ON THE FINITENESS OF ASSOCIATED PRIMES
OF LOCAL COHOMOLOGY MODULES

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(Communicated by Bernd Ulrich)

Abstract. Let $R$ be a Noetherian ring, $a$ be an ideal of $R$ and $M$ be a
finitely generated $R$-module. The aim of this paper is to show that if $t$ is
the least integer such that neither $H^i_a(M)$ nor $\text{supp}(H^i_a(M))$ is non-finite, then
$H^t_a(M)$ has finitely many associated primes. This combines the main results of
Brodmann and Faghani and independently of Khashyarmanesh and Salarian.

1. Introduction

Throughout this paper, $R$ is a Noetherian ring (with identity), $a$ is an ideal of
$R$ and $M$ is an $R$-module. For basic facts about commutative algebra see [3] and
[8]; for local cohomology refer to [2]. A module is finite if it is finitely generated
and a set is finite if it has finitely many elements. We use $\mathbb{N}_0$ to denote the set of
non-negative integers.

An interesting problem in commutative algebra is determining when the set of
associated primes of the $i$th local cohomology module $H^i_a(M)$ of $M$ is finite. If $R$ is a
regular local ring containing a field, then $H^t_a(R)$ has only finitely many associated
primes for all $i \geq 0$; cf. [4] (in positive characteristic) and [7] (in characteristic
zero). However, Katzman [5] has given an example of a Noetherian local ring and
an ideal $a$ such that $H^2_a(R)$ has infinitely many associated primes. But we have
many interesting results about the finiteness of $\text{Ass}_R(H^i_a(M))$. It is well known
that if $M$ is finite, then $\text{Ass}_R(H^i_a(M))$ is finite in either of the following cases:
(a) $H^i_a(M)$ is finite for all $i < t$; see [1] and [6];
(b) $\text{supp}(H^i_a(M))$ is finite for all $i < t$; see [6].

The aim of this paper is to combine (a) and (b). That is, if $M$ is finitely
generated, then $H^i_a(M)$ has only finitely many associated primes if, for all $i < t,$
$H^i_a(M)$ is finite or has finite support.

In section 2, we define: $M$ is an FSF module if there is a finite submodule $N$
of $M$ such that the quotient module $M/N$ has finite support, and we give some
properties of FSF modules.

In section 3, we will prove the following: Let $a$ be an ideal of the Noetherian ring
$R$, and let $M$ be an FSF $R$-module. Let $t \in \mathbb{N}_0$ be such that $H^i_a(M)$ is FSF for all
$i < t$. Then $\text{Hom}_R(R/a, H^i_a(M))$ is FSF. Therefore, $\text{Ass}_R(H^i_a(M))$ is finite. This
implies the main result as a consequence.

Received by the editors March 23, 2009, and, in revised form, October 1, 2009.
2010 Mathematics Subject Classification. Primary 13D45, 13E99.
Key words and phrases. Local cohomology, associated primes.
2. FSF Module

**Definition 2.1.** Let $R$ be a Noetherian ring and $M$ be an $R$-module. $M$ is called an FSF module if there is a finite submodule $N$ of $M$ such that support of the quotient module $M/N$ is finite.

**Proposition 2.2.** Let $M$ be an $R$-module. We have

(i) If $M$ is an FSF module, then $\text{Ass}_R(M)$ is finite.

(ii) Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of $R$-modules. Then $M$ is FSF iff $M_1$ and $M_2$ are FSF.

(iii) Let $M$ be an FSF module and $N$ be finite. Then $\text{Ext}_R^i(N, M)$ is FSF for all $i \geq 0$.

**Proof.** (i). This is trivial from the definition of FSF modules.

(ii). “$\Rightarrow$” If $M$ is an FSF module, it is easy to show that $M_1$ and $M_2$ are FSF. “$\Leftarrow$” Suppose that $M_1$ and $M_2$ are FSF. Let $N_1$ and $N_2$ be finitely generated submodules of $M_1$ and $M_2$, respectively, such that $\text{supp}(M_1/N_1)$ and $\text{supp}(M_2/N_2)$ are finite. We may assume that $M_1$ is a submodule of $M$ and that $M_2$ is a quotient module of $M$. Let $x_1, x_2, \ldots, x_n$ be generators of $N_1$ and $y_1, y_2, \ldots, y_m$ are generators of $N_2$ in $M_2 = M/M_1$. Let $N$ be a submodule of $M$ generated by $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$, so $N$ is finite, and it is not difficult to show that $\text{supp}(M/N)$ is finite. Hence, $M$ is FSF.

(iii) $M$ is FSF, so there exists an exact sequence

$$0 \to M_1 \to M \to M_2 \to 0,$$

with $M_1$ finitely generated and $\text{supp}(M_2)$ finite. This exact sequence induces exact sequences

$$\text{Ext}_R^i(N, M_1) \to \text{Ext}_R^i(N, M) \to \text{Ext}_R^i(N, M_2)$$

for all $i \in \mathbb{N}_0$. Since $N$ and $M_1$ are finitely generated modules and $\text{supp}(M_2)$ is finite, we have that $\text{Ext}_R^i(N, M_1)$ is finitely generated and $\text{supp}(\text{Ext}_R^i(N, M_2))$ is finite. Hence, $\text{Ext}_R^i(N, M)$ is FSF for all $i \in \mathbb{N}_0$.

3. The Main Result

**Proposition 3.1.** Let $a$ be an ideal of the Noetherian ring $R$, and let $M$ be an FSF $R$-module. Let $t \in \mathbb{N}_0$ be such that $H_a^t(M)$ is FSF for all $i < t$. Then

$$\text{Hom}_R(R/a, H_a^t(M))$$

is FSF. Therefore, $\text{Ass}_R(H_a^t(M))$ is finite.

**Proof.** The last assertion follows from the first, from Proposition 2.2(i) and from the fact that $\text{Ass}_R(H_a^t(M)) = \text{Ass}_R(\text{Hom}(R/a, H_a^t(M)))$.

We prove that $\text{Hom}_R(R/a, H_a^t(M))$ is FSF by induction on $t$. The case $t = 0$ is clear because $\text{Hom}_R(R/a, H_a^0(M)) \subseteq M$.

So, let $t > 0$ and set $\overline{M} = M/H_a^0(M)$. Then $\overline{M}$ is FSF, $H_a^0(\overline{M}) = 0$, and

$$H_a^k(\overline{M}) \cong H_a^k(M)$$

for all $k > 0$. Thus $H_a^t(\overline{M})$ is FSF for all $i < t$ and $H_a^t(\overline{M}) \cong H_a^t(M)$. Replace $M$ by $\overline{M}$ and assume henceforth that $H_a^0(M) = 0$. By Proposition 2.2(i), we have that $\text{Ass}_R(M)$ is finite. Combining this with $H_a^0(M) = 0$ implies that there exists $a \in a$ such that $a$ is an $M$-regular element. So, we have the short exact sequence

$$0 \to M \xrightarrow{\alpha} M \xrightarrow{\beta} M/aM \to 0,$$
where \( p \) is natural projection. This yields the exact cohomology sequences

\[
H^i_a(M) \longrightarrow H^i_a(M/aM) \longrightarrow H^{i+1}_a(M) \quad (\forall i \in \mathbb{N}_0).
\]

Hence, \( H^i_a(M/aM) \) is FSF for all \( i < t - 1 \). It is clear that \( M/aM \) is FSF, so by induction, we have that \( \text{Hom}_R(R/a, H^{i-1}_a(M/aM)) \) is FSF.

We consider the long exact sequence

\[
\cdots \longrightarrow \text{Hom}_R(R/a, H^{i-1}_a(M/aM)) \longrightarrow \text{Hom}_R(R/a, H^{i}_a(M)) \longrightarrow \text{Hom}_R(R/a, H^{i+1}_a(M)) \longrightarrow \cdots
\]

\( \cdots \) for \( i < t - 1 \). It is clear that \( \text{Hom}_R(R/a, H^{i-1}_a(M/aM)) \) is FSF. Furthermore, \( \text{Hom}_R(R/a, H^{i}_a(M)) \) is FSF, so by Proposition 2.2(iii); therefore, \( \text{Hom}_R(R/a, H^{i+1}_a(M)) \) is FSF. Therefore, \( \text{Hom}_R(R/a, H^{i}_a(M)) \) is FSF, as desired.

Finally, we have

**Theorem 3.2.** Let \( a \) be an ideal of the Noetherian ring \( R \), and let \( M \) be a finitely generated \( R \)-module. Let \( t \in \mathbb{N}_0 \) be such that either \( H^i_a(M) \) is finite or \( \text{supp}(H^i_a(M)) \) is finite for all \( i < t \). Then \( \text{Ass}_R(H^i_a(M)) \) is finite.

**References**


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