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TWO BOUNDS FOR THE NILPOTENCE CLASS OF AN ALGEBRA

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ABSTRACT. Supercharacters, which mimic the irreducible characters of certain p-groups, yield bounds on the nilpotence class of an algebra. Specifically, if an algebra J has either n superdegrees or n superclass sizes, then $J^{n+1} = 0$.

1. Introduction

In [1], Diaconis and Isaacs introduce supercharacters of algebra groups. The ultimate motivation for such a construction is that, as in ordinary character theory, a superficially slight amount of information can reveal much about the object to be studied. The results below can be viewed as further vindication for this view. As will be shown, the number of the superdegrees and the number of superclass sizes of an algebra group each provide a strong bound for the nilpotence class of the underlying algebra.

These bounds can be viewed as analogous to the significant work that has been done to discover what structure irreducible degrees or conjugacy class sizes impose on a group. See, for two relatively recent examples, [3] and [4]. As noted at the end of this section, however, the naive analog of the results below need not hold in the ordinary character theory of p-groups.

Throughout, let F be a finite field, and let J be a finite-dimensional, nilpotent F-algebra. Then the set

$$G = \{1 + x \mid x \in J\}$$

is a group with multiplication defined by

$$(1 + x)(1 + y) = 1 + (x + y + xy).$$

It is easy to see that

$$(1 + x)^{-1} = 1 + \sum_{i=1}^{\infty} (-x)^i,$$

where this expression is well-defined because J is nilpotent. A group of this form is called an algebra group.

The group G acts on the right and on the left of J in the expected way; for x and y in J, let

$$x(1 + y) = x + xy$$

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and

$$(1+y)x = x + yx,$$

noting that these actions commute with each other. Write GxG for the two-sided orbit of x. The subset $1 + GxG$ of G is called the superclass of $1 + x$. The first main result, Theorem 2.3, says that if there are exactly n different superclass sizes, then $J^{n+1} = 0$.

Write J^* for the dual space of J. The actions of the previous paragraph induce right and left actions of G on J^*; for $\lambda \in J^*$, $g \in G$, and $x \in J$, let

$$(\lambda \cdot g)(x) = \lambda(xg^{-1})$$

and

$$(g \cdot \lambda)(x) = \lambda(g^{-1}x).$$

Again, note that these actions commute with each other. Write λG and $G\lambda G$ for the right and two-sided orbits, respectively, of λ. The number $|\lambda G|$ is called a superdegree of J. The second main result, Theorem 3.3, says that if there are exactly n different superdegrees, then $J^{n+1} = 0$.

Before moving on, it is worth digressing to justify the names superdegree and superclass. Fix a non-trivial group homomorphism $\widetilde{\ }$ from the additive group of F to the group of non-zero complex numbers. Given any $\lambda \in J^*$, the supercharacter $\chi_\lambda : G \to \mathbb{C}$ is defined by

$$\chi_\lambda(1+x) = \frac{|\lambda G|}{|G\lambda G|} \sum_{\mu \in G\lambda G} \widetilde{\mu(x)}.$$

As it turns out, χ_λ is a character. Plainly, the degree of the supercharacter χ_λ is

$$\chi_\lambda(1) = |\lambda G|.$$

Superclasses are the supercharacter analog of conjugacy classes.

Although it is not understood why this should be so, there are results in which conjugacy class sizes and irreducible character degrees play similar roles. For this reason, these concepts are sometimes seen as dual to each other, and questions concerning one can motivate research concerning the other; for example, the introduction to [4] appeals to problems addressing character degrees in order to motivate the work that follows. It is unsurprising then that the number of superclass sizes and the number of superdegrees both yield the same bound on the nilpotence class of the underlying algebra.

What may be surprising is that any bound can be found. Bounding the nilpotence class of an algebra immediately gives a bound on the nilpotence class of the corresponding algebra group. However, supercharacters should be seen as a coarsening of the irreducible characters, and the number of irreducible degrees need not bound the nilpotence class of a p-group (see Example 3.11 in [2]).

One can show that, for any integer $n > 1$ and any integer $m > 0$, there is an algebra K where $K^{n+1} = 0$ and K has at least m superdegrees or superclass sizes (for example, take any finite, nilpotent algebra L for which $L^n \neq 0$, and then let K be the direct sum of m copies of L). In other words, the nilpotence class can be held fixed while the number of superdegrees or superclass sizes can be made arbitrarily large. On the other hand, given an integer $n > 0$, one can always create an algebra K such that $K^n \neq 0$, while K has exactly n superdegrees and superclass sizes (for example, take an algebra defined by a single generator x and single relation $x^{n+1} = 0$). In other words, there are always algebras for which this bound is as good as possible. Note that the deficiency described in the beginning of the paragraph would apply equally well to any bound that uses only the number of superdegrees or superclass sizes.

2. Superclass sizes

Before showing the superclass result, proving a pair of lemmas is necessary.

Lemma 2.1. *Let R be a nilpotent ring. Suppose*

$$x = ax + xb,$$

where $x, a, b \in R$. Then $x = 0$.

Proof. If $x \neq 0$, there is a maximal integer n such that $x \in R^n$. This contradicts that $x = ax + xb \in RR^n + R^nR \subseteq R^{n+1}$. □

For $x \in J$, the notation $Jx = \{zx \mid z \in J\}$ and $xJ = \{xz \mid z \in J\}$ is self-explanatory, but note that since J is nilpotent, a non-zero element x is in neither the subspace Jx nor xJ.

Lemma 2.2. *Suppose $x \in J$, and assume $|1 + GxG| > 1$. If $y \in J$, then*

$$|1 + GxG| > |1 + GxyG|.$$

Proof. By Corollary 3.2 of [1], the superclass $1 + GzG$ of an element $1 + z$ has the same size as the subspace $Jz + zJ$. The following proof will thus compare the sizes of the subspaces $Jx + xJ$ and $Jxy + xyJ$.

Clearly, $Jxy + xJ \supseteq Jxy + xyJ$. If $Jxy + xJ = Jxy + xyJ$, then, since $xy \in xJ$, Lemma 2.1 guarantees that $xy = 0$. The result is trivial in this case, so one may assume that $Jxy + xyJ$ is a proper subspace of $Jxy + xJ$. It will thus suffice to show that $|Jx + xJ| \geq |Jxy + xJ|$.

Fix a subspace complement M of $Jx \cap xJ$ in Jx; in other words, M is a subspace such that $Jx = M \oplus (Jx \cap xJ)$. Then $Jx + xJ = M \oplus xJ$.

Define the map $\theta : Jx + xJ \to Jxy + xJ$ as follows. Given $z \in Jx + xJ$, uniquely write $z = m + a$, where $m \in M$ and $a \in xJ$. Then set $\theta(z) = my + a$. Clearly, θ is a linear transformation.

It will now suffice to verify that θ is surjective. Let $b, c \in J$. Since $bx \in Jx$, there are elements $m \in M$ and $a \in Jx \cap xJ$ such that

$$bx = m + a.$$

Since $ay \in (Jx \cap xJ)y \subseteq xJy \subseteq xJ$, one sees that

$$\begin{aligned} \theta\big(m + (ay + xc)\big) &= my + (ay + xc) \\ &= (m + a)y + xc \\ &= bxy + xc. \end{aligned}$$

But b and c were arbitrary, so θ is surjective.

Putting this all together yields

$$|1 + GxG| = |Jx + xJ| \geq |Jxy + xJ| > |Jxy + xyJ| = |1 + GxyG|,$$

as needed. □

It is easy to check that $|1 + GxG| = 1$ iff x annihilates J on both the left and the right. Since $x = 0$ certainly satisfies this condition, the integer 1 is always a superclass size.

Theorem 2.3. *Suppose the finite, nilpotent algebra J has exactly n superclass sizes. Then $J^{n+1} = 0$.*

Proof. Let $1 = a_1 < a_2 < \cdots < a_n$ be the superclass sizes of J. Define M_i to be the ideal of J generated by all elements x such that $|1 + GxG| \leq a_i$. Clearly, $M_n = J$. As noted prior to the theorem, the ideal M_1 annihilates J.

Lemma 2.2 implies that $M_i J \subseteq M_{i-1}$ whenever $2 \leq i \leq n$. In particular, $J^n \subseteq M_1$. But $M_1 J = 0$, so $J^{n+1} = 0$. □

3. Superdegrees

Again, a bit of preparation is necessary.

For $\lambda \in J^*$, define

$$R_\lambda = \{y \in J \mid Jy \subseteq \ker \lambda\}.$$

Note that R_λ is a subalgebra of J. In particular, the set

$$1 + R_\lambda = \{1 + x \mid x \in R_\lambda\}$$

is a subgroup of G, and one can check that it is the stabilizer of λ under the right action of G. Thus, the superdegree $|\lambda G|$ is exactly $|G|/|1 + R_\lambda| = |J|/|R_\lambda|$.

Also, for each $\lambda \in J^*$, define the map $\varphi_\lambda : J \to J^*$ by setting $\varphi_\lambda(y)(z) = \lambda(zy)$.

Lemma 3.1. *Let $\lambda \in J^*$. Then φ_λ is a linear transformation with kernel R_λ, and*

$$\varphi_\lambda(y) \cdot g = \varphi_\lambda(g^{-1}y)$$

for all $y \in J$ and $g \in G$.

Proof. That φ_λ is linear is clear. Note that $\varphi_\lambda(y) = 0$ iff $\lambda(zy) = 0$ for all $z \in J$ iff y is in R_λ; that is, the subspace $\ker \varphi_\lambda$ is exactly the subspace R_λ.

For the latter claim, let $z \in J$, and check that

$$\begin{aligned}(\varphi_\lambda(y) \cdot g)(z) &= \varphi_\lambda(y)(zg^{-1}) \\ &= \lambda(zg^{-1}y) \\ &= \varphi_\lambda(g^{-1}y)(z).\end{aligned}$$

Since z was arbitrary, the claim holds. □

Lemma 3.2. *Suppose $\lambda \in J^*$, and assume $|\lambda G| > 1$. Let $y \in J$, and write $\mu = \varphi_\lambda(y)$. Then*

$$|\lambda G| > |\mu G|.$$

Proof. If $\mu = 0$, then $|\mu G| = 1$ and the result is trivial. So suppose $\mu \neq 0$.

Lemma 3.1 demonstrates that $\mu G = \varphi_\lambda(y)G = \varphi_\lambda(Gy) \subseteq \varphi_\lambda(J)$. But $\mu \neq 0$, so $0 \notin \mu G$. Thus,

$$|\mu G| < |\varphi_\lambda(J)| = |J|/|R_\lambda| = |\lambda G|,$$

as claimed. □

The second main result can now be shown. Note that the functional 0 has a right orbit of size 1, so 1 is always a superdegree.

Theorem 3.3. *Suppose the finite, nilpotent algebra J has exactly n superdegrees. Then $J^{n+1} = 0$.*

Proof. Let $1 = a_1 < a_2 < \cdots < a_n$ be the superdegrees. The proof will show by induction on i that $J^{i+1} \subseteq \ker\lambda$ whenever $|\lambda G| \le a_i$. Since $|\lambda G| \le a_n$ for all λ in J^*, it will then be clear that $J^{n+1} \subseteq \bigcap_{\lambda\in J^*}\ker\lambda = 0$.

For $i = 1$, suppose $|\lambda G| = 1$. Then $J = R_\lambda$, and by Lemma 3.1, the functional $\varphi_\lambda(y)$ is 0 for each $y \in J$. By the definition of φ_λ, this is exactly the statement that $\lambda(zy) = 0$ whenever $y, z \in J$. That is, $J^2 \subseteq \ker\lambda$.

So assume the claim holds for the positive integer $i-1$, and suppose $|\lambda G| \le a_i$. By the previous paragraph, there is no loss in assuming $|\lambda G| > 1$. Let $y \in J$ and let $z \in J^i$. It now suffices to show that $\lambda(zy) = 0$.

Set $\mu = \varphi_\lambda(y)$. By Lemma 3.2,

$$|\mu G| < |\lambda G| \le a_i,$$

and so

$$|\mu G| \le a_{i-1}.$$

The induction hypothesis implies that $\mu(J^i) = 0$, so

$$\begin{aligned} 0 &= \mu(z) \\ &= \lambda(zy), \end{aligned}$$

as needed. $\square$

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