd Ulrich)

Abstract. Let I be a proper nonnilpotent ideal in a local (Noetherian) ring (R, M) and let J be a reduction of I; that is, $J \subseteq I$ and $JI^n = I^{n+1}$ for some nonnegative integer n. We prove that there exists a finite free local unramified extension ring S of R such that the ideal IS has a minimal reduction $K \subseteq JS$ with the property that the number of elements in a minimal generating set of K is equal to the analytic spread of K and thus also equal to the analytic spread of I.

1. Introduction

Let I be an ideal in a Noetherian ring R. Northcott and Rees in [6] introduced the concept of a reduction of I, where an ideal J of R is a **reduction of** I if $J \subseteq I$ and $JI^n = I^{n+1}$ for all large integers n. If J is a reduction of I, then I is integral over J. As remarked by Swanson and Huneke in [9, page 5], reductions are an extremely useful tool for integral closure.

Let I be an ideal in a local ring (R, M) and let t be an indeterminate. The **analytic spread** $\ell(I)$ of the ideal I is the dimension of the graded ring

$$R[It]/MR[It] = R/M \oplus I/MI \oplus I^2/MI^2 \oplus \cdots .$$

Northcott and Rees prove in [6] that every reduction of I requires at least $\ell(I)$ generators; moreover, if R/M is infinite, then there exist reductions of I generated by $\ell(I)$ elements.

In the case where (R, M) is a local ring and R/M is finite, Northcott and Rees prove in [7] that if $\dim R = d > 0$, if $\operatorname{char}(R) = \operatorname{char}(R/M)$, and if I is M-primary, then there always exist positive integers r_i, $i = 1, \dots, d$, and elements $b_i \in I^{r_i}$ such that $I^m = b_1 I^{m-r_1} + \cdots + b_d I^{m-r_d}$ for at least one positive integer $m \geq \max\{r_i \mid i = 1, \dots, d\}$. The elements $b_1, \dots, b_d$ are called a **reduction of** I **of type** $[r_1, \dots, r_d]$. Northcott and Rees prove in [7, Theorem 4] that there exists a positive integer n, depending on I, such that I^n has a reduction $c_1, \dots, c_d$ of type $[1, \dots, 1]$, so $(c_1, \dots, c_d)R$ is a reduction of I^n.

Another appoach often used for dealing with reductions of an ideal I in a local ring (R, M) in the case where R/M is finite is to replace (R, M) and I

Received by the editors February 13, 2009, and, in revised form, May 26, 2009.
2000 *Mathematics Subject Classification.* Primary 13A15, 13E05, 13H10.
Key words and phrases. Minimal reduction, analytic spread, finite free local unramified extension, Rees valuation rings, projective equivalence of ideals.

with $(R(X), MR(X))$ and $IR(X)$, where X is an indeterminate and $R(X) = R[X]_{MR[X]}$; cf. [9, Section 8.4]. Then $R(X)/MR(X)$ is isomorphic to the infinite field $(R/M)(X)$, so $IR(X)$ has a minimal reduction that is generated by $\ell(IR(X))$ elements. Since $\ell(IR(X)) = \ell(I)$, this is often helpful for deducing certain things about I and R.

In the present paper, we consider a different approach to reductions in the case where R/M is finite. We use the classical concept of unramified extensions as in the following definition.

Definition 1.1. A quasi-local ring (S, N) is **unramified** over a quasi-local ring (R, M) in the case where R is a subring of S, $N = MS$, and S/N is separable over R/M. If S is an extension ring of a ring R and q is a prime ideal of S, then q is **unramified** over $q \cap R$ in the case where S_q is unramified over $R_{q\cap R}$.

Let I be a nonnilpotent ideal in a local ring (R, M). We show in Theorem 3.1 that there exists a finite free local extension ring (S, N) of R such that S is unramified over R and such that IS has a minimal reduction that is generated by $\ell(IS) = \ell(I)$ elements. In Example 2.3 we exhibit, for each finite field F, a Cohen-Macaulay local ring (R, M) of dimension two with $R/M = F$ such that the associated graded ring $G(M)$ is Cohen-Macaulay and such that M has no reductions generated by analytically independent elements.

We were motivated to consider reductions of ideals in local rings with finite residue fields by our work in [4] on generating sets of ideals and Rees valuation rings. By considering unramified finite integral extensions we have been able to extend results obtained in the papers [2] and [3] in the case of infinite residue fields to the case of finite residue fields. The main result here, Theorem 3.1, gives another instance where some desirable behavior in a local ring (R, M) which usually depends on R/M being infinite can be recovered in the finite residue field case by passing to a finite free local unramified extension ring S of R. We present in Theorem 4.1 an application of Theorem 3.1.

Our notation is mainly as in [5] and [10].

2. Analytic independence and a motivating example

We recall several definitions from [6].

Definition 2.1. Let I be an ideal of a local ring (R, M). Then:

(2.1.1) Elements $b_1, \ldots, b_g$ in I are **analytically independent in** I in the case when the following holds: if $f(X_1, \ldots, X_g) \in R[X_1, \ldots, X_g]$ is a form of arbitrary positive degree n such that $f(b_1, \ldots, b_g) \in MI^n$, then $f(X_1, \ldots, X_g) \in M[X_1, \ldots, X_g]$; that is, all the coefficients of $f(X_1, \ldots, X_g)$ are in M.

(2.1.2) Elements $b_1, \ldots, b_g$ in R are **analytically independent** in the case when the following holds: if $f(X_1, \ldots, X_g) \in R[X_1, \ldots, X_g]$ is a form of arbitrary positive degree n such that $f(b_1, \ldots, b_g) = 0$, then $f(X_1, \ldots, X_g) \in M[X_1, \ldots, X_g]$; that is, all the coefficients of $f(X_1, \ldots, X_g)$ are in M.

(2.1.3) An ideal $J \subseteq I$ is said to be a **minimal reduction** of I if J is a reduction of I and no ideal strictly contained in J is a reduction of I.

(2.1.4) Let $\mu(I)$ denote the number of elements in a minimal generating set of the ideal I.

We collect in Remark 2.2 several facts about analytic independence and reductions.

Remark 2.2. Let (R, M) be a local ring and let I be an ideal of R.

(2.2.1) Elements $b_1, \ldots, b_g$ in R are analytically independent if and only if they are analytically independent in the ideal $(b_1, \ldots, b_g)R$ that they generate.

(2.2.2) If $b_1, \ldots, b_g$ are analytically independent in I, then $\{b_i, \ldots, b_g\}$ is a minimal generating set of the ideal $(b_1, \ldots, b_g)R$.

(2.2.3) In general there is no unique minimal reduction of the ideal I, but if J is a reduction of I, then there exists at least one ideal $K \subseteq J$ such that K is a minimal reduction of I [9, Theorem 8.3.6].

(2.2.4) If J is a minimal reduction of I, then elements in a minimal generating set for J are part of a minimal generating set for I [9, Proposition 8.3.3].

(2.2.5) If J is a reduction of I and $\mu(J) = \ell(I)$, then J is a minimal reduction of I and every minimal generating set of J consists of analytically independent elements [9, Corollary 8.3.5].

(2.2.6) There exists a positive integer n such that I^n has a reduction generated by $\ell(I)$ elements [9, Proposition 8.3.8].

(2.2.7) If (R, M) has an infinite residue field, then every minimal reduction J of I is generated by $\ell(I) = \ell(J)$ elements and these elements are analytically independent [6, Lemma 2, p. 149].

With Remark 2.2.7 in mind, in Example 2.3 we exhibit a Cohen-Macaulay local ring (R, M) of dimension two, so $\ell(M) = 2$, such that M has no (minimal) reduction generated by two elements.

Example 2.3. Let F be an arbitrary finite field. There exists a Cohen-Macaulay local ring (R, M) of dimension two with $R/M = F$ such that the associated graded ring

$$G(M) = R/M \oplus M/M^2 \oplus M^2/M^3 \oplus \cdots$$

is Cohen-Macaulay and such that M has no reductions generated by analytically independent elements.

Proof. Let n denote the number of nonzero elements in F, let $\{X_i\}_{i=1}^{n+3}$ be indeterminates and let S denote the polynomial ring $F[\{X_i\}_{i=1}^{n+3}]$. For each pair of integers j and k with $1 \le j < k \le n+3$, let P_{jk} denote the prime ideal of S generated by the $n+1$ variables $X_i \in \{X_i\}_{i=1}^{n+3}$ with $i \notin \{j, k\}$. Let I denote the intersection of the $\binom{n+3}{2}$ prime ideals P_{jk} and let $T = S/I$. Regarding the polynomial ring S as a graded ring with each X_i of degree one, it is clear that

$$T = \bigoplus_{i \ge 0} T_i = F[T_1]$$

is a homogeneous graded ring of dimension two. Let $N = \bigoplus_{i \ge 1} T_i$ denote the graded maximal ideal of T and let $R = T_N$ and $M = NT_N$. Then (R, M) is a local ring of dimension two and the associated graded ring $G(M) \cong T$. To prove that R is Cohen-Macaulay, it suffices to show that T is Cohen-Macaulay, and to prove that M has no reductions generated by analytically independent elements, it suffices to prove that T has no homogeneous system of parameters consisting of linear forms.

Notice that the minimal prime ideals of T are precisely the prime ideals $P_{jk}T$ and that $T/P_{jk}T = S/P_{jk} \cong F[X_j, X_k]$ and $M/P_{jk}R$ is the maximal ideal of the localized polynomial ring $F[X_j, X_k]_{(X_j,X_k)}$. Therefore M has $\binom{n+3}{2}$ Rees valuation rings with exactly one Rees valuation ring corresponding to each minimal prime of R or, equivalently, of T. The Rees valuation ring of M corresponding to P_{jk} is described by the order valuation on the polynomial ring $F[X_j, X_k]$. The ring T is the Stanley-Reisner ring associated to the simplicial complex Δ on a vertex set $V = \{v_1, \ldots, v_{n+3}\}$, where $\dim \Delta = 1$ and where each edge $\{v_j, v_k\}$ with $1 \le j < k \le n+3$ is in Δ. It is clear that Δ is connected. Since $\dim \Delta = 1$, it follows that Δ is shellable and $T = F[\Delta]$ is a Cohen-Macaulay ring; cf. [1, Exercise 5.1.26, page 222]. Let x_i denote the image of X_i in T_1. Notice that T_1 is a vector space over F of dimension $n+3$ and the x_i form an F-basis for T_1. Suppose there exist $f, g \in T_1$ that form a system of parameters for T. Since T is Cohen-Macaulay, f, g is a regular sequence on T, and for each nonzero $a \in F$, the element $f - ag$ is regular on T. Therefore $f - ag$ is not in any minimal prime of T. This means that

$$f - ag = b_1 x_1 + \cdots + b_{n+3} x_{n+3},$$

where the $b_i \in F$ and at most one of the $b_i = 0$. Both f and g also have this property. Therefore f and g both have a nonzero coefficient of x_i for at least $n+1$ of the x_i, and we may assume this is true for $x_1, \ldots, x_{n+1}$. Then for each $i \in \{1, \ldots, n+1\}$ there exists a nonzero $a_i \in F$ such that $f - a_i g$ has zero as the coefficient of x_i. If c_i is the coefficient of x_i for f and d_i is the coefficient of x_i for g, then $f - \frac{c_i}{d_i} g$ has zero as the coefficient of x_i. Since F has only n nonzero elements, there exists a nonzero element $a \in F$ such that $f - ag$ has zero as the coefficient of x_i for at least two i with $1 \le i \le n+1$. This implies that $f - ag$ is in some minimal prime of T, which contradicts the fact that $f - ag$ is regular on T. $\square$

3. Reductions in local rings with finite residue fields

Examples such as Example 2.3 motivate our consideration in Theorem 3.1 of reductions of ideals in a local ring (R, M) that have a finite residue field.

Theorem 3.1. *Let I be a nonnilpotent ideal of a local (Noetherian) ring (R, M). Let X be an indeterminate, and assume that $\ell(IR[X]_{MR[X]}) = g$. Let J be a reduction of I and let $\mu(J) = h$. Then $h \ge g$, and there exists a simple finite free local unramified extension ring (S, N) of R such that IS has a minimal reduction $K \subseteq JS$ with $\mu(K) = \ell(K) = g = \ell(KS(X))$.*

Proof.[1] If R/M is infinite, then Remark 2.2.7 implies the conclusion holds with $S = R$. Also, if $h = g$, then Remark 2.2.5 implies the conclusion holds with $S = R$. Thus it may be assumed that the field $R/M = F$ is finite and that $h > g$.

Let $R^* = R[X]_{MR[X]}$. Then $R^*/(MR^*) \cong F(X)$ is an infinite field, so there exist elements $c_1(X), \ldots, c_g(X)$ in JR^* such that $L = (c_1(X), \ldots, c_g(X))R(X)$ is a minimal reduction of $IR(X)$, say $I^{n+1}R^* = LI^nR^*$. It may be assumed that $c_1(X), \ldots, c_g(X) \in R[X]$. There exists a polynomial $r(X) \in R[X] \setminus MR[X]$ such that

$$(*) \qquad r(X) I^{n+1} R[X] \subseteq L I^n R[X] \subseteq I^{n+1} R[X].$$

[1] We thank the referee for suggesting the proof given here. It is shorter and more direct than our original proof.

Let an overbar denote residue classes modulo M. Then $\overline{r}(X)$ is a nonzero polynomial in $F[X]$. Let x be an element in the algebraic closure of the finite field F with the property that the degree of the field extension $F(x)/F$ is greater than the degree of the polynomial $\overline{r}(X)$. Thus $\overline{r}(x) \neq 0$. Let $G = F[x] = F[X]/(\overline{f}(X)R[X])$, where $\overline{f}(X) \in F[X]$ is the minimal monic polynomial of x over F.

Let $f(X)$ be a monic pre-image of $\overline{f}(X)$ in $R[X]$ and let $S = R[X]/(f(X)R[X])$, so $S = R[y]$ with $y = X + (f(X))R[X]$. Then $S/(MS) \cong F[x] = G$. It follows that S is a simple finite free extension ring of R with MS its unique maximal ideal and S is unramified over R as in Definition 1.1.

Let $K = (c_1(y), \dots, c_g(y))S$, where the $c_i(X)$ are as in the second paragraph of this proof, so $K = L/(f(X)R[X])$. It follows from (*) that K is a minimal reduction of IS contained in JS and that $\mu(K) = \ell(K) = g = \ell(KS(X))$. $\square$

4. An application concerning projective equivalence

Let I be a regular proper ideal of the Noetherian ring R; that is, I contains a regular element of R and $I \neq R$. An ideal J of R is **projectively equivalent** to I if there exist positive integers m and n such that $(I^m)_a = (J^n)_a$, where K_a is the integral closure in R of an ideal K of R. Let $\mathbf{P}(I)$ denote the set of integrally closed ideals that are projectively equivalent to I.

The set Rees$I$ of Rees valuation rings of I is a finite set of rank one discrete valuation rings (DVRs) that determine the integral closure $(I^n)_a$ of I^n for every positive integer n and is the unique minimal set of DVRs having this property; cf. [9, Section 10.1]. If $(V_1, N_1), \dots, (V_n, N_n)$ are the Rees valuation rings of I, then the integers $(e_1, \dots, e_n)$, where $IV_i = N_i^{e_i}$, are the **Rees integers** of I.

Theorem 4.1 illustrates how Theorem 3.1 may be applied. The issues considered in Theorem 4.1 are developed further in [2], [3] and [4].

Theorem 4.1. *Let I be an M-primary regular ideal of a quasi-unmixed local ring (R, M) of dimension d.*

(4.1.1) *If the greatest common divisor of the Rees integers of I is a unit in R, then there exists a finite free extension ring A of R and an ideal J of A such that $\mathbf{P}(IA) = \mathbf{P}(J) = \{(J^n)_a \mid n \geq 1\}$. Further, if R is an integral domain and z is a minimal prime ideal of A, then the ideal $J_1 = (J + z)/z$ in the integral domain $A_1 = A/z$ is such that $\mathbf{P}(IA_1) = \mathbf{P}(J_1) = \{({J_1}^n)_a \mid n \geq 1\}$.*

(4.1.2) *If the least common multiple of the Rees integers of I is a unit in R, then there exists a finite free extension ring A of R and a radical ideal J of A such that $\mathbf{P}(IA) = \mathbf{P}(J) = \{(J^n)_a \mid n \geq 1\}$ and $JV = N$ for all Rees valuation rings (V, N) of J. Further, if R is an integral domain and z is a minimal prime ideal of A, then $J_1 = (J + z)/z$ is a radical ideal in the integral domain $A_1 = A/z$ and $\mathbf{P}(IA_1) = \mathbf{P}(J_1) = \{({J_1}^n)_a \mid n \geq 1\}$ and $J_1V_1 = N_1$ for all Rees valuation rings (V_1, N_1) of J_1.*

Proof. For (4.1.1), by Theorem 3.1 there exists a finite free local unramified extension ring (S, N) of R such that IS has a minimal reduction $K \subseteq IS$ with $\mu(K) = \ell(K) = d = \ell(KS(X))$. Then $\mathbf{P}(IS) = \mathbf{P}(K)$ and $K = (b_1, \cdots, b_d)S$, where the b_i are analytically independent. Since S is unramified over R, it follows that IS has the same Rees integers as I, with possibly some of the Rees integers of IS occurring more times. Also, IS and K have the same Rees valuation rings and

Rees integers, since K is a reduction of IS. Further, S is quasi-unmixed, by [8, Corollary 2.14 and Theorem 3.1], since R is quasi-unmixed and S is a finite free extension ring of R. Thus we may replace (R, M) by (S, N) and I by K. Since (S, N) is quasi-unmixed, $b_i V = IV$ for each $V \in \text{Rees}K = \text{Rees}IS$ and each $i = 1, \ldots, d$ by [9, Theorem 10.26]. Therefore (4.1.1) follows from [2, Theorem 2.5].

The proof of (4.1.2) is similar, but use [3, Theorem 3.7] in place of [2, Theorem 2.5]. $\square$

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