PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 5, May 2010, Pages 1751–1764
S 0002-9939(10)10194-4
Article electronically published on January 6, 2010

PERIODIC POINT FREE HOMEOMORPHISMS OF THE OPEN ANNULUS: FROM SKEW PRODUCTS TO NON-FIBRED MAPS

T. JÄGER

(Communicated by Yingfei Yi)

ABSTRACT. The aim of this paper is to study and compare the dynamics of two classes of periodic point free homeomorphisms of the open annulus, homotopic to the identity. First, we consider skew products over irrational rotations (often called quasiperiodically forced monotone maps) and derive a decomposition of the phase space that strengthens a classification given by J. Stark. There exists a sequence of invariant essential embedded open annuli on which the dynamics are either topologically transitive or wandering (from one of the boundary components to the other). The remaining regions between these annuli are densely filled by so-called *invariant minimal strips*, which serve as natural analogues for fixed points of one-dimensional maps in this context.

Secondly, we study homeomorphisms of the open annulus which have neither periodic points nor wandering open sets. Somewhat surprisingly, there are remarkable analogies to the case of skew product transformations considered before. Invariant minimal strips can be replaced by a class of objects which we call *invariant circloids*, and using this concept we arrive again at a decomposition of the phase space. There exists a sequence of invariant essential embedded open annuli with transitive dynamics, and the remaining regions are densely filled by invariant circloids. In particular, the dynamics on the whole phase space are transitive if and only if there exists no invariant circloid and if and only if there exists an orbit which is unbounded both above and below.

1. Introduction

We say that a homeomorphism f of the open annulus $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$, homotopic to the identity, is a *quasiperiodically forced (qpf) monotone map* if it has the skew product structure

$$f(\theta, x) \;=\; (\theta + \omega, f_\theta(x)), \quad \omega \in \mathbb{T}^1 \setminus \mathbb{Q}\ . \tag{1.1}$$

The strictly increasing maps $f_\theta : \mathbb{R} \to \mathbb{R}$ are called *fibre maps*. There are several motivations to studying maps of this type and quasiperiodically forced systems in general. First, there are many models in physics with such a skew product structure, occurring basically whenever a system is influenced by two periodic external factors with incommensurate frequencies (see, for example, [2, 3]). Secondly, qpf

Received by the editors February 11, 2009, and, in revised form, September 3, 2009.
2010 *Mathematics Subject Classification.* Primary 37E30.
This work was supported by a research fellowship (Ja 1721/1-1) and the Emmy Noether program (Ja 1721/2-1) of the German Research Foundation (DFG).

©2010 American Mathematical Society
Reverts to public domain 28 years from publication

systems exhibit interesting dynamical phenomena, like the existence of so-called strange non-chaotic attractors, which are not easy to find elsewhere [4, 5]. However, the point we want to emphasize here is a third one, which has not yet received much attention. Qpf monotone maps present the simplest class of annular homeomorphisms in which the existence of periodic orbits is *a priori* excluded. They thus provide simple models whose study might allow a deeper understanding of more general periodic point free dynamics on the annulus (and likewise on the torus). While the existence of fixed and periodic points is already quite well-understood [6, 7, 8], the description of the possible behaviour of systems without periodic orbits still presents a substantial problem.

However, it is evident that the existence of a skew product structure is a very strong assumption. It is therefore far from being obvious that the concepts developed in this setting can be carried over to more general situations. Our aim here is to demonstrate that this is indeed possible by providing analogous results on the two classes of systems mentioned above. It will even turn out that the theory runs more smoothly in the non-fibred setting, since the non-wandering assumption we make there allows us to avoid a number of difficulties. Hence, on the technical side our main achievement will be the improved classification of qpf monotone maps, whereas the results on the non-fibred case are rather interesting from the conceptual point of view.

An important notion in the study of qpf monotone maps is that of a *strip*, which is defined as a compact set $A \subseteq \mathbb{A}$ whose intersection with any fibre $\{\theta\} \times \mathbb{R}$ is a single non-empty compact interval (possibly reduced to a point). A good way to think about these objects is to view them as generalisations of curves $\Gamma = \{(\theta, \gamma(\theta)) \mid \theta \in \mathbb{T}^1\}$, with $\gamma : \mathbb{T}^1 \to \mathbb{R}$ continuous, where each point of the curve is replaced by a vertical interval. A strip is called *pinched* if it intersects generic fibres in a single point, and it is called *minimal* if it does not strictly contain any smaller strip. In order to avoid confusion with the dynamical definition of minimality, we will also call minimal strips *m-strips*. (Note that an invariant minimal strip is not necessarily a minimal set.) One reason for the significance of invariant strips is their intimate relation to minimal sets. Any pinched invariant strip contains a unique minimal set, and conversely one can assign an invariant m-strip to any minimal set [1]. Furthermore, Stark showed that any compact transitive set is either contained in a pinched invariant strip or it is contained in the region between two invariant m-strips which are the two boundary components of the transitive set [1, Theorems 4 and 5]. As a consequence of our results, we will obtain that in the second case the transitive set is actually not only contained in but equal to the region between the two m-strips. The key observation is the following.

Proposition 1.1. *Suppose f is a qpf monotone map and there exists no invariant strip. Then either f is topologically transitive on $\mathbb{A}$ or there exists a curve $\Gamma = \{(\theta, \gamma(\theta)) \mid \theta \in \mathbb{T}^1\}$, with $\gamma : \mathbb{T}^1 \to \mathbb{R}$ continuous, that is disjoint from its image.*

Given two functions $\varphi, \psi : \mathbb{T}^1 \to \mathbb{R} \cup \{\pm\infty\}$, we let

$$(\varphi, \psi) \ := \ \{(\theta, x) \in \mathbb{A} \mid x \in (\varphi(\theta), \psi(\theta))\} \ . \tag{1.2}$$

Half-open and closed intervals are defined similarly. It is easy to see that φ is upper semi-continuous and ψ is lower semi-continuous if and only if (φ, ψ) is homeomorphic to the open annulus. (In this case, the homeomorphism can even be chosen such that it preserves the first coordinate.)

Theorem 1.2. *Suppose f is a qpf monotone map. Then there exists a sequence of invariant open annuli $(A_n = (\varphi_n, \psi_n))_{n\in\Lambda}$, where Λ is a (possibly infinite) interval of integers,*[1] *with $\varphi_n < \psi_n \leq \varphi_{n+1}$ $\forall n \in \Lambda \setminus \{\sup\Lambda\}$, such that either $f_{|A_n}$ is topologically transitive or there exists a curve $\Gamma \subseteq A_n$ which is disjoint from its image. The regions $S_n = [\psi_n, \varphi_{n+1}]$ between these annuli are densely filled by invariant m-strips.*

If the annulus A_n contains a curve Γ which is disjoint from its image, then we say that the dynamics on A_n are wandering. In this case, it is possible to give some more information. Given a function $\gamma : \mathbb{T}^1 \to \mathbb{R}$, we let $f^n_{\theta-n\omega} = f_{\theta-\omega} \circ \cdots \circ f_{\theta-n\omega}$ and define $f^n\gamma$ by

$$f^n\gamma(\theta) \ := \ f^n_{\theta-n\omega}(\gamma(\theta - n\omega)) \ . \tag{1.3}$$

Addendum 1.3. If Γ is a curve in A_n which is disjoint from its image and $f\gamma > \gamma$, then there exists a residual subset $\Theta \subseteq \mathbb{T}^1$ such that on Θ the sequence $f^n\gamma$ converges pointwise to ψ_n as $n \to \infty$ and to φ_n as $n \to -\infty$. The converse statements hold if $f\gamma < \gamma$.

We then turn to study the class of non-wandering homeomorphisms[2] of the open annulus, homotopic to the identity, which we denote by $\mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$. For such maps, an extension of the Poincaré-Birkhoff Theorem due to Franks [6] yields a very general criterium for the existence of periodic orbits. Not much is known, however, about the dynamics in the periodic point free case. One exception is the so-called Arc Translation Theorem by Kwapisz [9] (see also [10]), which concerns the finite-time dynamics 'in the horizontal direction'. It states that if f is a homeomorphism of the compact annulus, homotopic to the identity, with uniquely defined irrational rotation number, then for any $n \in \mathbb{N}$ there exists a simple arc Γ_n that joins the two boundary components and is disjoint from its first n iterates. It therefore shows exactly the same combinatorics as an orbit of the respective irrational rotation on the circle. This result was extended to the case of periodic point free area-preserving homeomorphisms of the open annulus by Béguin, Crovisier and Le Roux [11]. They also showed that an area-preserving homeomorphism of $\mathbb{A}$, homotopic to the identity, is periodic point free if and only if its rotation set is reduced to a single irrational number. (As the definition of the rotation set on the open annulus requires some care and we will not further be concerned with it, we refrain from stating it here and just refer to the very lucid exposition in [12].)

In this note, we want to pursue a complementary direction by considering the dynamical behaviour 'in the vertical direction'. Our aim is to derive analogues of the above statements on qpf monotone maps. Evidently, the first thing which is needed for this purpose is a suitable non-fibred version of the concept of m-strips. Now, an essential feature of an m-strip $A = [\varphi, \psi]$ is that it *'cuts'* the annulus into exactly two open parts $(-\infty, \varphi)$ and (ψ, ∞), and that it is minimal with respect to inclusion amongst such sets. This motivates the following definition.

Definition 1.4. A subset $C \subseteq \mathbb{A}$ is called a *circloid* if it satisfies the following:

(i) C is compact and connected.

[1] Of course, it suffices to consider the three cases $\Lambda = \mathbb{Z}, \Lambda = \mathbb{N}$ or $\Lambda = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$.

[2] Homeomorphisms without wandering open sets.

(ii) The complement of C consists of exactly two connected components, which are both unbounded.[3]

(iii) C does not strictly contain a smaller set with properties (i) and (ii).

While this class of objects does not seem to have been studied systematically before, it should be mentioned that there are many examples of invariant circloids in the literature and even some interesting general results which apply to these objects. The simplest circloids are surely essential simple closed curves. However, circloids may also have a much more complicated structure, as for example the pseudo-circle introduced by Bing [13], which was later described as an invariant set of smooth surface diffeomorphisms by Handel [14] and Herman [15]. Another type of sophisticated examples is given by Birkhoff attractors, which occur in the context of dissipative twist maps [16]. Last but not least, any strip in the above sense is also a circloid (see Lemma 3.1).

If a set $A \subseteq \mathbb{A}$ has properties (i) and (ii) above, we call it an *annuloid*. Such sets have been studied by Franks and Le Calvez in the context of generic annular diffeomorphisms [17] (under the name 'filled essential continua'). Using Carathéodory prime ends, they associated two rotation numbers to an invariant annuloid, one for the prime ends in the upper and one for those in the lower connected component of the complement. If A is an invariant annuloid of an area-preserving annular homeomorphism and does not contain any periodic points, then these two rotation numbers are both equal to the same irrational number, which is the unique rotation number on the annuloid.

Finally, circloids play a crucial role in the proof of the fact that an area-preserving homeomorphism of the two-torus f with lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ is semi-conjugate to the irrational rotation R_ρ on $\mathbb{T}^2$ if and only if there exists a constant $C > 0$ with $|F^n(z) - z - n\rho| \leq C \ \forall n \in \mathbb{N},\ z \in \mathbb{R}^2$ [18]. This can be seen as an extension of Poincaré's celebrated Classification Theorem for orientation-preserving circle homeomorphisms. The main step in the proof in [18] consists of constructing a dynamical lamination of the phase space consisting of pairwise disjoint circloids that are permuted by f in the same way as the horizontal/vertical lines by the respective irrational rotation. Not surprisingly, the result had a precursor in the theory of qpf maps [19].

In our context, we view circloids as 'smallest units' in a decomposition of the phase space. The second type of building blocks appearing in this decomposition are, similar as above, invariant essential embedded open annuli with transitive dynamics. We start again with a preliminary result.

Proposition 1.5. *Suppose $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ has no periodic orbits. Then f is topologically transitive if and only if there exists no invariant circloid and if and only if there exists an orbit which is unbounded both above and below.*

We call a subset of $\mathbb{A}$ *essential* if its complement does not contain a connected component which is unbounded in both directions.[4] If $A \subseteq \mathbb{A}$ is bounded above and has essential closure, we denote by $\mathcal{U}^+(A)$ the connected component of $\overline{A}^c$ which is unbounded above. Similarly, if $A \subseteq \mathbb{A}$ is bounded below and has essential closure,

[3]Note that this implies that one of the two connected components of C^c is bounded below and unbounded above, whereas the other is unbounded below and bounded above.

[4]If A is compact, this just means that it is not contained in any embedded open topological disk.

we denote by $\mathcal{U}^-(A)$ the connected component of $\overline{A}^c$ which is unbounded below. If A is connected, then $\mathcal{U}^+(A) \cup \{\infty\}$, respectively $\mathcal{U}^- \cup \{-\infty\}$, is simply connected and therefore a topological open disk. We call $\mathcal{U}^+(A)$ an *upper hemisphere* and $\mathcal{U}^-(A)$ a *lower hemisphere* in this case.[5] Given two circloids C_1, C_2, we write $C_1 \preccurlyeq C_2$ if $C_1 \cap \mathcal{U}^+(C_2) = \emptyset$ and $C_1 \prec C_2$ if $C_1 \subseteq \mathcal{U}^-(C_2)$. Further, we let $[C_1, C_2] = \mathbb{A} \setminus (\mathcal{U}^-(C_1) \cup \mathcal{U}^+(C_2))$ and $(C_1, C_2) = \mathcal{U}^+(C_1) \cap \mathcal{U}^-(C_2)$. Note that if $C_1 \prec C_2$, then (C_1, C_2) is an essential embedded open annulus.

Theorem 1.6. *Suppose $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ has no periodic orbits. Then there exists a sequence of invariant essential embedded open annuli $(A_n = (C_n^-, C_n^+))_{n \in \Lambda}$, where Λ is a (possibly infinite) interval of integers, with $C_n^- \prec C_n^+ \preccurlyeq C_{n+1}^-$ $\forall n \in \Lambda \setminus \{\sup \Lambda\}$, such that the dynamics on A_n are transitive and the remaining regions $S_n = [C_n^+, C_{n+1}^-]$ between A_n and A_{n+1} are densely filled by invariant circloids.*

Remark 1.7. (a) In analogy to the theory of twist maps, it seems appropriate to call the transitive annuli in the above statement *'instability regions'* (see also [17]).

(b) A similar decomposition statement holds on the torus. A non-wandering toral homeomorphism which is homotopic to the identity and has no periodic orbits is either topologically transitive or it has an invariant circloid [18]. In the second case, the complement of this circloid is homeomorphic to an open annulus, and consequently Theorem 1.6 yields a decomposition into transitive annuli and invariant circloids.

(c) The simplest examples of transitive but non-minimal homeomorphisms of $\mathbb{A}$ without periodic orbits are given by skew translations (rotation cocycles) of the form $f(\theta, x) = (\theta + \omega, x + \alpha(x))$, where $\alpha : \mathbb{T}^1 \to \mathbb{R}$ is continuous. The Gottschalk-Hedlund Theorem states that when α has no continuous coboundary (that is, f has no continuous invariant curves), then there exists a residual set of orbits which are unbounded above and below and f is therefore transitive. Such examples were given, for example, by Furstenberg (see [20]). Real-analytic functions α without continuous coboundary can be constructed via the Anosov-Katok (or fast-approximation) method [21, Section 12.6b].

The construction of non-fibred annular homeomorphisms with transitive but non-minimal dynamics is more intricate, but again real-analytic (and also area-preserving) examples can be produced via the Anosov-Katok method. This is done e.g. in [22], where the authors show in addition that in their examples the set of points with dense orbit has zero Lebesgue measure.

(d) An issue that we do not go into here but which will be a natural topic for future research is the description of the dynamics on invariant circloids. There are many open questions that can be asked, in particular in analogy to results on invariant strips for fibred systems. For example, how many minimal sets can a periodic point free invariant circloid contain? Does every invariant circloid contain an almost automorphic minimal set? Under what assumptions are the dynamics on an invariant circloid semi-conjugate to an irrational rotation?

[5] Formally we should say 'punctured hemispheres', but we prefer this terminology for the sake of brevity.

Some interesting results on invariant circloids can be found in [17], where the authors use Carathéodory prime ends to study rotation numbers and the existence of periodic orbits for circloids and annuloids.

2. Quasiperiodically forced monotone maps

2.1. **Notation and preliminaries on invariant strips.** Suppose f is a qpf monotone map. Then a map $\varphi : \mathbb{T}^1 \to \mathbb{R}$ is called an f-invariant graph if

$$f_\theta(\varphi(\theta)) \;=\; \varphi(\theta+\omega) \quad \forall \theta \in \mathbb{T}^1 \; . \tag{2.1}$$

If $C \subseteq \mathbb{A}$, we let $C_\theta := \{x \in \mathbb{R} \mid (\theta, x) \in C\}$. Given any subset $C \subseteq \mathbb{A}$ with $\pi_1(C) = \mathbb{T}^1$, we define its *upper* and *lower bounding graphs* as

$$\varphi^+_C(\theta) \;:=\; \sup C_\theta \quad \text{and} \quad \varphi^-_C(\theta) \;:=\; \inf C_\theta \; . \tag{2.2}$$

When C is compact, then φ^+_C is upper semi-continuous (u.s.c.) and φ^-_C is lower semi-continuous (l.s.c.). If C is compact and invariant, then the graphs $\varphi^\pm_C$ are invariant and $A = [\varphi^-_C, \varphi^+_C]$ is an invariant strip. Note that more generally we can apply definition (2.2) to any bounded F-invariant set A in order to obtain invariant (but not necessarily semi-continuous) bounding graphs.

For any graph $\mathbb{T}^1 \to \mathbb{R}$ we denote its point set by the corresponding capital letter, for example $\Phi := \{(\theta, \varphi(\theta)) \mid \theta \in \mathbb{T}^1\}$. Further, we let $\varphi^+ := \varphi^+_{\overline{\Phi}}$ and $\varphi^- := \varphi^-_{\overline{\Phi}}$. For simplicity, we denote $\varphi^{+-} = (\varphi^+)^-$, $\varphi^{-+} = (\varphi^-)^+$. An important property of this construction is the following.

Lemma 2.1 ([1])**.** *Given any u.s.c. graph φ, the strip $[\varphi^-, \varphi]$ is pinched. The analogous statement holds for l.s.c. graphs.*

The next lemma describes the relations between minimal sets, invariant m-strips and their bounding graphs.

Lemma 2.2 ([1, 19])**.** *If M is a minimal set, then $A = [\varphi^-_M, \varphi^+_M]$ is an m-strip. Conversely, if $A = [\varphi^-, \varphi^+]$ is an m-strip, then it contains a unique minimal set $M = \overline{\Phi^-} = \overline{\Phi^+}$ and $\varphi^{-+} = \varphi^+$ and $\varphi^{+-} = \varphi^-$ hold.*

There is a simple procedure to produce m-strips, starting with semi-continuous invariant graphs.

Lemma 2.3 ([19])**.** *Suppose φ is a u.s.c. invariant graph. Then $C = [\varphi^-, \varphi^{-+}]$ is an m-strip. If φ is an invariant graph, then C is invariant.*

Finally, if A and B are two strips, then we say that $A \preccurlyeq B$ if and only if $\varphi^-_A \leq \varphi^-_B$ and $\varphi^+_A \leq \varphi^+_B$. We say that $A \prec B$ if $\varphi^+_A < \varphi^-_B$.

2.2. **Decomposition of the phase space for qpf monotone maps.** We start by proving Proposition 1.1, which is a direct consequence of the following two lemmas.

Lemma 2.4. *Suppose f is a qpf monotone map without invariant strips. Then either f is topologically transitive or there exists a wandering open set.*

Proof. A direct proof is given implicitly in [23] (as Part 2 of the proof of Thm. 4.4). Here, we simply invoke Proposition 1.5 (which is proved in the next section). Suppose f has no wandering open set. Then it meets the assumptions of Proposition 1.5, such that it is either topologically transitive or has an invariant circloid C. However, in the second case, $[\varphi^-_C, \varphi^+_C]$ would be an invariant strip, contradicting the assumptions. Hence, f must be topologically transitive. □

Lemma 2.5. *Suppose f is a qpf monotone map without invariant strips. If there exists a wandering open set W, then there exists a curve $\Gamma = \{(\theta, \gamma(\theta)) \mid \theta \in \mathbb{T}^1\}$, with $\gamma : \mathbb{T}^1 \to \mathbb{R}$ continuous, which is disjoint from its image.*

Proof. We construct the point set Γ of the curve. Note that we may allow Γ to contain vertical segments: the property we are interested in is open w.r.t. Hausdorff distance. Therefore any vertical parts can be slightly tilted in order to obtain a curve that can be represented as a graph over $\mathbb{T}^1$. See Figure 1.

By going over to a suitable iterate, we may assume w.l.o.g. that W contains a ball of diameter larger than $d(0,\omega)$. Let $\Lambda_0 \subseteq W$ be a straight horizontal line segment of length ω. Denote the endpoints of Λ_0 by $a_0 = (\theta_0, x_0)$ and $b_0 = (\theta_0+\omega, x_0)$. Further, let $\Lambda_k := f^k(\Lambda_0)$, $x_k := f^k_{\theta_0}(x_0)$ and $y_k := f^k_{\theta_1}(x_0)$, such that $a_k := (\theta_k, x_k) = f^k(a_0)$ and $b_k := (\theta_{k+1}, y_k) = f^k(b_0)$. Denote the closed vertical line segment between b_k and a_{k+1} by $[b_k, a_{k+1}]$. Finally, let $\Gamma_k := \bigcup_{i=0}^{k} \Lambda_i \cup \bigcup_{i=0}^{k-1} [b_i, a_{i+1}]$. Note that Γ_k is a curve that joins a_0 and b_k and contains k vertical segments. Further, by construction,

$$f(\Gamma_k) \setminus \Gamma_k \;=\; \Lambda_{k+1} \cup (b_k, a_{k+1}]\ . \tag{2.3}$$

Now let $m := \max\{k \in \mathbb{N} \mid k\omega < 1\}$ (here, we exceptionally view ω as an element of $(0,1) \subseteq \mathbb{R}$), and define Γ as the union of $\Gamma_m \setminus \pi_1^{-1}([0,\omega))$, Λ_0 and the vertical line segment between a_0 and Λ_m. This defines a closed curve that winds exactly once around the torus and contains $m+1$ vertical segments. Further, if we assume w.l.o.g. that $m \geq 2$, then $\Gamma_1 \subseteq \Gamma$. From now on we assume that a_1 lies above b_0, such that a_{k+1} lies above b_k for all $k \in \mathbb{N}_0$ by monotonicity. (The case that a_1 lies below b_0 is completely symmetric.) In order to show that Γ (or a modification thereof) is mapped either above or below itself, we have to distinguish three different cases (see Figure 2).

Case 1. First, assume there exists $n \in \mathbb{N}$ such that b_n lies above Λ_0.[6] W.l.o.g. we can assume $n = m$; otherwise we lift the system to the j-fold cover $(\mathbb{R}/j\mathbb{Z}) \times \mathbb{T}^1$ of $\mathbb{T}^2$, where j is the integer part of $n\omega$, and repeat the construction of Γ as above. (It easy to see that if there exists a curve with the required property on this j-fold cover, then the same is true for the original system.) If b_m lies above Λ_0, then by the assumption made above, the same holds for a_{m+1}. Further, by monotonicity of the fiber maps, b_{m+1} lies above Λ_1. As Λ_{m+1} joins a_{m+1} and b_{m+1} and cannot intersect $\Lambda_0 \cup \Lambda_1$ (recall that Λ_0 is contained in the wandering set W), we obtain that $(b_m, a_{m+1}] \cup \Lambda_{m+1}$ lies strictly above Γ_1. Together with (2.3) this implies $f(\Gamma) \succcurlyeq \Gamma$ and $f^N(\Gamma) \succ \Gamma$ for sufficiently large N. Γ can now easily be modified in order to obtain $f(\Gamma) \succ \Gamma$.

Case 2. Secondly, assume there exists $n \in \mathbb{N}$ such that b_n lies below Λ_0 and Λ_{n+1} does not intersect $[b_0, a_1]$. Again, we can assume $n = m$. As Λ_{m+1} cannot intersect $\Lambda_0 \cup \Lambda_1$, it is disjoint from Γ_1 in this case. Since b_{m+1} lies below $\Lambda_1 \subseteq \Gamma_1$ by monotonicity, this implies that the whole curve $\Lambda_{m+1} \cup (b_m, a_{m+1}]$ is below Γ_1. Similarly to the above, we obtain $f(\Gamma) \preccurlyeq \Gamma$ and $f^N(\Gamma) \prec \Gamma$ for sufficiently large N. Again replacing Γ by a slight modification if necessary, we obtain $F(\Gamma) \prec \Gamma$.

[6] Suppose Λ is the point set of a curve λ defined on a subinterval I of $\mathbb{T}^1$ and $b \in \mathbb{T}^2$. Then by saying $b = (\theta, x)$ is above Λ, we mean that $x > \lambda(\theta)$, implicitly assuming $\theta \in I$. In the following we will use similarly obvious terminology without further explanation.

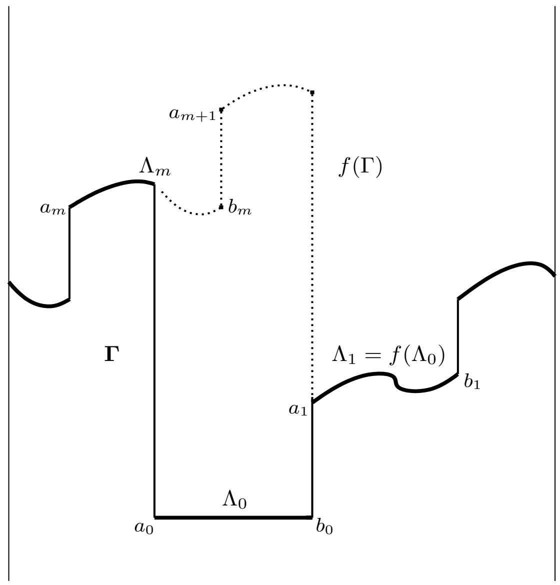

FIGURE 1. The construction of the curve Γ (thick line) and its image $f(\Gamma)$ (dotted line). The picture corresponds to Case 1 in Figure 2.

Case 3. Finally, we have to address the case where b_n lies below Λ_0 and Λ_{n+1} intersects $[b_0, a_1]$ whenever $\pi_1(b_n) \in \pi_1(\Lambda_0)$. We show that this implies the existence of an invariant strip, contradicting the assumptions. Let $\Omega := \bigcap_{k\geq 0} \overline{\bigcup_{j=k}^{\infty} \Gamma_j}$. Clearly Ω is a closed f-invariant set. We claim that for a sufficiently small $\delta > 0$ its upper bounding graph satisfies

$$x_0 + \delta \ \leq \ \varphi_\Omega^+(\theta) \ \leq \ x_1 + \delta \quad \forall \theta \in I := (\theta_1 - \delta, \theta_1). \tag{2.4}$$

By invariance, this implies immediately that φ_Ω^+ is bounded. Thus the closed set Ω is bounded above, and its upper bounding graph φ_Ω^+ is upper semi-continuous. The inequalities in (2.4) then hold for φ_Ω^{+-} as well, such that this graph is also bounded and $[\varphi_\Omega^+, \varphi_\Omega^{+-}]$ defines an invariant strip.

It remains to prove (2.4). As Γ_0 is contained in the wandering set W, there exist small boxes $W_0 := B_\delta(\theta_1) \times B_\delta(x_0)$ around b_0 and $W_1 := B_\delta(\theta_1) \times B_\delta(x_1)$ around a_1 which no curve Λ_j with $j \geq 2$ can intersect. We fix $\delta \in (0, \omega)$ with this property. Now, whenever $\pi_1(b_n) \in \pi_1(\Lambda_0)$, the curve $\Lambda_n \cup [b_n, a_{n+1}] \cup \Lambda_{n+1} = f^n(\Gamma_1)$ has to pass through below the set $\widehat{W}_1 := I \times B_\delta(x_1)$. This holds for Λ_{n+1} as this curve must intersect $[b_0, a_1]$ and cannot intersect $\widehat{W}_1$, and Λ_n lies below Λ_0 anyway as this is true for its right endpoint b_n. Consequently none of the sets $\bigcup_{j\geq k} \Gamma_j$ intersect the region $\{(\theta, x) \mid \theta \in I,\ x > x_1 - \delta\}$, and from this the upper bound in (2.4) follows easily.

For the lower bound, note that there are infinitely many $n \in \mathbb{N}$ such that $\pi_1(W_1) \subseteq \pi_1(\Lambda_{n+1})$. For such an n, the curve Λ_{n+1} has to pass through between the boxes W_0 and W_1 on their whole width. Therefore the upper bounding graphs of the sets $\overline{\bigcup_{j\geq k} \Gamma_j}$ are always above W_0, and consequently the same is true for their pointwise limit φ_Ω^+. $\square$

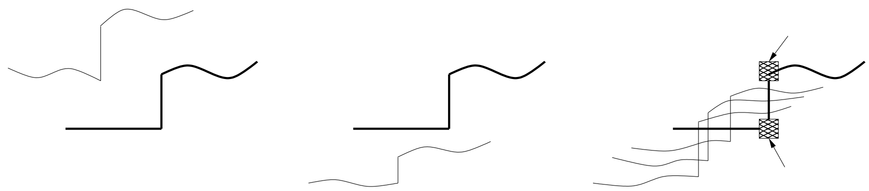

FIGURE 2. Distinction of the three cases in the construction of Γ. Case 1 implies $F(\Gamma) \succcurlyeq \Gamma$, and Case 2 implies $F(\Gamma) \preccurlyeq \Gamma$. Case 3 is excluded since it leads to the existence of an invariant strip.

Proof of Theorem 1.2 . Given a qpf monotone map f, we denote by $\mathcal{M}$ the set of all f-invariant m-strips and let $\mathcal{C} = \mathrm{cl}\left(\bigcup_{C \in \mathcal{M}} C\right)$. Suppose that A is a connected component of $\mathcal{C}^c$. Then there exists an interval $I \subseteq \mathbb{T}^1$ and $x_0 \in \mathbb{R}$ such that $I \times \{x_0\} \subseteq A$. Let

$$\varphi(\theta) := \sup\left\{\varphi_C^- \mid C \in \mathcal{M},\ \varphi_C^-(\theta') < x_0\ \forall \theta' \in I\right\} \ , \tag{2.5}$$

$$\psi(\theta) := \inf\left\{\varphi_C^+ \mid C \in \mathcal{M},\ \varphi_C^+(\theta') > x_0\ \forall \theta' \in I\right\} \ . \tag{2.6}$$

(For simplicity we assume that these graphs are bounded, but the cases $\varphi \equiv -\infty$ or $\psi \equiv \infty$ can be treated similarly.) Then φ is l.s.c. as the supremum over a family of l.s.c. functions, and similarly ψ is u.s.c. Lemma 2.3 implies that $C^- = [\varphi^{+-}, \varphi^+]$ and $C^+ = [\psi^-, \psi^{-+}]$ are invariant m-strips. Clearly, $I \times \{x_0\}$ lies above C^- and below C^+. Further, by definition of φ and ψ there are no invariant m-strips contained in the invariant annulus $\tilde{A} = (\varphi^+, \psi^-)$. Consequently the latter is a connected component of $\mathcal{C}^c$, and since $I \times \{x_0\} \subseteq \tilde{A}$ we must have $\tilde{A} = A$.

We thus obtain that all connected components of $\mathcal{C}^c$ are embedded essential open annuli, and labelling them according to their order on $\mathbb{A}$ yields the required decomposition of the phase space. Since each of the annuli A_n is homeomorphic to $\mathbb{A}$, Proposition 1.1 yields the dichotomy for the dynamics on the A_n. □

Proof of Addendum 1.3. Given a curve $\Gamma = \{(\theta, \gamma(\theta)) \mid \theta \in \mathbb{T}^1\}$ with $f(\Gamma) \succ \Gamma$, the sequence of continuous functions $f^n\gamma$ is monotonically increasing and bounded by ψ_n. (The case $\psi_n \equiv \infty$ is treated similarly.) Hence, they have to converge pointwise to an l.s.c. and invariant graph $\psi \leq \psi_n$. Since the strip $[\psi^{+-}, \psi^+]$ is an invariant m-strip and there are no such strips in A_n, we must have $\psi^+ = \psi_n^+$, which implies that ψ and ψ_n^+ coincide on a residual set Θ by Lemma 2.1 . Since $\psi \leq \psi_n \leq \psi_n^+$, this implies that ψ and ψ_n are equal on Θ as well. □

Remark 2.6. There are no known examples of a qpf monotone map admitting a free curve where one has $\psi \neq \psi_n$ in the proof of the addendum. However, examples of this type exist for so-called *pinched skew products* [4, 24], which are qpf interval maps with non-decreasing but not strictly monotone fibre maps.

3. PERIODIC POINT FREE NON-WANDERING HOMEOMORPHISMS OF THE OPEN ANNULUS

3.1. **Invariant circloids.** The aim of this section is twofold. On the one hand, we provide some preliminary results on circloids, which will entail Proposition 1.5 and Theorem 1.6 rather easily in the next section. On the other hand, we will also mention some facts, mostly taken from [18], that will not further be used in the present paper. The motivation for this is the following: While the analogy between qpf and periodic point free maps has already led to some advances [18, 25] and further progress in this direction is expected, it will be rare that results on both classes of maps can be presented in the same article. Hence, we want to take advantage of this opportunity and discuss some relations between circloids and strips in more detail, which shows that even on a technical level the two types of objects can often be treated in a very similar way. We start with the following.

Lemma 3.1. *Any m-strip* $C = [\varphi_C^-, \varphi_C^+]$ *is a circloid.*

Proof. Suppose C is an m-strip. Then the bounding graphs φ_C^- and φ_C^+ can be approximated from below, respectively above, by monotone sequences of continuous functions φ_n^-, φ_n^+. Thus $C = \bigcap_{n\in\mathbb{N}}[\varphi_n^-, \varphi_n^+]$ is the intersection of a nested sequence of compact annuli. This shows that C is connected and essential. Further, its complement consists exactly of the two connected components $(-\infty, \varphi_C^-) = \mathcal{U}^-(C)$ and $(\varphi_C^+, \infty) = \mathcal{U}^+(C)$. Hence C is an annuloid, and it remains to show that it does not strictly contain a smaller annuloid.

In order to do so, suppose that $C' \subsetneq C$ is compact and let $(\theta, x) \in C \setminus C'$. Then there is a small box $B_\delta(\theta) \times B_\delta(x)$ disjoint from C'. Since we have $\Phi_C^+ \subseteq \overline{\Phi_C^-}$ and $\Phi_C^- \subseteq \overline{\Phi_C^+}$ by Lemma 2.2 and since $x \in [\varphi_C^-(\theta), \varphi_C^+(\theta)]$, we can find $\theta_1, \theta_2 \in B_\delta(\theta)$ with $\varphi_C^-(\theta_1) > x - \delta/2$ and $\varphi_C^+(\theta_2) < x + \delta/2$. This implies that the infinite line obtained as the union of the vertical half-lines from $(\theta_1, -\infty)$ to $(\theta_1, x - \delta/2)$ and from $(\theta_2, x + \delta/2)$ to (θ_2, ∞) with the straight line segment from $(\theta_1, x - \delta/2)$ to $(\theta_2, x + \delta/2)$ is disjoint from C'. Hence C' is not essential and therefore there is no annuloid. $\square$

In the fibred setting, an important operation is to assign bounding graphs to a strip. Obviously, due to the lack of a fibred structure this does not make sense anymore for annuloids. (One may of course define such bounding graphs, but these neither determine the annuloid uniquely nor inherit its invariance.) However, if C is a strip, then the connected components $\mathcal{U}^-(C) = (-\infty, \varphi_C^-)$ and $\mathcal{U}^+(C) = (\varphi_C^+, \infty)$ encode exactly the same information as the graphs $\varphi_C^\pm$. Hence, it is natural to consider the hemispheres $\mathcal{U}^-(C)$ and $\mathcal{U}^+(C)$ as the natural substitutes for bounding graphs in the non-fibred setting. It turns out that this analogy carries surprisingly far. For example, the minimality of a strip C is characterised by the fact that $\varphi_C^{+-} = \varphi_C^-$ and $\varphi_C^{-+} = \varphi_C^+$ (Lemma 2.2). Furthermore, any upper semi-continuous graph φ generates an associated m-strip $[\varphi^-, \varphi^{-+}]$. These statements have their counterparts for circloids. We write $\mathcal{U}^{+-}(C)$ instead of $\mathcal{U}^-(\mathcal{U}^+(C))$, etc.

Lemma 3.2 ([18])**.** *An annuloid C is a circloid if and only if $\mathcal{U}^{-+}(C) = \mathcal{U}^+(C)$ and $\mathcal{U}^{+-}(C) = \mathcal{U}^-(C)$. Further, given any upper hemisphere*[7] *U, the set $C = \mathbb{A} \setminus (\mathcal{U}^-(U) \cup \mathcal{U}^{-+}(U))$ is a circloid.*

Given an annuloid A, we can apply Lemma 3.2 to $\mathcal{U}^+(A)$ to obtain a circloid $C^+ = \mathbb{A} \setminus (\mathcal{U}^{+-}(A) \cup \mathcal{U}^{+-+}(A))$. As it is easy to see that $C^+ \subseteq \mathbb{A}$ and that when A is invariant then this is inherited by C^+, we obtain the following.

Corollary 3.3. *Any annuloid contains a circloid. If $f \in \mathrm{Homeo}_0(\mathbb{A})$, then any f-invariant annuloid contains an f-invariant circloid.*

[7]We call U an upper hemisphere if $U \cup \{+\infty\}$ is simply connected and bounded below and contains a neighbourhood of $+\infty$.

Another important notion for qpf maps is that of a pinched strip, which could equivalently be defined as a strip with empty interior [1]. Hence, if we call an annuloid with empty interior *thin*, then this should be the analogue of 'pinched'. A crucial property of pinched strips is that they only contain one minimal strip [1], and the analogous statement is again true for thin annuloids.

Lemma 3.4 ([18])**.** *Any thin annuloid contains exactly one circloid.*

However, there are also some differences to the fibred situation which should be mentioned. First, contrary to a pinched strip which always contains a unique minimal set [1], a thin annuloid may contain several of these. This is the case for some Birkhoff attractors of dissipative twist maps which contain different periodic orbits (e.g. [16]). Secondly, while any m-strip is pinched [1], a circloid is not necessarily thin (see Figure 3). However, this becomes true under some additional conditions.

Lemma 3.5 ([18])**.** *If $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ and C is an f-invariant circloid, then C has empty interior.*

Similarly, two invariant circloids do not always have to be disjoint, but again this is true in the non-wandering case.

Lemma 3.6 ([18])**.** *Suppose $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$. Then any two f-invariant circloids are either equal or disjoint.*

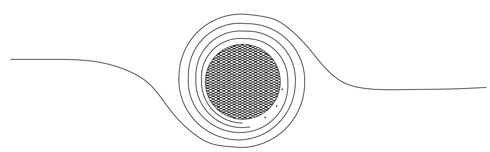

FIGURE 3. Example of a circloid with non-empty interior.

In fact, the two preceding statements are more or less direct consequences of the following technical lemma. Since it plays a crucial role in the next section, we repeat the short proof given in [18] for the convenience of the reader.

Lemma 3.7 ([18])**.** *Suppose $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ has no periodic points. Then any open f-invariant set contains an essential simple closed curve.*

Proof. Suppose that $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ has no periodic points and $V \subseteq \mathbb{A}$ is an open f-invariant set. Fix a small open ball $B \subseteq V$. Since B is non-wandering, there exists some $k \geq 1$ with $f^k(B) \cap B \neq \emptyset$. Choose a lift $G : \mathbb{R}^2 \to \mathbb{R}^2$ of f^k and a connected component $\hat{B}$ of $\pi^{-1}(B)$ such that $G(\hat{B}) \cap \hat{B} \neq \emptyset$ (here $\pi : \mathbb{R}^2 \to \mathbb{A}$ denotes the canonical projection). Since G has no periodic points, a sufficiently small ball $D \subseteq \hat{B}$ will satisfy $G(D) \cap D = \emptyset$. It follows from a result by Franks [6, Prop. 1.3] that $G^n(D) \cap D = \emptyset \ \forall n \in \mathbb{Z}$. Thus, as $\pi(D)$ is non-wandering for f^k (Lemma A.1), the G-orbit of D has to intersect one of its integer translates. The same then certainly holds for $\hat{B}$. Since $\bigcup_{n \in \mathbb{Z}} G^n(\hat{B}) \subseteq \pi^{-1}(V)$ is connected, this shows that V contains an essential closed curve, which can be chosen to be simple. $\square$

3.2. **Decomposition of the phase space.** As mentioned, the statements given in the introduction now follow quite easily.

Proof of Proposition 1.5. It is obvious that transitivity implies the existence of an orbit which is unbounded in both directions, and this in turn excludes the existence of an invariant circloid. Hence, it remains to show that if $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ has no periodic orbits and is not topologically transitive, then it has an invariant circloid.

Due to the lack of transitivity, there exist open sets $U, V \subseteq A$ with $f^n(U) \cap V = \emptyset\ \forall n \in \mathbb{N}$. Since f is non-wandering, we even have $f^n(U) \cap V = \emptyset\ \forall n \in \mathbb{Z}$. By going over to a higher iterate if necessary, we may assume that $f(U) \cap U \neq \emptyset$ and $f(V) \cap V \neq \emptyset$. (Here, we use the fact that all iterates of f are non-wandering by Lemma A.1.) It follows that the two disjoint open invariant sets $\tilde{U} = \bigcup_{n\in\mathbb{Z}} f^n(U)$ and $\tilde{V} = \bigcup_{n\in\mathbb{Z}} f^n(V)$ are connected. By Lemma 3.7 both sets contain an essential simple closed curve and, consequently, one of them must lie below the other, say $\tilde{V} \subseteq \mathcal{U}^-(\tilde{U})$. Then $\mathcal{U}^{-+}(\tilde{U}) \subseteq \mathcal{U}^+(\tilde{V})$, which implies that $\mathcal{U}^{-+}(\tilde{U})$ and $\mathcal{U}^{+-}(\tilde{V})$ are disjoint. It follows that the set $A = \mathbb{A} \setminus (\mathcal{U}^{-+}(\tilde{U}) \cup \mathcal{U}^{+-}(\tilde{V}))$ is an invariant annuloid and Corollary 3.3 therefore yields the existence of an invariant circloid. □

Proof of Theorem 1.6 . Given $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ without periodic orbits, let $\mathcal{M}$ be the set of all f-invariant circloids and $\mathcal{C} = \mathrm{cl}\left(\bigcup_{C\in\mathcal{M}} C\right)$. Suppose A is a connected component of $\mathcal{C}^c$. As f is non-wandering, the open set A has to be periodic; that is, there exists some $n \in \mathbb{N}$ such that A is f^n-invariant. Due to Lemma 3.7, A contains an essential simple closed curve Γ. Further, any Jordan curve J in A which is nullhomotopic in $\mathbb{A}$ is also nullhomotopic in A. This follows from the fact that no circloid can intersect the interior of the disk bounded by J, since it would otherwise have to be contained in this disk by connectedness. Hence, A is an embedded essential open annulus. This implies in particular that A is not only periodic but invariant. (An essential embedded annulus that is mapped above or below itself can never return.) We let $C^+ = \mathbb{A} \setminus (\mathcal{U}^{+-}(A) \cup \mathcal{U}^{+-+}(A))$. By Lemma 3.2, C^+ is a circloid. In the same way we define $C^- = \mathbb{A} \setminus (\mathcal{U}^{-+}(A) \cup \mathcal{U}^{-+-}(A))$. Both C^- and C^+ inherit the invariance of A and are therefore contained in $\mathcal{C}$. Hence $A \subseteq (C^-, C^+)$. Conversely, there is no other circloid C contained in (C^-, C^+), since this would imply that either $A \subseteq (C^-, C)$ or $A \subseteq (C, C^+)$. It follows that $A = (C^-, C^+)$. The fact that $f_{|A}$ is topologically transitive follows from Proposition 1.5 . Labelling all such annuli according to their ordering on $\mathbb{A}$ now yields the statement of the theorem. □

Appendix A. Iterates of non-wandering homeomorphisms

The following lemma, which was used in the proof of Proposition 1.5, is surely well-known folklore. However, as we do not know a suitable reference, it is included for the convenience of the reader.

Lemma A.1. *Suppose X is a locally connected Hausdorff space and $f \in \mathrm{Homeo}(X)$ has no wandering open sets. Then no iterate of f has wandering open sets.*

Proof. Given any set $A \subseteq X$ and $q \in \mathbb{N}$, we always let $\tilde{A} := \bigcup_{n\in\mathbb{Z}} f^{nq}(A)$. Suppose for a contradiction that U is an f^q-wandering open set, $q > 1$. We start by excluding two simple cases:

(a) If $\tilde{U} \cap f^j(\tilde{U}) = \emptyset\ \forall j = 1, \ldots, q-1$, then U is obviously f-wandering, contradicting the assumptions.

(b) Suppose $f(\tilde{U}) = \tilde{U}$. Let $W \subseteq U$ be open and connected. Then for any $j = 1,\ldots,q-1$ there exists some $n_j \in \mathbb{Z}$ with $f^j(W) \subseteq f^{n_j q}(U)$. (Note that since the sets $f^{nq}(U),\ n \in \mathbb{Z}$, are pairwise disjoint, connected subsets of $\tilde{U}$ are always contained in precisely one of these sets.) Since $\tilde{U}$ contains no periodic points, we can reduce W further if necessary to ensure that $f^j(W) \cap f^{n_j q}(W) = \emptyset\ \forall j = 1,\ldots,q-1$. Hence $f^j(W) \cap \tilde{W} = \emptyset\ \forall j = 1,\ldots,q-1$. Since $\tilde{W}$ is f-invariant, we obtain $\tilde{W} \cap f^j(\tilde{W}) = \emptyset\ \forall j = 1,\ldots,q-1$ such that we arrive at case (a).

In order to treat the general case, we first remark that by proceeding by induction on q, we may always assume q to be prime. For if the statement holds for all $q' \leq q$ and $q+1 = mn$ with $m,n > 1$, then by the induction assumption, $f^m \in \mathrm{Homeo}(X)$ has no wandering open sets, and therefore, again by the induction assumption, the same is true for its n-th iterate $f^{mn} = f^{q+1}$. If $1 \leq l \leq q$, let

$$A_l \ := \ \left\{(\alpha_1,\ldots,\alpha_l) \in \mathbb{N}_0^l \mid 0 \leq \alpha_1 < \cdots < \alpha_l \leq q-1\right\} \ .$$

Given $\alpha \in A_l$ and $j \in \mathbb{Z}$, we let $\alpha + j$ denote the vector in A_l whose entries are $\alpha_i + j \bmod q$ $(i = 1,\ldots,l)$. (The order of the entries may have to be rearranged.) Further, for $\alpha \in A_l$, we define the set

$$C_\alpha \ := \ \bigcap_{i=1}^{l} f^{\alpha_i}(\tilde{U}) \ .$$

Let $k := \max\{l \in \{1,\ldots,q\} \mid \exists \alpha \in A_l : C_\alpha \neq \emptyset\}$. If $l = q$, then the set

$$V \ := \ U \cap \bigcap_{i=1}^{q-1} f^i(\tilde{U})$$

is non-empty, f^q-wandering and obviously satisfies $\tilde{V} = \bigcap_{j=0}^{q-1} f^j(\tilde{U}) = f(\tilde{V})$. This leads to case (b). Hence, we may assume $k < q$. Now, choose some $\alpha \in A_k$ with $C_\alpha \neq \emptyset$. Without loss of generality, we may assume $\alpha_1 = 0$ (otherwise replace α by $\alpha - \alpha_1$). Let

$$W \ := \ \bigcup_{j=0}^{q-1} f^j(U) \cap f^{\alpha_2+j}(\tilde{U}) \cap \ldots \cap f^{\alpha_k+j}(\tilde{U}) \ \neq \ \emptyset \ .$$

Then $\tilde{W} = \bigcup_{j=0}^{q-1} f^j(C_\alpha)$ satisfies $f(\tilde{W}) = \tilde{W}$ (note that C_α is f^q-invariant), and W is non-empty and open. Furthermore, we claim that W is f^q-wandering.

In order to see this, suppose $f^{nq}(W) \cap W \neq \emptyset$. Then there exist j,m such that

$$\left(f^{nq+j}(U) \cap \bigcap_{i=2}^{k} f^{\alpha_i+j}(\tilde{U})\right) \cap \left(f^m(U) \cap \bigcap_{i=2}^{k} f^{\alpha_i+m}(\tilde{U})\right) \ \neq \ \emptyset \ .$$

$j = m$ is not possible, since U is f^q-wandering. However, if $j \neq m$, then the above set is contained in the intersection $C_{\alpha+j} \cap C_{\alpha+m}$. Since q is a prime number, it follows that $\alpha + j \neq \alpha + m$, and hence $C_{\alpha+j} \cap C_{\alpha+m} = \emptyset$ by the definition of k. Thus, we must have $f^{-q}(W) \cap W = \emptyset$, and the claim is proved. Now W satisfies all the assumptions of case (b), and we arrive at a contradiction. $\square$

References

1. J. Stark. Transitive sets for quasi-periodically forced monotone maps. *Dyn. Syst.*, 18(4):351–364, 2003. MR2021504 (2004k:37086)
2. S. Aubry and G. André. Analyticity breaking and Anderson localization in incommensurate lattices. Proc. 8th Int. Coll. on Group Theoretical Methods in Physics, Kiryat Anavim, Israel, 1979 (Bristol: Hilger). *Ann. Isr. Phys. Soc.*, 3:133–164, 1980. MR626837 (83b:82076)
3. F.J. Romeiras and E. Ott. Strange nonchaotic attractors of the damped pendulum with quasiperiodic forcing. *Phys. Rev. A*, 35(10):4404–4413, 1987. MR892453 (88i:58115)
4. C. Grebogi, E. Ott, S. Pelikan, and J.A. Yorke. Strange attractors that are not chaotic. *Physica D*, 13:261–268, 1984. MR775290 (86g:58089)
5. A. Prasad, S.S. Negi, and R. Ramaswamy. Strange nonchaotic attractors. *Int. J. Bif. Chaos*, 11(2):291–309, 2001. MR1830343 (2002b:37044)
6. J. Franks. Generalizations of the Poincaré-Birkhoff theorem. *Ann. of Math. (2)*, 128(1):139–151, 1988. MR951509 (89m:54052)
7. J. Franks. Recurrence and fixed points of surface homeomorphisms. *Ergodic Theory Dyn. Syst.*, 8*:99–107, 1988. MR967632 (90d:58124)
8. J. Franks. Realizing rotation vectors for torus homeomorphisms. *Trans. Amer. Math. Soc.*, 311(1):107–115, 1989. MR958891 (89k:58239)
9. J. Kwapisz. A priori degeneracy of one-dimensional rotation sets for periodic point free torus maps. *Trans. Amer. Math. Soc.*, 354(7):2865–2895, 2002. MR1895207 (2003d:37058)
10. F. Béguin, S. Crovisier, and F. Le Roux. Pseudo-rotations of the closed annulus: Variation on a theorem of J. Kwapisz. *Nonlinearity*, 17(4):1427–1453, 2004. MR2069713 (2005d:37084)
11. F. Béguin, S. Crovisier, and F. Le Roux. Pseudo-rotations of the open annulus. *Bull. Braz. Math. Soc.*, 37(2):275–306, 2006. MR2266384 (2008b:37074)
12. P. Le Calvez. Rotation numbers in the infinite annulus. *Proc. Amer. Math. Soc.*, 129(11):3221–3230, 2001. MR1844997 (2002h:37069)
13. R.H. Bing. A homogeneous indecomposable plane continuum. *Duke Math. J.*, 15:729–742, 1948. MR0027144 (10:261a)
14. M. Handel. A pathological area preserving C^∞ diffeomorphism of the plane. *Proc. Amer. Math. Soc.*, 86(1):163–168, 1982. MR663889 (84f:58040)
15. M. Herman. Construction of some curious diffeomorphisms of the Riemann sphere. *J. Lond. Math. Soc.*, 34:375–384, 1986. MR856520 (87m:58128)
16. P. Le Calvez. Propriétés dynamiques des difféomorphismes de l'anneau et du tore. *Astérisque*, 204, 1991. MR1183304 (94d:58092)
17. J. Franks and P. Le Calvez. Regions of instability for non-twist maps. *Ergodic Theory Dyn. Syst.*, 23(1):111–141, 2003. MR1971199 (2003m:37053)
18. T. Jäger. Linearisation of conservative toral homeomorphisms. *Invent. Math.*, 76(3):601–616, 2009.
19. T. Jäger and J. Stark. Towards a classification for quasiperiodically forced circle homeomorphisms. *J. Lond. Math. Soc.*, 73(3):727–744, 2006. MR2241977 (2008f:37087)
20. R. Mañé. *Ergodic Theory and Differentiable Dynamics*. Springer-Verlag, 1987. MR889254 (88c:58040)
21. A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1997. MR1326374 (96c:58055)
22. B. Fayad and A. Katok. Constructions in elliptic dynamics. *Ergodic Theory Dyn. Syst.*, 24(5):1477–1520, 2004. MR2104594 (2008b:37031)
23. T. Jäger and G. Keller. The Denjoy type of argument for quasiperiodically forced circle diffeomorphisms. *Ergodic Theory Dyn. Syst.*, 26(2):447–465, 2006. MR2218770 (2007b:37081)
24. G. Keller. A note on strange nonchaotic attractors. *Fundam. Math.*, 151(2):139–148, 1996. MR1418993 (97j:58088)
25. T. Jäger. The concept of bounded mean motion for toral homeomorphisms. *Dyn. Syst.*, 24(3):277–297, 2009.

Institut für Analysis, Technische Universität Dresden, Zellescher Weg 14–16, 01219 Dresden, Germany

E-mail address: `Tobias.Oertel-Jaeger@tu-dresden.de`