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H_∞-CALCULUS FOR HYPOELLIPTIC PSEUDODIFFERENTIAL OPERATORS

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Abstract. We establish the existence of a bounded H_∞-calculus for a large class of hypoelliptic pseudodifferential operators on $\mathbb{R}^n$ and closed manifolds.

Introduction

Maximal regularity has proven to be a highly efficient concept in the theory of nonlinear parabolic evolution equations as it can be used to obtain existence and regularity results for nonlinear equations by studying their linearizations.

One way of establishing maximal regularity for a linear evolution equation $\partial_t u + Au = f$ is to prove that A admits a bounded H_∞-calculus in the sense of McIntosh [21]. For excellent surveys on maximal regularity and H_∞-calculus, see Denk, Hieber, Prüss [9] or Kunstmann and Weis [20].

In this short article we will show how a few basic functional analytic facts about algebras of pseudodifferential operators, combined with classical techniques developed by Seeley [23] and Kumano-go [18] (see also Kumano-go-Tsutsumi [19]) imply the existence of a bounded H_∞-calculus for a large class of sectorially hypoelliptic pseudodifferential operators.

A key point is to focus on the symbols and to establish the calculus on the symbolic level. This will then imply the existence of a bounded H_∞-calculus for the associated operators on each Banach space in which zero-order pseudodifferential operators of the considered type act continuously.

We will first prove our result for symbols in the Hörmander classes $S^m_{\rho,\delta}$ as most readers will be familiar with that calculus. Then we will sketch the changes necessary for symbols in the Beals-Fefferman classes $S^\mu_{\Phi,\varphi}$ and the Weyl-Hörmander classes $S(m,g)$.

Let us recall some basic facts about the H_∞-calculus. In the sequel, Λ will denote the sector

$$\Lambda = \Lambda(\theta) = \left\{ re^{i\varphi} : r \geq 0,\ \theta \leq \varphi \leq 2\pi - \theta \right\}, \qquad 0 < \theta < \pi, \tag{1}$$

in the complex plane. By H_∞ we denote the space of all bounded holomorphic functions $f : \mathbb{C}\backslash\Lambda \to \mathbb{C}$, equipped with the supremum norm, and by H the subspace consisting of all functions f for which $|f(z)| \leq c(|z|^d + |z|^{-d})^{-1}$ for suitable $c, d > 0$, depending on f. For every $f \in H_\infty$ there exists a sequence (f_j) in H converging to

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f uniformly on compact subsets of $\mathbb{C}\setminus\Lambda$ such that $\|f_j\|_\infty \le c\|f\|_\infty$ with a constant c independent of j and f. Moreover, each $f \in H_\infty$ has nontangential boundary values on $\partial\Lambda$ in L^∞.

Let $A : \mathcal{D}(A) \subset E \to E$ be a closed and densely defined operator in a Banach space E such that

(H1) $\Lambda \setminus \{0\}$ is contained in the resolvent set of A,
(H2) $\|\lambda(\lambda - A)^{-1}\|_{\mathscr{L}(E)}$ is uniformly bounded on $\Lambda \setminus \{0\}$, and
(H3) A is injective with dense range.

Then

$$f(A) := \frac{i}{2\pi}\int_{\partial\Lambda} f(\lambda)(A-\lambda)^{-1}\,d\lambda, \qquad f \in H, \tag{2}$$

defines an element in $\mathscr{L}(E)$. Given $f \in H_\infty$, choose a sequence (f_j) in H converging to f as described above. Then the limit

$$f(A)x = \lim f_j(A)x$$

exists for x in $\mathcal{D}(A) \cap \operatorname{ran}(A)$, which can be shown to be dense in E. The limit is independent of the choice of the sequence and defines a closable operator $f(A) : \mathcal{D}(A) \to E$. Its closure is again denoted by $f(A)$.

We say that the operator A admits a bounded H_∞-calculus with respect to $\mathbb{C}\setminus\Lambda$ if $f(A)$ extends to a bounded operator on E and

$$\|f(A)\|_{\mathscr{L}(E)} \le M\,\|f\|_\infty \qquad \text{for all } f \in H_\infty \tag{3}$$

with a constant M independent of f. In view of the Banach-Steinhaus theorem, it is sufficient to prove estimate (3) for $f \in H$. The key to the proof is a thorough understanding of the resolvent of A for estimating the operator norm of (2).

A particular choice of a bounded holomorphic function is $f(z) = z^{it}$, $t \in \mathbb{R}$. It implies the boundedness of the purely imaginary powers: $\|A^{it}\|_{\mathscr{L}(E)} \le Me^{|t|\theta}$ for all $t \in \mathbb{R}$. It had been shown before by Dore and Venni [12] that this implies maximal regularity, provided $\theta > \pi/2$.

The existence of a bounded H_∞-calculus has been established for many situations. Amann, Hieber, Simonett [2] and Duong, Simonett [13] treat differential operators on $\mathbb{R}^n$ and on compact manifolds with little regularity in the coefficients; Duong in [14] considers boundary value problems on smooth manifolds, extending Seeley's work [24] on bounded imaginary powers; Denk, Dore, Hieber, Prüss, and Venni [10] investigate boundary value problems of little regularity; Abels [1] studies the Stokes operator; Escher and Seiler [15] consider the Dirichlet-Neumann operator for domains of low regularity; Denk, Saal, and Seiler study Douglis-Nirenberg systems [11]. Coriasco, Schrohe, and Seiler show boundedness of the imaginary powers for differential operators on manifolds with conical singularities [7] and the existence of an H_∞-calculus for boundary value problems on manifolds with conical singularities [8].

1. Functional analytic preliminaries

We recall the concept of Ψ-algebras introduced by Gramsch [16].

Definition 1.1. Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{B}$ a Fréchet subalgebra with a stronger topology and the same unit. We call $\mathcal{B}$ a Ψ-subalgebra of $\mathcal{A}$ if it is spectrally invariant in $\mathcal{A}$, i.e., if $\mathcal{B}^{-1} = \mathcal{B} \cap \mathcal{A}^{-1}$.

Remark 1.2. A Ψ-algebra has an open group of invertible elements. According to a result of Waelbroeck [26], inversion is then continuous.

1.3. Symbol algebras. By $S^m_{\rho,\delta}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$ we denote the Hörmander class of pseudodifferential symbols on $\mathbb{R}^n$ for which all seminorms $q_{\alpha,\beta}$ defined by

$$q_{\alpha,\beta}(a) = \sup_{x,\xi} |D^\alpha_\xi D^\beta_x a(x,\xi)| \langle \xi \rangle^{-m+\rho|\alpha|-\delta|\beta|} \qquad (\alpha, \beta \in \mathbb{N}^n_0)$$

are finite. This gives $S^m_{\rho,\delta}$ a Fréchet topology. Moreover, $S^0_{\rho,\delta}$ is a Fréchet algebra with the Leibniz product $\#$, associating to two symbols a and b the symbol $a\#b$ of the composition $\operatorname{op} a \circ \operatorname{op} b$.

The following theorem is due to R. Beals [4]; see also Ueberberg [25].

Theorem 1.4. *$S^0_{\rho,\delta}$ is a Ψ-subalgebra of $\mathscr{L}(L^2(\mathbb{R}^n))$.*

2. The resolvent as a pseudodifferential operator

2.1. General hypoellipticity assumption. Let $a \in S^m_{\rho,\delta}$ for some $m \ge 0$ and $0 \le \delta < \rho \le 1$, possibly matrix-valued. Assume that there exist constants $c, C > 0$ such that for $x, \xi \in \mathbb{R}^n$, $|\xi| \ge C$, the spectrum of $a(x,\xi)$ lies outside $\Lambda \cup \{|\lambda| \le c\}$ and, for $\lambda \in \Lambda$,

$$|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)| \; |(a(x,\xi) - \lambda)^{-1}| \le c_{\alpha\beta} \langle \xi \rangle^{-\rho|\alpha|+\delta|\beta|}. \tag{4}$$

Remark 2.2. (*a*) For $|\xi| \ge C$, $a(x,\xi)$ thus is invertible, and its spectrum lies in

$$\Omega_{x,\xi} = \{z \in \mathbb{C} \setminus \Lambda : |z| < 2|a(x,\xi)|\}.$$

(*b*) Estimate (4) continues to hold outside $\Omega_{x,\xi}$: For $|\lambda| \ge 2|a(x,\xi)|$, we have $|\lambda|^{-1} \le (2|a(x,\xi)|)^{-1} \le |a(x,\xi)^{-1}|/2$. Hence

$$\begin{aligned} |\partial^\alpha_\xi \partial^\beta_x a(x,\xi)||(a(x,\xi) - \lambda)^{-1}| &= |\partial^\alpha_\xi \partial^\beta_x a(x,\xi)||\lambda|^{-1} \Big| \Big(\frac{a(x,\xi)}{\lambda} - 1 \Big)^{-1} \Big| \\ &\le |\partial^\alpha_\xi \partial^\beta_x a(x,\xi)||a(x,\xi)^{-1}|, \end{aligned}$$

and the last term can be estimated using (4) for $\lambda = 0$.

(*c*) Since $(a-\lambda)^{-1} = -\lambda^{-1}(1 - a(a-\lambda)^{-1})$ we deduce from (b) and the fact that 0 is not in the spectrum of $a(x,\xi)$, that, for some $c_0 > 0$,

$$|(a(x,\xi) - \lambda)^{-1}| \le c_0 \langle \lambda \rangle^{-1}, \quad |\xi| \ge C, \lambda \notin \Omega_{x,\xi}. \tag{5}$$

Following Seeley's classical idea, we now construct a parameter-dependent parametrix to $a - \lambda$. In fact, a coarse parametrix will be sufficient for our purposes.

Definition 2.3. For $x, \xi \in \mathbb{R}^n, |\xi| \ge C$ and $\lambda \notin \Omega_{x,\xi}$, we define the sequence (b_j) recursively by $b_0(x,\xi;\lambda) = (a(x,\xi) - \lambda)^{-1}$ and

$$b_{j+1}(x,\xi;\lambda) = -b_0(x,\xi;\lambda) \sum_{\substack{|\alpha|+k=j+1,\\ 0 \le k \le j}} \frac{1}{\alpha!} \partial^\alpha_\xi a(x,\xi) D^\alpha_x b_k(x,\xi;\lambda).$$

Note that $\alpha \ne 0$ in the summation, so that $\partial^\alpha_\xi (a - \lambda) = \partial^\alpha_\xi a$.

2.4. Key observation. $\partial^\alpha_\xi D^\beta_x b_j$ is a linear combination of terms of the form

$$b_0(x,\xi;\lambda) \; \partial^{\alpha_1}_\xi D^{\beta_1}_x a(x,\xi) \; b_0(x,\xi;\lambda) \ldots \partial^{\alpha_r}_\xi D^{\beta_r}_x a(x,\xi) b_0(x,\xi;\lambda)$$

with suitable r and $|\alpha_1|+\cdots+|\alpha_r| = j+|\alpha|$ and $|\beta_1|+\cdots+|\beta_r| = j+|\beta|$. Indeed this follows from the iteration process together with the fact that

$$\partial(a(x,\xi)-\lambda)^{-1} = (a(x,\xi)-\lambda)^{-1}\ \partial a(x,\xi)\ (a(x,\xi)-\lambda)^{-1}$$

for an arbitrary derivative $\partial = \partial_{x_j}$ or $\partial = \partial_{\xi_j}$. Note that for $j+|\alpha|+|\beta|>0$, we have at least three factors b_0 in $\partial_\xi^\alpha D_x^\beta b_j$.

Definition 2.5. We fix a smooth zero-excision function φ, vanishing for $|\xi|\le C$ and let for $x,\xi\in\mathbb{R}^n, \lambda\notin\Omega_{x,\xi}$, $N=1,2,\dots,$

$$b^N(x,\xi;\lambda) = \sum_{j<N}\varphi(\xi)b_j(x,\xi;\lambda).$$

Moreover, we define the symbols $r^N(\lambda)$, $\lambda\in\Lambda$, by

$$r^N(\lambda) = (a-\lambda)\#b^N(\lambda)-1.$$

Lemma 2.6. *(a) $\lambda\mapsto\langle\lambda\rangle b^N(\lambda)$ is bounded continuous from Λ to $S^0_{\rho,\delta}$.*
(b) $\lambda\mapsto\langle\lambda\rangle r^N(\lambda)$ is a bounded continuous map from Λ to $S^{m-N(\rho-\delta)}_{\rho,\delta}$.

Proof. (a) Boundedness is an immediate consequence of observation 2.4 and (5); the resolvent identity $b_0(\lambda)-b_0(\lambda_0) = (\lambda_0-\lambda)b_0(\lambda)b_0(\lambda_0)$ implies continuity.

For (b) write

$$r^N(\lambda) = \big((a-\lambda)\#b^N(\lambda)-q_N(\lambda)\big)+(q_N(\lambda)-1)$$

with $q_N(\lambda) = \sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_\xi^\alpha(a-\lambda)D_x^\alpha b^N(\lambda)$. First consider

$$q_N(\lambda)-1 = \Big(\sum_{j+|\alpha|<N} + \sum_{j,|\alpha|<N,j+|\alpha|\ge N}\Big)\partial_\xi^\alpha(a-\lambda)D_x^\alpha(\varphi b_j)(\lambda)\ -\ 1.$$

By construction, the first sum equals φ and thus differs from 1 by a regularizing symbol. In the second sum we have $j\neq 0$ and $\alpha\neq 0$. Hence it is a linear combination of terms with the structure in observation 2.4. Again continuity follows from the resolvent identity and the boundedness from (4) and (5).

Finally recall that the difference $\langle\lambda\rangle\big((a-\lambda)\#b^N(\lambda)-q_N(\lambda)\big)$ is given by an oscillatory integral. Its symbol seminorms in $S^{m-N(\rho-\delta)}_{\rho,\delta}$ can be estimated in terms of those for $\partial_\xi^\gamma a$ in $S^{m-N\rho}_{\rho,\delta}$ and those for $\langle\lambda\rangle D_x^\gamma b^N$ in $S^{N\delta}_{\rho,\delta}$ for $|\gamma|=N$. As both are bounded and the dependence on λ is continuous, the assertion follows. □

Remark 2.7. In the same way we can construct $\tilde b^N(\lambda)$ such that

$$\lambda\mapsto\langle\lambda\rangle\,\tilde r^N(\lambda) = \langle\lambda\rangle\big(\tilde b^N(\lambda)\,\#\,(a-\lambda)-1\big)$$

is bounded and continuous from Λ to $S^{m-N(\rho-\delta)}_{\rho,\delta}$.

Corollary 2.8. *Fix N so large that $m-N(\rho-\delta)\le 0$. Then $1+r^N(\lambda)$ tends to 1 in $S^0_{\rho,\delta}$ as $|\lambda|\to\infty$. For large R, it is thus invertible with respect to the Leibniz product on $\Lambda_R = \{\lambda\in\Lambda : |\lambda|\ge R\}$, since the group of invertibles is open by Remark 1.2. As inversion is continuous, the inverse also tends to 1; its seminorms stay bounded.*

Repeating the argument with $\tilde{r}^N$ we find that also $a-\lambda$ is invertible on Λ_R for a possibly larger R. Writing $(\cdot)^{-\#}$ for the inverse with respect to the Leibniz product,

$$(a-\lambda)^{-\#} = b^N(\lambda) \# (1+r^N(\lambda))^{-\#}, \quad \lambda \in \Lambda_R,$$

and its seminorms in $S^0_{\rho,\delta}$ decay like $\langle\lambda\rangle^{-1}$.

Moreover, the identity $(1+r^N(\lambda))^{-\#} = 1 - r^N(\lambda) \# (1+r^N(\lambda))^{-\#}$ shows that

$$(a-\lambda)^{-\#} - b^N(\lambda) = s^N(\lambda), \quad \lambda \in \Lambda_R,$$

with $\lambda \mapsto \langle\lambda\rangle^2 s^N(\lambda)$ bounded and continuous from Λ_R to $S^{m-N(\rho-\delta)}_{\rho,\delta}$.

Corollary 2.9. *Replacing a by $a+c$ for some $c > R$ we obtain the invertibility of $a-\lambda$ for all $\lambda \in \Lambda$.*

Remark 2.10. In fact, $\lambda \to (a-\lambda)^{-\#}$ is a holomorphic function on Λ_R with values in $S^0_{\rho,\delta}$. This follows from the resolvent identity and the continuity:

$$\begin{aligned}\lim_{\lambda\to\lambda_0} \frac{(a-\lambda)^{-\#} - (a-\lambda_0)^{-\#}}{\lambda-\lambda_0} &= \lim_{\lambda\to\lambda_0} (a-\lambda)^{-\#} \# (a-\lambda_0)^{-\#} \\ &= (a-\lambda_0)^{-\#} \# (a-\lambda_0)^{-\#}.\end{aligned}$$

3. H_∞-CALCULUS

3.1. Functional calculus for symbols. Let a satisfy the assumptions in Section 2.1. Assume, moreover, that $a-\lambda$ is invertible with respect to the Leibniz product for all $\lambda \in \Lambda$. As pointed out in Corollary 2.9 this will always be the case after a suitable shift.

For $f \in H$ define the function $f(a)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$f(a)(x,\xi) = \frac{i}{2\pi} \int_{\partial\Lambda} f(\lambda)(a-\lambda)^{-\#}(x,\xi)\, d\lambda.$$

As before, $(a-\lambda)^{-\#}$ denotes the symbol of the Leibniz inverse to $a-\lambda$. The integral converges in $S^0_{\rho,\delta}$ due to the decay property of f and since $\lambda \mapsto (a-\lambda)^{-\#}$ is continuous and decays like $\langle\lambda\rangle^{-1}$ in all seminorms by Corollary 2.8.

Theorem 3.2. *Under these assumptions, $f(a)$ is a symbol in $S^0_{\rho,\delta}$, and for each symbol seminorm q there is a constant M_q, independent of f, such that $q(f(a)) \le M_q \, \|f\|_\infty$.*

Proof. Let q be the $S^0_{\rho,\delta}$-seminorm given by $q(p) = \sup_{x,\xi} |D^\alpha_\xi D^\beta_x p(x,\xi)| \langle\xi\rangle^{\rho|\alpha|-\delta|\beta|}$. For large N write

$$f(a) = \frac{i}{2\pi} \int_{\partial\Lambda} f(\lambda) b^N(\lambda)\, d\lambda + \frac{i}{2\pi} \int_{\partial\Lambda} f(\lambda) s^N(\lambda)\, d\lambda =: b^N_f + s^N_f.$$

For the first term, we note that $b^N(x,\xi;\lambda)$ is a holomorphic function of λ away from the spectrum of $a(x,\xi)$ and $O(\langle\lambda\rangle^{-1})$ in all seminorms of $S^0_{\rho,\delta}$. Moreover, $f(\lambda)$ decays like $O(\langle\lambda\rangle^{-d})$ for some $d > 0$ as $|\lambda| \to \infty$. For estimating $b^N_f(x,\xi)$ we therefore can replace the contour $\partial\Lambda$ by $\partial\Omega_{x,\xi}$. Then

$$q(b^N_f) \le \frac{1}{2\pi} \|f\|_\infty \cdot \sup_{x,\xi} \Big(\text{Length } \partial\Omega_{x,\xi} \times \sum_{j<N} \sup_{\partial\Omega_{x,\xi}} |D^\alpha_\xi D^\beta_x b_j(x,\xi;\lambda)| \langle\xi\rangle^{\rho|\alpha|-\delta|\beta|} \Big).$$

The length of the contour is bounded by a constant times $|a(x,\xi)|$. The specific form of $D_\xi^\alpha D_x^\beta b_j$ observed in Section 2.4 together with estimate (4) implies that

$$|D_\xi^\alpha D_x^\beta b_j(x,\xi;\lambda)|\langle\xi\rangle^{\rho|\alpha|-\delta|\beta|} \le c\ |b_0(x,\xi;\lambda)|$$

for a suitable constant $c>0$. Again estimate (4) shows that the right hand side is bounded by a constant times $|a(x,\xi)|^{-1}$. Hence $q(b_f^N)\le c_q\|f\|_\infty$ with suitable c_q.

For the second term, Corollary 2.8 implies that

$$q(s_f^N)\le \frac{1}{2\pi}\|f\|_\infty \sup_{\lambda\in\Lambda}\langle\lambda\rangle^2 q(s^N(\lambda))\ \left|\int_{\partial\Lambda}\langle\lambda\rangle^{-2}\,d\lambda\right| \le d_q\ \|f\|_\infty$$

for a suitable d_q. Together, these two estimates show the assertion. □

Corollary 3.3. *Under the above assumptions we have*

$$\mathrm{op}(f(a)) = \frac{i}{2\pi}\int_{\partial\Lambda} f(\lambda)(\mathrm{op}(a)-\lambda)^{-1}\,d\lambda,$$

where we consider the pseudodifferential operators on, say, the Schwartz space $\mathcal{S}$.

3.4. Functional calculus for operators. Let E be a Banach space of tempered distributions on $\mathbb{R}^n$ which contains $\mathcal{S}$ as a dense subspace and for which the mapping

$$\mathrm{op}: S^0_{\rho,\delta}\to \mathscr{L}(E)$$

is continuous. Let a be a symbol which meets the assumptions in Section 2.1. Then $\mathrm{op}\,a:\mathcal{S}\to\mathcal{S}$ has a unique closed extension A in E. We assume that

(H1′) Λ is contained in the resolvent set of A.
(H2′) $\|\langle\lambda\rangle(A-\lambda)^{-1}\|_{\mathcal{L}(E)}$ is uniformly bounded in $\lambda\in\Lambda$.

We have sharpened condition (H1) (see the introduction) in that we assume that 0 belongs to the resolvent set. In this case, (H2′) is equivalent to (H2), and (H3) is automatically fulfilled. According to Corollary 2.9, (H1′) and (H2′) will always hold upon replacing A by $A+c$ with a suitably large positive constant c.

For f in H we define the operator $f(A)$ by the Dunford integral (2). Assumption (H2′) implies that the integral converges in $\mathscr{L}(E)$ due to the decay of f near infinity.

Theorem 3.5. *Under these assumptions,*

$$\|f(A)\|_{\mathscr{L}(E)}\le M\ \|f\|_\infty$$

for a suitable constant M independent of f.

Proof. As $\lambda\to(A-\lambda)^{-1}$ is a continuous function on Λ with values in $\mathscr{L}(E)$, it suffices to find an estimate for the integral over $\partial\Lambda\cap\{|\lambda|>R\}$ for large R. On this set,

$$(A-\lambda)^{-1} = B^N(\lambda)+S^N(\lambda)$$

with $B^N=\mathrm{op}\,b^N$ and $S^N=\mathrm{op}\,s^N$. Then $(A-\lambda)^{-1}$, B^N and S^N depend continuously on λ as operators in $\mathscr{L}(E)$, and the norms of $B^N(\lambda)$ and $(A-\lambda)^{-1}$ in $\mathscr{L}(E)$ are $O(\langle\lambda\rangle^{-1})$, and that of $S^N(\lambda)$ is $O(\langle\lambda\rangle^{-2})$. Hence the norm of

$$\int_{\partial\Lambda\cap\{|\lambda|\ge R\}} f(\lambda)S^N(\lambda)\,d\lambda$$

is bounded by a constant times $\|f\|_\infty$.

In order to estimate the integral involving $B^N(\lambda) = \sum_{j<N} \operatorname{op} b_j(\lambda)$, it is sufficient to treat

$$\int_{\partial\Lambda\cap\{|\lambda|\geq R\}} f(\lambda) \operatorname{op} b_j(\lambda)\, d\lambda = \operatorname{op}\Big(\int_{\partial\Lambda\cap\{|\lambda|\geq R\}} f(\lambda) b_j(\lambda)\, d\lambda\Big).$$

In view of the fact that $b_j(x,\xi;\lambda)$ is an analytic function of λ for $|\lambda| > |a(x,\xi)|$ and that $b_j(x,\xi;\lambda)$ decays like $\langle\lambda\rangle^{-1}$ while $f(\lambda)$ decays like $\langle\lambda\rangle^{-d}$ for some $d > 0$, we can replace the contour by the contour $\mathcal{C}_{x,\xi}$ which runs from $Re^{i\theta}$ to $2|a(x,\xi)|e^{i\theta}$ along the straight ray, then clockwise about the origin on a circular arc to the point $2|a(x,\xi)|e^{i(2\pi-\theta)}$ and then along the ray to $Re^{i(2\pi-\theta)}$.

Now the same argument as in the proof of Theorem 3.2 shows that each seminorm for $\int_{\mathcal{C}_{x,\xi}} f(\lambda) b_j(\lambda)\, d\lambda$ in the topology of $S^0_{\rho,\delta}$ is bounded by a multiple of $\|f\|_\infty$. As this topology is stronger than that of $\mathscr{L}(E)$, we obtain that

$$\Big\| \int_{\partial\Lambda\cap\{|\lambda|\geq R\}} f(\lambda) B^N(\lambda)\, d\lambda \Big\| \leq c\|f\|_\infty.$$

This completes the argument. □

Remark 3.6. Although many of the arguments can be carried out for general choices of ρ and δ, the requirement that op maps $S^0_{\rho,\delta}$ to $\mathscr{L}(E)$ will in general impose rather strict conditions on ρ. If E is an L^p-space of Sobolev, Besov or Triebel-Lizorkin type, for example, we will have to choose $\rho = 1$ whenever $p \neq 2$.

4. The manifold case

An operator $A : C^\infty(M,F) \to C^\infty(M,F)$ acting between sections of a vector bundle F on a smooth closed manifold M is called a pseudodifferential operator with local symbols in $S^m_{\rho,\delta}$ if all localizations of A to coordinate neighborhoods are of the form $\operatorname{op} a$ for a suitable symbol $a \in S^m_{\rho,\delta}$. One has to assume $1-\rho \leq \delta \leq \rho$ for the pseudodifferential calculus on manifolds to make sense. The pseudodifferential operators with local symbols in $S^0_{\rho,\delta}$ endowed with the $S^0_{\rho,\delta}$-seminorms on the local symbols then form a Fréchet algebra which is a Ψ-algebra in $\mathscr{L}(L^2(M,F))$; cf. [22].

Suppose the local symbols of A satisfy the assumptions of Section 2.1. We then construct the parameter-dependent parametrix in each coordinate chart. The operators associated to the local parametrices are patched to a global parameter-dependent pseudodifferential operator $B^N(\lambda)$ on the manifold.

Similarly to what was shown by Seeley in [23], the $S^0_{\rho,\delta}$ symbol seminorms for $B^N(\lambda)$ are $O(\langle\lambda\rangle^{-1})$ and $(A-\lambda)B^N(\lambda) = 1 + R^N(\lambda)$ for an operator family R^N whose symbol seminorms in $S^0_{\rho,\delta}$ decay like $\langle\lambda\rangle^{-1}$ as $\lambda \to \infty$. In the same way, a right inverse is obtained, and the invertibility of $A-\lambda$ follows. Just as above, $1 + R^N(\lambda)$ tends to 1 in the $S^0_{\rho,\delta}$-topology. It is therefore invertible in $S^0_{\rho,\delta}$ for large λ, and

$$(A-\lambda)^{-1} = B^N(\lambda) + S^N(\lambda)$$

with an operator family S^N whose symbol seminorms in $S^0_{\rho,\delta}$ are $O(\langle\lambda\rangle^{-2})$ on Λ.

Assuming that $A-\lambda$ is invertible for all λ in Λ, we can define $f(A)$ by the Dunford integral (2). The decay of the resolvent shows that the integral converges in the topology of $S^0_{\rho,\delta}$. Moreover, the same analysis as for Theorem 3.2 shows that for each seminorm q on $S^0_{\rho,\delta}$, we have $q(f(A)) \leq M_q\|f\|_\infty$.

Next fix a Banach space E of distributions on M in which $C^\infty(M)$ is dense. Assume that for each pseudodifferential operator P on M with local symbols in $S^0_{\rho,\delta}$, the operator norm of P in $\mathscr{L}(E)$ can be estimated in terms of the seminorms of its local symbols. Then the analysis for Theorem 3.5 shows that A has a bounded H_∞-calculus.

5. Symbols of Beals-Fefferman type

Let Φ, φ be a pair of weight functions on $\mathbb{R}^n \times \mathbb{R}^n$ in the sense of Beals and Fefferman, satisfying the standard conditions (equations (1.1) through (1.6) in [3]). Let $\mu \in O(\Phi, \varphi)$; i.e., for suitable constants c, C,

$$\begin{aligned} &|\mu(x,\xi) - \mu(y,\eta)| \le C \text{ if } |x-y| \le c\varphi(x,\xi) \text{ and } |\xi - \eta| \le c\Phi(x,\xi) \\ &\text{for some real } k, K, m: \quad c(\varphi\Phi)^{-m} \le e^\mu \varphi^{-k}\Phi^{-K} \le C(\varphi\Phi)^m. \end{aligned}$$

Simple examples are the functions $(K,k) := K\log\Phi + k\log\varphi$ for arbitrary $K, k \in \mathbb{R}$. We denote by $S^\mu_{\Phi,\varphi}$ the associated symbol class on $\mathbb{R}^n$, consisting of all $a = a(x,\xi)$ such that for all multi-indices α, β,

$$|D^\alpha_\xi D^\beta_x a(x,\xi)| \le C_{\alpha,\beta} e^\mu \Phi^{-|\alpha|}\varphi^{-|\beta|}.$$

The associated seminorms give $S^\mu_{\Phi,\varphi}$ a Fréchet topology. We recover the Hörmander class $S^m_{\rho,\delta}$ for $\Phi = \langle\xi\rangle^\rho$, $\varphi = \langle\xi\rangle^{-\delta}$, and $\mu = \ln\langle\xi\rangle^m$.

Each symbol $a \in S^0_{\Phi,\varphi}$ induces a bounded pseudodifferential operator on $L^2(\mathbb{R}^n)$, and the symbol topology is stronger than the operator topology. In [4], Beals showed:

Theorem 5.1. *$S^0_{\Phi,\varphi}$ is a Ψ-subalgebra of $\mathscr{L}(L^2(\mathbb{R}^n))$.*

We now modify the hypoellipticity condition 2.1 to

Definition 5.2. Let $a \in S^\mu_{\Phi,\varphi}$ for some $\mu \ge 0$, possibly matrix-valued. Assume there exist constants $c, C > 0$ such that for all $x, \xi \in \mathbb{R}^n$, with $(\varphi\Phi)(x,\xi) \ge C$, the spectrum of $a(x,\xi)$ lies outside $\Lambda \cup \{|\lambda| \le c\}$, and, for $\lambda \in \Lambda$,

$$|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)|\ |(a(x,\xi) - \lambda)^{-1}| \le c_{\alpha\beta}\Phi^{-|\alpha|}\varphi^{-|\beta|}. \tag{6}$$

We define $\Omega_{x,\xi}$ as before and note that (6) extends to $\mathbb{C} \setminus \Omega_{x,\xi}$ for $(\varphi\Phi)(x,\xi) \ge C$.

While the b_j are as before, we make a small change in the construction of b^N in order to account for the new hypoellipticity condition. Instead of the function φ employed there, we use a zero-excision function $\tilde\varphi$ defined as follows: we choose a function $\psi \in C^\infty(\mathbb{R})$ with $\psi(t) = 0$ for $t \le 1$ and $\psi(t) = 1$ for $t \ge 2$ and then let $\tilde\varphi(x,\xi) = \psi((\Phi\varphi)(x,\xi)/C)$ with the above constant C.

The results of 2.6–2.9 then follow in an analogous way, replacing in Lemma 2.6 and Remark 2.7 the space $S^0_{\rho,\delta}$ by $S^0_{\Phi,\varphi}$ and $S^{m-N(\rho-\delta)}_{\rho,\delta}$ by $S^{\mu-(N,N)}_{\Phi,\varphi}$ with the above definition of (K,k). Note that the condition in Corollary 2.8 can be fulfilled as a consequence of the assumptions on μ.

We then obtain the statements of Theorem 3.2 (with $f(a)$ now in $S^0_{\Phi,\varphi}$), Corollary 3.3 and Theorem 3.5 in the same way as before.

6. Symbols in Weyl-Hörmander classes

Let σ be the canonical symplectic form on $\mathbb{R}^{2n}$ and g a σ-temperate metric; see Hörmander [17] for details. We denote by $|\cdot|_k^g$, $k \in \mathbb{N}_0$, the associated seminorm system. Let m be a (σ, g)-temperate weight function and h, H be defined by

$$h(x,\xi)^2 = \sup_{(y,\eta)} \frac{g_{(x,\xi)}(y,\eta)}{g^\sigma_{(x,\xi)}(y,\eta)}, \quad H(x,\xi)^2 = \sup_{(y,\eta)} \frac{g^\sigma_{(x,\xi)}(y,\eta)}{g_{(x,\xi)}(y,\eta)},$$

where g^σ is the dual metric to g. It is required that $h \le 1$. Note that h is a (σ, g)-temperate weight function and, by suitable regularization [17, Section 2], h can be assumed to be smooth. The symbol space $S(m,g)$ then consists of all functions p on $\mathbb{R}^n \times \mathbb{R}^n$ for which $|p|_k^g(x,\xi)/m(x,\xi)$ is bounded for each k. A symbol $p \in S(m,g)$ defines the Weyl pseudodifferential operator $\operatorname{op}^w p : \mathcal{S} \to \mathcal{S}$ by

$$(\operatorname{op}^w p)u(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} p((x+y)/2,\xi)u(y)\,dy\,d\xi.$$

For $p \in S(m_1,g)$ and $q \in S(m_2,g)$ we have $\operatorname{op}^w p \circ \operatorname{op}^w q = \operatorname{op}^w r$ for some $r \in S(m_1m_2,g)$, and the symbol seminorms for r can be estimated in terms of those for p and q. We write $r = p\#^w q$. Furthermore, the remainder

$$R_N(p,q) = (p\#^w q)(x,\xi) - \sum_{j<N} \frac{\sigma(\partial_x,\partial_\xi;\partial_y,\partial_\eta)^j}{(2i)^j j!} p(x,\xi)q(y,\eta)|_{(y,\eta)=(x,\xi)}$$

is an element of $S(m_1m_2h^N, g)$. The corresponding mapping is continuous.

For p in $S(1,g)$, $\operatorname{op}^w p$ is bounded on $L^2(\mathbb{R}^n)$, so that $S(1,g)$ can be considered a Fréchet subalgebra of $\mathscr{L}(L^2(\mathbb{R}^n))$ with the product $\#^w$. It is not clear whether this algebra is always spectrally invariant. This is, however, true under mild restrictions, as shown by Bony [5, Corollaire 4.4]:

Theorem 6.1. *$S(1,g)$ is a Ψ-subalgebra of $\mathscr{L}(L^2(\mathbb{R}^n))$ whenever g is geodesically temperate and of reinforced slowness ('lenteur renforcée').*

Here, the metric g is called geodesically temperate if there exist constants $c > 0$ and $N \in \mathbb{N}$ such that for all (x,ξ) and (y,η),

$$\Big(g^\sigma_{(x,\xi)}(\cdot)/g^\sigma_{(y,\eta)}(\cdot)\Big)^{\pm 1} \le c\Big(1 + d_\sigma\big((x,\xi),(y,\eta)\big)\Big)^N,$$

where $d_\sigma((x,\xi),(y,\eta))$ is the geodesic distance of (x,ξ) and (y,η) with respect to the metric g^σ. Moreover, g is said to be of reinforced slowness, if there exists a constant $c > 0$ such that

$$\big(g_{(x,\xi)}(\cdot)/g_{(y,\eta)}(\cdot)\big)^{\pm 1} \le c \ \text{ whenever } \ g^\sigma_{(x,\xi)}((x,\xi)-(y,\eta)) \le c^{-1}H^2(x,\xi).$$

6.2. Hypoellipticity assumption. Let a be a complex-valued symbol in $S(m,g)$, with g geodesically temperate and of reinforced slowness and, moreover,

$$c_m h^L \le m \le c_m^{-1} h^{-L} \text{ for suitable } c_m, L > 0. \tag{7}$$

Assume that there exist constants c, c' such that $a(x,\xi)$ lies outside $\Lambda \cup \{|\lambda| \le c\}$ whenever $h(x,\xi) \le c'$ and that, for $\lambda \in \Lambda$ and suitable constants c_k,

$$|a|_k^g(x,\xi)\ |(a(x,\xi)-\lambda)^{-1}| \le c_k, \quad \lambda \in \Lambda, k \in \mathbb{N}_0. \tag{8}$$

6.3. Parametrices and inverses in the Weyl calculus. For (x,ξ) with $h(x,\xi) \le c'$ we define $\Omega_{x,\xi}$ as before and note that estimate (8) extends to $\mathbb{C}\setminus\Omega_{x,\xi}$.

Next we construct coarse parameter-dependent right and left parametrices b^N and $\tilde{b}^N$ to $a-\lambda$ with respect to the Weyl symbol product. For $\lambda \notin \Omega_{x,\xi}$ and all (x,ξ) with $h(x,\xi)\le c'$ we determine $b_j = b_j(x,\xi;\lambda)$ iteratively as follows:

$$b_0(x,\xi;\lambda) = (a(x,\xi)-\lambda)^{-1},$$

$$b_j(x,\xi;\lambda) = -b_0(x,\xi;\lambda)\sum_{k+l=j,\,k>0}\frac{\sigma^k}{(2i)^k k!}a(x,\xi)b_l(y,\eta)|_{(y,\eta)=(x,\xi)},$$

$j=1,2,\dots$, where for better legibility we wrote σ instead of $\sigma(\partial_x,\partial_\xi;\partial_y,\partial_\eta)$.

We choose a smooth function ψ on $\mathbb{R}$ with $\psi(t)=1$ for $t<1/2$ and $\psi(t)=0$ for $t\ge 1$ and let $\varphi(x,\xi)=\psi(h(x,\xi)/c')$ with the above constant c'. We then define

$$b^N(x,\xi;\lambda) = \varphi(x,\xi)\sum_{j<N} b_j(x,\xi;\lambda).$$

Next we recall that, for two smooth functions p, q,

$$\begin{aligned}&|(\sigma(\partial_x,\partial_\xi;\partial_y,\partial_\eta)^j p(x,\xi)q(y,\eta)|_{(y,\eta)=(x,\xi)}|_k^g\\ &\qquad\le (2n)^j\sum_{l=0}^{k}\binom{k}{l}|p|^g_{j+l}|q|^g_{j+k-l}h^j;\end{aligned}\tag{9}$$

see e.g. Buzano and Nicola [6, Lemma 2.3]. It follows from (8) and observation 2.4 that

$$|\varphi b_0(\lambda)|_k^g(x,\xi)\le c_k|a(x,\xi)-\lambda|^{-1}$$

for suitable constants c_k. This shows that $\langle\lambda\rangle\varphi b_0(\lambda)$ is bounded in $S(1,g)$. Moreover, we infer from (8), (9), and the iteration process that

$$|\varphi b_j(\lambda)|_k^g(x,\xi)\le c_k|a(x,\xi)-\lambda|h(x,\xi)^j\tag{10}$$

with (different) constants c_k, uniformly for all x,ξ and $\lambda\notin\Omega_{x,\xi}$.

Lemma 6.4. *Let $r^N(\lambda)=(a-\lambda)\#^w b^N(\lambda)-1$. Then $\lambda\mapsto\langle\lambda\rangle r^N(\lambda)$ is a bounded continuous map from Λ to $S(mh^N,g)$.*

Proof. The proof is analogous to that of Lemma 2.6. We write $r_N(\lambda)$ as

$$R_N\left(a-\lambda,b^N(\lambda)\right)+\Big(\sum_{j,k<N}\frac{\sigma^k}{(2i)^k k!}(a(x,\xi)-\lambda)(\varphi b_j)(y,\eta;\lambda)|_{(y,\eta)=(x,\xi)}-1\Big).\tag{11}$$

Then $\langle\lambda\rangle R_N\left(a-\lambda,b^N(\lambda)\right)=R_N\left(a,\langle\lambda\rangle b^N(\lambda)\right)$ is bounded and continuous in λ with values in $S(mh^N,g)$. Next we consider the second summand. We split the summation into those terms, where $j+k<N$ and those, where $j+k\ge N$. By construction, the former sum equals φ. The difference $\varphi-1$ belongs to $S(h^M,g)$ for any M, and hence to $S(mh^N,g)$. In the latter, we have $j,k\ne 0$ and can therefore replace $a-\lambda$ by a. We conclude from (8), (9) and (10) that, even after multiplication by $\langle\lambda\rangle$, the $S(mh^N,g)$-seminorms of all these terms are finite. Continuity follows from the resolvent identity as before. □

Similarly, we construct a left parametrix $\tilde{b}^N$ with remainder $\tilde{r}^N$ such that $\lambda\mapsto\langle\lambda\rangle\tilde{r}^N(\lambda)$ is a bounded continuous map from Λ to $S(mh^N,g)$.

Since, by assumption, $S(mh^N,g)\hookrightarrow S(1,g)$ for large N, the results of Corollaries 2.8 and 2.9 then follow as above.

6.5. Functional calculus. We obtain the statements of Theorem 3.2 with $f(a)$ now in $S(1,g)$, of Corollary 3.3 and of Theorem 3.5 in the same way as before.

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