ON ADDITIVE COMPLEMENTS

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Abstract. Two infinite sequences $A$ and $B$ of non-negative integers are called additive complements if their sum contains all sufficiently large integers. Let $A(x)$ and $B(x)$ be the counting functions of $A$ and $B$. For additive complements $A$ and $B$, Sárközy and Szemerédi proved that if $\limsup_{x \to \infty} \frac{A(x)B(x)}{x^2} \leq 1$, then $A(x)B(x) - x \to +\infty$. In this paper, we prove that for additive complements $A$ and $B$, if $\limsup_{x \to \infty} \frac{A(x)B(x)}{x^2} < \frac{5}{4}$ or $\limsup_{x \to \infty} \frac{A(x)B(x)}{x^2} > 2$, then $A(x)B(x) - x \to +\infty$.

1. Introduction

Two infinite sequences $A$ and $B$ of non-negative integers are called additive complements if their sum contains all sufficiently large integers. Let $A(x)$ and $B(x)$ be the counting functions of $A$ and $B$, namely,

$$A(x) = \sum_{a \leq x \atop a \in A} 1 \quad \text{and} \quad B(x) = \sum_{b \leq x \atop b \in B} 1.$$

Motivated by a problem of Hanani and Erdős [2], [3], Danzer [1] conjectured that for additive complements $A$ and $B$, if

$$\limsup_{x \to \infty} \frac{A(x)B(x)}{x} \leq 1,$$

then

(1) $A(x)B(x) - x \to +\infty$ as $x \to +\infty$.

(See also [4, p. 10], [5, p. 75] and [6].) In [8], Sárközy and Szemerédi proved this conjecture.

In this paper, we prove the following result.

Theorem. For additive complements $A$ and $B$, if

$$\limsup_{x \to \infty} \frac{A(x)B(x)}{x} > 2$$

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or

\[
\limsup_{x \to \infty} \frac{A(x)B(x)}{x} < \frac{5}{4},
\]

then (1) must hold.

For the construction of additive complements \( A \) and \( B \) with \( A(x)B(x) \sim x \) one may refer to [1] and [7].

2. Proof of the Theorem

Let \( f(n) \) be the number of solutions of \( a + b = n \), for \( a \in A, b \in B \). For additive complements \( A \) and \( B \), there exists a constant \( n_0 \) such that

\[
f(n) \geq 1 \quad \text{for} \quad n > n_0.
\]

Hence \( A(x)B(x) \geq [x] - n_0 \).

If (1) does not hold, then \(-\infty < \liminf_{x \to \infty} (A(x)B(x) - x) < +\infty\). Assume that

\[
\liminf_{x \to \infty} (A(x)B(x) - x) = L.
\]

By the same arguments as in Sárközy and Szemerédi [8, p. 238], there exists an integer \( n_1 \) with

\[
f(n) \equiv 1 \quad \text{for} \quad n \geq n_1.
\]

Then

\[
A(x)B(x) \leq \sum_{n \leq 2x} f(n) \leq \sum_{n \leq n_1} f(n) + (2x - n_1) = 2x + O(1).
\]

Hence

\[
\limsup_{x \to \infty} \frac{A(x)B(x)}{x} \leq 2.
\]

Thus, if

\[
\limsup_{x \to \infty} \frac{A(x)B(x)}{x} > 2,
\]

then

\[
\liminf_{x \to \infty} (A(x)B(x) - x) = +\infty.
\]

Now we assume that the additive complements \( A \) and \( B \) satisfy

\[
\limsup_{x \to \infty} \frac{A(x)B(x)}{x} = M.
\]

Then \( A(x)B(x) \leq x(M + o(1)) \). Since \( A \) and \( B \) are infinite, we have \( A(x) = o(x) \) and \( B(x) = o(x) \).

Let \( x_1 < x_2 < \cdots \) be all positive integers with \( A(x_k)B(x_k) - x_k = L, b^{(k)} \) be the largest integer in \( (A \cup B) \cap [0, x_k] \) and \( y_k = x_k - b^{(k)} \). By the same discussion as in [8], we know that \( y_k \to +\infty \). Since

\[
x_k + L = A(x_k)B(x_k) = A(b^{(k)})B(b^{(k)})
\]

\[
= A(x_k - y_k)B(x_k - y_k) \leq (M + o_k(1))(x_k - y_k),
\]

we have

\[
y_k \leq (1 - \frac{1}{M})x_k + o_k(x_k).
\]

As in [8] we know that

\[
A(x_k) - A(y_k) \leq L + n_0
\]
and

\[ A(x_k) - A(2y_k) = 0, \quad k \geq k_0. \]

Define \( D, D_1, D_1^+ \) and \( D_1^- \) in the same way as in \( S \), namely,
\[
D = \{(b, a) : b \in B, a \in A, b \leq x_k - y_k, a \leq x_k - y_k, b - a > y_k\},
\]
\[
D_1 = \{(b, a) : b \in B, a \in A, 2y_k < b \leq x_k - y_k, b - a > y_k\},
\]
\[
D_1^+ = \{(b, a) : b \in B, a \in A, 2y_k < b \leq x_k - y_k, a \leq 2y_k\},
\]
\[
D_1^- = \{(b, a) : b \in B, a \in A, 2y_k < b \leq x_k - y_k, a \geq b - y_k\}.
\]

By \( A(x_k) = A(2y_k) \) for \( k \geq k_0 \), we have \( |D_1| = |D_1^+| - |D_1^-| \). As in \( S \), it suffices to show that for large \( k \), \( |D| > x_k - 2y_k \).

Fix an integer \( t \) with
\[ t > \frac{1}{2} - M. \]

Let
\[ \alpha_i = 1 + \frac{i - 1}{t} \quad (1 \leq i \leq t + 1) \]
and
\[ A(\alpha_i y_k)B(\alpha_i y_k) = M_i \alpha_i y_k. \]

Obviously, \( 1 - \alpha_i, i(1) \leq M_i \leq M + \alpha_i, i(1) \). For \( D_1^+ \) we have
\[
|D_1^+| = (B(x_k - y_k) - B(2y_k))A(2y_k) - B(2y_k)A(2y_k)
= B(x_k)A(x_k) - B(2y_k)A(2y_k) = (x_k + O_k(1)) - 2M_{t+1}y_k.
\]

For \( D_1^- \), as in \( S \) we have \( |D_1^-| = o(y_k) \). Hence
\[
|D_1| = x_k - 2M_{t+1}y_k + o(y_k).
\]

We redefine \( D_2 \) as
\[
D_2 = \{(b, a) : b \in B, a \in A, b \leq 2y_k, b - a > y_k\}.
\]

Obviously, \( D_1 \cap D_2 = \emptyset \) and \( D = D_1 \cup D_2 \). Hence \( |D| = |D_1| + |D_2| \).

Now we are going to estimate \( |D_2| \). It follows from the definition of \( D_2 \) that
\[
|D_2| = \sum_{y_k < b \leq 2y_k} \sum_{1 \leq i \leq t} \sum_{\alpha_i y_k < b \leq \alpha_{i+1} y_k} A(b - y_k)
\geq \sum_{i=1}^t A(\alpha_i - 1)y_k)(B(\alpha_{i+1} y_k) - B(\alpha_i y_k)).
\]

Since
\[
A(\alpha_i - 1)y_k \geq \frac{A(\alpha_i - 1)y_k B(\alpha_i - 1)y_k}{A(y_k) B(y_k)} A(y_k) \geq \frac{\alpha_i - 1}{M} - o_k(1) A(y_k)
\]

and
\[
B(\alpha_{i+1} y_k) - B(\alpha_i y_k) = \frac{B(\alpha_{i+1} y_k) A(y_k) - B(\alpha_i y_k) A(y_k)}{A(y_k)}
= \frac{B(\alpha_{i+1} y_k)(A(\alpha_{i+1} y_k) + O_k(1)) - B(\alpha_i y_k)(A(\alpha_i y_k) + O_k(1))}{A(y_k)}
= \frac{M_{i+1} \alpha_{i+1} - M_i \alpha_i + o_k(1)}{A(y_k)} y_k.
\]
we have

\[ |D_2| \geq \sum_{i=1}^{t} \left( \frac{\alpha_i - 1}{M} (M_{i+1} \alpha_{i+1} - M_i \alpha_i) \right) y_k + o_t(y_k) \]

\[ = \frac{1}{M} \left( - \sum_{i=2}^{t+1} M_i \alpha_i (\alpha_i - \alpha_{i-1}) + M_{i+1} \alpha_{i+1} (\alpha_{i+1} - 1) \right) y_k + o_t(y_k) \]

\[ \geq \frac{1}{M} \left( -(M + o_t(1)) \sum_{i=2}^{t+1} \alpha_i (\alpha_i - \alpha_{i-1}) + 2M_{t+1} \right) y_k + o_t(y_k) \]

\[ = \frac{1}{M} \left( -\frac{3}{2} M + 2M_{t+1} - \frac{M}{2t} \right) y_k + o_t(y_k). \]

Thus by \( t > \frac{1}{M} \) we have

\[ |D| \geq x_k + \left( -2M_{t+1} - \frac{3}{2} + \frac{2M_{t+1}}{M} - \frac{1}{2t} \right) y_k + o_t(y_k) \]

\[ \geq x_k + \left( -2M_{t+1} (1 - \frac{1}{M}) - \frac{3}{2} - \frac{1}{2t} \right) y_k + o_t(y_k) \]

\[ \geq x_k + \left( -2(M + o_t(1)) (1 - \frac{1}{M}) - \frac{3}{2} - \frac{1}{2t} \right) y_k + o_t(y_k) \]

\[ \geq x_k + \left( -2M + 1 - \frac{1}{2t} \right) y_k + o_t(y_k) > x_k - 2y_k, \quad k > k_1. \]

By the same discussion as in [5], we obtain a contradiction, which completes the proof of the theorem.

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REFERENCES


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