PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 6, June 2010, Pages 2001–2014
S 0002-9939(10)10255-X
Article electronically published on February 16, 2010

AN ALEKSANDROV TYPE ESTIMATE FOR α-CONVEX FUNCTIONS

CRISTIAN E. GUTIÉRREZ AND FEDERICO TOURNIER

(Communicated by Matthew J. Gursky)

Abstract. In the context of α-convexity, using an operator similar to the Monge-Ampère operator based on the notion of normal mapping, we estimate the difference between a function u and the solution of the homogeneous problem U in terms of the measure of the normal mapping of u and a power of the distance to the boundary.

1. Introduction

In the theory of the Monge-Ampère equation the following estimate due to Aleksandrov is of great importance: if u is convex in Ω, an open bounded convex subset of $\mathbb{R}^n$, and $u \in C(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$, then

$$|u(x)|^n \leq C \operatorname{dist}(x, \partial\Omega) \operatorname{diam}(\Omega)^{n-1} |Du(\Omega)|, \tag{1.1}$$

for all $x \in \Omega$, where

$$Du(\Omega) = \{p \in \mathbb{R}^n : \exists\, \overline{x} \in \Omega \quad \text{such that } u(x) \geq u(\overline{x}) + p \cdot (x - \overline{x}) \ \ \forall x \in \Omega\}, \tag{1.2}$$

with a constant C depending only on n and independent of u. The estimate plays a crucial role in the theory of sections of solutions to the Monge-Ampère equation and consequently in regularity theory; see [Caf90], [Gut01], [GH00]. More generally, if u is not necessarily convex but satisfies $u(x_0) \leq 0$ at some $x_0 \in \Omega$, then (1.1) holds at $x = x_0$. Indeed, taking v to be the convex function defining a cone with base in $\partial\Omega$ and vertex at the point $(x_0, u(x_0))$, and following the argument in [Gut01, Lemma 1.4.1], we obtain $Dv(x_0) \subset Du(\Omega)$. Then the proof of [Gut01, Theorem 1.4.2] applies in this case.

The purpose of this paper is to prove this estimate in the context of α-convex functions, $\alpha > 1$; see Definition 2.3. In the language of optimal mass transportation these are functions that are convex with respect to the cost function $c(x, y) = |x - y|^\alpha$. In our estimate, the subdifferential $Du(\Omega)$ on the right hand side of (1.1) is replaced by the quantity

$$F_u(\Omega) = \{y \in \mathbb{R}^n : \exists\, \bar{x} \in \Omega \text{ such that } u(x) \geq u(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha \forall x \in \Omega\}, \tag{1.3}$$

and $|u(x)|$ on left hand side of (1.1) gets replaced by $U(x) - u(x)$, where U is the solution of the homogeneous problem $|F_U(\Omega)| = 0$ with $U = 0$ on $\partial\Omega$. The case $\alpha = 2$ is related to standard convexity since $F_u(x) = x + 2Du(x)$; see the

Received by the editors October 27, 2008.
2010 *Mathematics Subject Classification.* Primary 35-XX.
The first author was partially supported by NSF grant DMS–0610374.

end of the proof of Lemma 3.4. The structure of the set F_u is related to the condition (A3w) introduced in [TW09] for general cost functions, and it is proven there, in Example 4, that $|x-y|^\alpha$ satisfies this condition only when $\alpha = 2$ or when $-2 \le \alpha < 1$. Consequently, from the results of Loeper [Loe09, Proposition 2.11 and Theorem 3.1], the set $F_u(\bar{x})$ defined in Definition 2.1 is in general not connected. We refer to the paper [GN07] for results on Monge-Ampère type equations arising in optimal mass transportation for general cost functions and properties of the subdifferential F_u. Optimal mass transportation has recently become a very active area of research; we mention, in particular, the fundamental work of Ma, Trudinger and Wang [MTW05], for smooth cost functions. For further references see [Vil07].

The main estimate proved in this paper is the following:

Let $\alpha > 1$ and let Ω be an open, bounded, convex domain in $\mathbb{R}^n$. If $u \in C(\bar{\Omega})$ with $u = 0$ on $\partial\Omega$ and such that $0 \le u \le U$ in Ω, then for all $x \in \Omega$ we have

$$(U(x) - u(x))^n \le C(\mathrm{dist}(x, \partial\Omega))^{\frac{n+1}{2}} \mathrm{diam}(\Omega)^{\frac{n(2\alpha-3)-1}{2}} |F_u(\Omega)| \tag{1.4}$$

whenever $n(2\alpha - 3) - 1 \ge 0$, and

$$(U(x) - u(x))^n \le C(\mathrm{dist}(x, \partial\Omega))^{n(\alpha-1)} |F_u(\Omega)| \tag{1.5}$$

whenever $n(2\alpha - 3) - 1 \le 0$. The constant C depends only on n and α, and U is the solution of the homogeneous problem as stated above. Depending on the value of α, the hypothesis $u \ge 0$ is essential for the validity of the estimates. Indeed, if $\alpha > 2n/(n-1)$ and $u < 0$, then it is not possible to give an estimate of $|u|$ by any positive power of the distance; see Remark 5.3. However, if $\alpha \le 2n/(n-1)$ and $u < 0$, then such an estimate holds; see Theorem 5.2.

The paper is organized as follows. Section 2 contains preliminary results. In Section 3, we solve the homogeneous Dirichlet problem giving an explicit characterization of the solution. In Section 4, we find the solution u to the Dirichlet problem when the right hand side is a multiple β of the Dirac delta function at a point $\bar{x} \in \Omega$, and we estimate the size of β in terms of $U(\bar{x}) - u(\bar{x})$ and $\mathrm{dist}(\bar{x}, \partial\Omega)$. The whole Section 4 is devoted to proving Theorem 4.1, and Lemma 4.3, and the main estimates are finally proved in Section 5.

2. Definitions and preliminary results

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, convex set, and $\alpha > 1$.

Definition 2.1. Let $u : \Omega \to \mathbb{R}$ and $\bar{x} \in \Omega$. We define

$$F_u(\bar{x}) := \{y \in \mathbb{R}^n : u(x) \ge u(\bar{x}) + |\, \bar{x} - y \,|^\alpha - |\, x - y \,|^\alpha \ \ \forall x \in \Omega\}. \tag{2.1}$$

If $E \subseteq \Omega$, we define $F_u(E) = \bigcup_{x \in E} F_u(x)$.

Remark 2.2. If $u \in C(\overline{\Omega})$ and $\bar{x} \in \partial\Omega$, then we say that $y \in F_u(\bar{x})$ if there exists $\hat{x} \in \Omega$ such that $y \in F_u(\hat{x})$ and $u(x) \ge u(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha$ for all $x \in \Omega$.

Definition 2.3. We say that $u : \Omega \longrightarrow R$ is α-convex in Ω if $F_u(x) \ne \emptyset$, $\forall x \in \Omega$.

Lemma 2.4. *If $u : \Omega \longrightarrow R$ is α-convex in Ω, then u is locally Lipschitz continuous in Ω.*

Proof. First we check the boundedness of u. That u is bounded below is trivial. We show that u is locally bounded above in Ω. Let $K \Subset \Omega$ be compact and suppose there exist $x_0 \in K$ and $\{x_n\} \subset K$ with $x_n \to x_0$ and $u(x_n) \to +\infty$. If

$y_n \in F_u(x_n)$, then $u(x) \geq u(x_n) + |x_n - y_n|^\alpha - |x - y_n|^\alpha \ \forall x \in \Omega$. If $|y_n| \leq M$, then clearly $\lim_{n\to\infty} u(x_n) + |x_n - y_n|^\alpha - |x - y_n|^\alpha = +\infty$ for each $x \in \Omega$ (passing through a subsequence if necessary) which yields $u(x) = +\infty$ for $x \in \Omega$. So, we can assume that $|y_n| \to +\infty$. Let $I = \bigcup_{k=1}^\infty \bigcap_{n=k}^\infty B_{|x_n - y_n|}(y_n)$. If $x \in I$, then $\lim_{n\to\infty} u(x_n) + |x_n - y_n|^\alpha - |x - y_n|^\alpha = +\infty$ and so $I \cap \Omega = \emptyset$.

Letting $\xi_n = \dfrac{y_n - x_n}{|y_n - x_n|}$, we may assume that $\xi_n \to e_1$. Consider the cone $\mathcal{C} = \{x : e_1 \cdot \left(\frac{x - x_0}{|x - x_0|}\right) \geq \delta\}$. We claim that $\mathcal{C} \subset I$. If we let $x \in \mathcal{C}$, there exists N such that $\xi_n \cdot \left(\dfrac{x - x_n}{|x - x_n|}\right) \geq \delta/2$ for all $n \geq N$. Since $|x_n - y_n| \to \infty$, there exists N' such that $\dfrac{2}{\delta}|x - x_n| \leq \dfrac{2}{\delta}\mathrm{diam}(\Omega) \leq |x_n - y_n|$ for all $n \geq N'$. Thus, $\dfrac{\delta}{2} \leq \dfrac{x_n - y_n}{|x_n - y_n|} \cdot \dfrac{x - x_n}{|x - x_n|} \leq \dfrac{\delta}{2}\dfrac{|x_n - y_n|^2}{|x_n - y_n||x - x_n|}$ for all n sufficiently large and the claim is proved. Since $x_0 \in \Omega$ there exists $x' \in \Omega \cap \mathcal{C}$, and therefore $x' \in \Omega \cap I$, a contradiction. Therefore, u is bounded on compact subsets of Ω.

We next show that u is locally Lipschitz. Let B be a ball with $2B \Subset \Omega$. Then we show first that the set $F_u(B)$ is bounded. Otherwise, there exist $y_n \in F_u(x_n)$, with $x_n \in B$ and $|y_n| \to \infty$. Since u is bounded above in $2B$ and bounded below in Ω, we get that $|x_n - y_n|^\alpha - |x - y_n|^\alpha \leq M$ for all $x \in 2B \Subset \Omega$. Let $x = x_n + \beta\xi_n$ with $\xi_n = \dfrac{y_n - x_n}{|y_n - x_n|}$. We have $x \in 2B$ for β small, and therefore $|x_n - y_n|^\alpha - |x - y_n|^\alpha = |x_n - y_n|^\alpha - |x_n + \beta\xi_n - y_n|^\alpha = \alpha\beta|x_n + \overline{\beta}\xi_n - y_n|^{\alpha-1}$ for some $0 < \overline{\beta} < \beta$ from the mean value theorem. But the last expression tends to $+\infty$ as $n \to \infty$, which yields a contradiction. Finally, let $B \Subset \Omega$ and let $x_0, x_1 \in B$ and $y_0 \in F_u(x_0), y_1 \in F_u(x_1)$. Then $u(x) \geq u(x_0) + |x_0 - y_0|^\alpha - |x - y_0|^\alpha$ and $u(x) \geq u(x_1) + |x_1 - y_1|^\alpha - |x - y_1|^\alpha$ $\forall x \in \Omega$. Thus, $|x_0 - y_0|^\alpha - |x_1 - y_0|^\alpha \leq u(x_1) - u(x_0) \leq |x_0 - y_1|^\alpha - |x_1 - y_1|^\alpha$. Again from the mean value theorem it follows that $\alpha|\hat{x} - y_0|^{\alpha-2}(\hat{x} - y_0)\cdot(x_0 - x_1) \leq u(x_1) - u(x_0) \leq \alpha|\bar{x} - y_1|^{\alpha-2}(\bar{x} - y_1)\cdot(x_0 - x_1)$ for some $\hat{x}, \bar{x} \in [x_0, x_1]$. Consequently, $-\alpha|\hat{x} - y_0|^{\alpha-1}|x_0 - x_1| \leq u(x_1) - u(x_0) \leq \alpha|\bar{x} - y_1|^{\alpha-1}|x_0 - x_1|$ and the Lipschitz continuity of u in B follows. □

Remark 2.5. If u is α-convex in Ω, then

$$|\{y \in \mathbb{R}^n : y \in F_u(x_1) \cap F_u(x_2), x_1 \neq x_2 \in \Omega\}| = 0. \tag{2.2}$$

Indeed, the function $u^\star(z) = \inf_{x\in\Omega} u(x) + |x - z|^\alpha$ is locally Lipschitz in $\mathbb{R}^n$. Suppose $y_1 \in F_u(x_1)$, where $x_1 \in \Omega$. Then $u^\star(z) \leq u(x_1) + |x_1 - z|^\alpha = u(x_1) + |x_1 - y_1|^\alpha - |x_1 - y_1|^\alpha + |x_1 - z|^\alpha = u^\star(y_1) - |x_1 - y_1|^\alpha + |x_1 - z|^\alpha$, $\forall z \in \mathbb{R}^n$. Therefore, we see that if $y \in F_u(x_1) \cap F_u(x_2)$ for some $x_1 \neq x_2 \in \Omega$, then $u^\star$ cannot be differentiable at y. This proves the remark.

Remark 2.6. The conclusion in the previous remark also holds if $u \in C(\overline{\Omega})$ is α-convex and the y's are taken so that $y \in F_u(x_1) \cap F_u(x_2)$, where $x_1 \in \Omega$ and $x_2 \in \partial\Omega$.

For each $y_0 \in \Omega$ consider the set

$$A_{y_0} = \{x \in \partial\Omega : \mathrm{dist}(y_0, \partial\Omega) = |y_0 - x|\}. \tag{2.3}$$

We have the following lemma, which will be used in the proof of Theorem 3.1.

Lemma 2.7. *For each $y_0 \in \Omega$ and for each $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, there exists $\bar{x} \in A_{y_0}$, a sequence $y_k = y_0 + \delta_k \xi$ with $\delta_k > 0$ and $\delta_k \to 0$, and $x_k \in A_{y_k}$ such that $x_k \to \bar{x}$.*

Proof. For each k, let $x_k \in A_{y_0+\frac{1}{k}\xi}$ and since $\{x_k\}$ is a bounded sequence, passing through a subsequence, we can assume $x_k \to \bar{x} \in \partial\Omega$. Then $\operatorname{dist}(y_0, \partial\Omega) = \lim_{k\to\infty} \operatorname{dist}(y_0 + \frac{1}{k}\xi, \partial\Omega) = \lim_{k\to\infty} |y_0 + \frac{1}{k}\xi - x_k| = |y_0 - \bar{x}|$, i.e. $\bar{x} \in A_{y_0}$. □

3. Homogeneous Dirichlet problem

In this section, we solve the Dirichlet problem with zero boundary data and give a characterization of the solution. This problem was considered in [GN07] for general cost functions, but in our case we need to have a more precise characterization of the solution; see [GN07, Theorem 6.7].

Theorem 3.1. *Let Ω be a bounded, open, convex domain in $\mathbb{R}^n$. Let $\alpha > 1$. Given $y \in \mathbb{R}^n$, let $v_{\lambda,y}(x) = \lambda - |x-y|^\alpha$ ($v_{\lambda,y}$ is α-convex in $\mathbb{R}^n$) and let*

$$U(x) = \sup\{v_{\lambda,y}(x) : v_{\lambda,y} \le 0 \text{ on } \partial\Omega\}.$$

We then have:

(1) *U is α-convex in Ω;*
(2) *$U \in C(\overline{\Omega})$ and $U = 0$ on $\partial\Omega$;*
(3) *$|F_U(\Omega)| = 0$;*
(4) *$U(x) = \sup\{\operatorname{dist}^\alpha(y, \partial\Omega) - |x-y|^\alpha : y \in \Omega\}$;*
(5) *if $U(x_0) = \operatorname{dist}^\alpha(y_0, \partial\Omega) - |x_0 - y_0|^\alpha$, then $x_0 \in \Lambda_{y_0}$ with*

$$\Lambda_{y_0} = \{x \in \Omega : \text{for each } \xi \text{ with } |\xi| = 1 \text{ there exists } \bar{x} \in A_{y_0} \text{ such that } \big(|\bar{x} - y_0|^{\alpha-2}(y_0 - \bar{x}) - |x - y_0|^{\alpha-2}(y_0 - x)\big) \cdot \xi \le 0\},$$

where the set A_{y_0} is defined in (2.3).

Proof. (1) To define U it is enough to consider the set of functions $v_{\lambda,y}$ with $y \in \Omega$. Because if $y \notin \Omega$, and $v_{\lambda,y}(x) = \lambda - |x-y|^\alpha$ satisfies $v_{\lambda,y} \le 0$ on $\partial\Omega$, then $v_{\lambda,y} \le 0$ in Ω. While if $y \in \Omega$, then $v(x) = \operatorname{dist}^\alpha(y, \partial\Omega) - |x-y|^\alpha$ satisfies $v \le 0$ on $\partial\Omega$, and hence $U(x) \ge v(x)\ \forall\ x \in \Omega$; in particular,

$$U(y) \ge \operatorname{dist}^\alpha(y, \partial\Omega)\ \forall\ y \in \Omega. \tag{3.1}$$

Let $x_0 \in \Omega$. From the definition of U, $U(x_0) = \lim_{n\to\infty} \lambda_n - |x_0 - y_n|^\alpha = \lim_{n\to\infty} v_n(x_0)$ with $v_n(x) = \lambda_n - |x - y_n|^\alpha$ and $v_n(x) \le 0$ on $\partial\Omega$. Since $y_n \in \Omega$, we may assume $y_n \to y_0 \in \overline{\Omega}$, and so also $\lambda_n \to \lambda_0$. Thus, $U(x_0) = \lambda_0 - |x_0 - y_0|^\alpha$ and $v_n \to v_0$ uniformly in $\overline{\Omega}$ where $v_0(x) = \lambda_0 - |x - y_0|^\alpha$. It follows that $v_0 \le 0$ on $\partial\Omega$ and since $v_0(x_0) = U(x_0) \ge \operatorname{dist}^\alpha(x_0, \partial\Omega) > 0$, we have $y_0 \in \Omega$. Since $U \ge v_n$, it follows that $U \ge v_0$ in Ω and therefore $y_0 \in F_U(x_0)$, so U is α-convex.

(2) Since U is α-convex in Ω, from Lemma 2.4, U is continuous in Ω. We show that U is continuous up to the boundary. Fix $\bar{x} \in \partial\Omega$ and let $\eta(\bar{x})$ be the unit inner normal to some supporting plane to $\partial\Omega$ at $\bar{x}$, and let $V(x) = 2\operatorname{diam}(\Omega)^{\alpha-1}\langle x - \bar{x}, \eta(\bar{x})\rangle$. Let $v_{\lambda,y}(x) = \lambda - |x-y|^\alpha$ with $v_{\lambda,y} \le 0$ on $\partial\Omega$. Since $|D(V - v_{\lambda,y})(x)| > 0$ for $x \in \Omega$ and $V - v_{\lambda,y} \ge 0$ on $\partial\Omega$, it follows that $V - v_{\lambda,y} \ge 0$ in Ω. Therefore $V(x) \ge U(x)$ for $x \in \Omega$. This together with (3.1) yields that $U \in C(\overline{\Omega})$ and $U = 0$ on $\partial\Omega$.

(3) Let $y \in F_U(\bar{x})$ for some $\bar{x} \in \Omega$, so $U(x) \ge U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha$ for all $x \in \Omega$ and we claim that there exists $\hat{x} \in \partial\Omega$ such that $U(\bar{x}) + |\bar{x} - y|^\alpha - |\hat{x} - y|^\alpha = 0$.

Otherwise, since $U = 0$ on $\partial\Omega$, we must have that $U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha < 0$ for all $x \in \partial\Omega$, and hence, there exists $\epsilon > 0$ such that $U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha + \epsilon < 0$ for all $x \in \partial\Omega$. Then by the definition of U, we must have $U(x) \geq U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha + \epsilon$ for all $x \in \Omega$, and in particular, $U(\bar{x}) \geq U(\bar{x}) + \epsilon$, a contradiction, and the claim is proved. Therefore, there exists $\hat{x} \in \partial\Omega$ such that $U(\bar{x}) + |\bar{x} - y|^\alpha - |\hat{x} - y|^\alpha = 0$ and since $U(\hat{x}) = 0$ this clearly implies that $y \in F_U(\hat{x})$. We have then proved that $F_U(\Omega) \subseteq \{y : \exists\, x_1 \in \Omega, x_2 \in \partial\Omega \text{ such that } y \in F_U(x_1) \cap F_U(x_2)\}$, which implies that $|F_U(\Omega)| = 0$.

We remark that if $\alpha = 2$, we have that $U(x) = |\hat{x} - y|^2 - |x - y|^2$ for all $x \in [\bar{x}, \hat{x}]$, and hence $y \in F_U(x)$ for all $x \in [\bar{x}, \hat{x}]$, where $\hat{x} \in \partial\Omega$ is as above.

(4) Let $\bar{x} \in \Omega$ be arbitrary and let $\bar{y} \in F_U(\bar{x})$. From the claim in (3), we conclude that $U(x) \geq |\hat{x} - y|^\alpha - |x - \bar{y}|^\alpha$ for all $x \in \Omega$ with equality at $\bar{x}$ and at $\hat{x} \in \partial\Omega$. Hence, $|x - \bar{y}|^\alpha \geq |\hat{x} - \bar{y}|^\alpha$, for all $x \in \partial\Omega$. This implies that $|\hat{x} - \bar{y}|^\alpha = \operatorname{dist}^\alpha(\bar{y}, \partial\Omega)$ and so $U(x) \geq \operatorname{dist}^\alpha(\bar{y}, \partial\Omega) - |x - \bar{y}|^\alpha$ for all $x \in \Omega$ with equality at $\bar{x}$. Therefore, we get $U(\bar{x}) = \operatorname{dist}^\alpha(\bar{y}, \partial\Omega) - |\bar{x} - \bar{y}|^\alpha \leq \sup\{\operatorname{dist}^\alpha(y, \partial\Omega) - |\bar{x} - y|^\alpha : y \in \Omega\}$.

The reverse inequality follows by noting that for any y fixed in Ω, the function $v(x) = \operatorname{dist}^\alpha(y, \partial\Omega) - |x - y|^\alpha$ satisfies $v \leq 0$ on $\partial\Omega$, v α-convex, and so $U \geq v$.

(5) Suppose $U(x_0) = \operatorname{dist}^\alpha(y_0, \partial\Omega) - |x_0 - y_0|^\alpha$, and we will show $x_0 \in \Lambda_{y_0}$. Otherwise, there exists $\bar{\xi}$ with $|\bar{\xi}| = 1$ such that for all $x \in A_{y_0}$,

$$\langle |x - y_0|^{\alpha-2}(y_0 - x) - |x_0 - y_0|^{\alpha-2}(y_0 - x_0), \bar{\xi}\rangle > 0. \tag{3.2}$$

From Lemma 2.7 applied to $\bar{\xi}$, we know that there exists $\bar{x} \in A_{y_0}$ and $\delta_k \to 0$, $\delta_k > 0$, and $x_k \in A_{y_0 + \delta_k\bar{\xi}}$ such that $x_k \to \bar{x}$. Using $x = \bar{x}$ in equation (3.2), by definition of U and the fact that $x_k \in A_{y_0+\delta_k\bar{\xi}}$, we have

(3.3)

$$\begin{aligned}
0 &\geq |x_k - (y_0 + \delta_k\bar{\xi})|^\alpha - |x_0 - (y_0 + \delta_k\bar{\xi})|^\alpha - U(x_0) \\
&= |\delta_k\bar{\xi} - (x_k - y_0)|^\alpha - |\bar{x} - y_0|^\alpha - (|\delta_k\bar{\xi} - (x_0 - y_0)|^\alpha - |x_0 - y_0|^\alpha) \\
&= |x_k - y_0|^\alpha - |\bar{x} - y_0|^\alpha + \alpha\delta_k \Big\{ |\bar{\delta}_k\bar{\xi} - (x_k - y_0)|^{\alpha-2}\langle \bar{\delta}_k\bar{\xi} + y_0 - x_k, \bar{\xi}\rangle \\
&\qquad\qquad - |\hat{\delta}_k\bar{\xi} - (x_0 - y_0)|^{\alpha-2}\langle \hat{\delta}_k\bar{\xi} + y_0 - x_0, \bar{\xi}\rangle \Big\},
\end{aligned}$$

by the mean value theorem for some $0 < \bar{\delta}_k, \hat{\delta}_k < \delta_k$. Notice that $\lim_{\delta_k \to 0}\{...\} = \langle |\bar{x} - y_0|^{\alpha-2}(y_0 - \bar{x}) - |x_0 - y_0|^{\alpha-2}(y_0 - x_0), \bar{\xi}\rangle > 0$ and since $x_k \in \partial\Omega$ and $\bar{x} \in A_{y_0}$ we also have $|x_k - y_0|^\alpha - |\bar{x} - y_0|^\alpha \geq 0$ and hence, we conclude that for δ_k small enough, (3.3) is positive, which is a contradiction, thus proving the claim. This completes the proof. □

Remark 3.2. We analyze in passing the case $\alpha = 2$. In this case, $\Lambda_{y_0} = \{x : \forall\, \xi, |\xi| = 1, \exists\, \bar{x} \in A_{y_0} \text{ such that } \langle x - \bar{x}, \xi\rangle \leq 0\} = \text{convex hull}(A_{y_0})$ and we have the following conclusions:

If $x_0 \in \Omega$ and $U(x_0) = \operatorname{dist}^2(y_0, \partial\Omega) - |x_0 - y_0|^2$, then $A_{y_0} = \{x \in \partial\Omega : \operatorname{dist}(y_0, \partial\Omega) = |y_0 - x|\}$ is not a singleton. Moreover, if $U(x_0) = \operatorname{dist}^2(y_0, \partial\Omega) - |x_0 - y_0|^2 = |\bar{x} - y_0|^2 - |x_0 - y_0|^2$, where $\bar{x} \in A_{y_0}$, then $U(x) = |\bar{x} - y_0|^2 - |x - y_0|^2$ for all $x \in \text{convex hull}(A_{y_0})$. Also, $x_0 \in B_{|y_0 - \bar{x}|/2}\left(\frac{\bar{x} + y_0}{2}\right)$ and $|x_0 - \bar{x}|^2 \leq U(x_0)$. This can be realized by taking $\xi = x_0 - y_0$ in the definition of Λ_{y_0} and $\bar{x}$ to be the corresponding point in A_{y_0}.

Now, let us look at the case $\alpha > 1$ and consider the set Λ_{y_0}. Set $p_{y_0} = (x - y_0)|x - y_0|^{\alpha-2}$. One can check that $\Lambda_{y_0} = p_{y_0}^{-1}(\text{convex hull}(p_{y_0}(A_{y_0})))$.

Let $\xi = x_0 - y_0$. Then there exists $\bar{x} \in A_{y_0}$ such that $\langle |\bar{x} - y_0|^{\alpha-2}(y_0 - \bar{x}) - |x_0 - y_0|^{\alpha-2}(y_0 - x_0), x_0 - y_0\rangle \leq 0$, which gives $|x_0 - y_0|^\alpha \leq |\bar{x} - y_0|^{\alpha-2}\langle \bar{x} - y_0, x_0 - y_0\rangle$. Taking for instance $y_0 = 0$ and $\bar{x}$ along e_1, we see that if $x_0 \in \Lambda_{y_0}$, then x_0 is on the set obtained by rotating the polar curve $r = R(\cos\theta)^{\frac{1}{\alpha-1}}$ around the e_1-axis, where $R = |\bar{x}|$.

3.1. **Regularity of U.** We prove the following theorem.

Theorem 3.3. *The function U in Theorem* 3.1 *is $C^1(\Omega)$.*

We first prove a lemma.

Lemma 3.4. *Let u be α-convex in Ω. Then $u \in C^1(\Omega)$ if and only if $F_u(x)$ is a singleton for each $x \in \Omega$.*

Proof. It is clear that if for some $x \in \Omega$, $F_u(x)$ has more than one point, then u is not differentiable at x.

To prove the other implication, fix $\bar{x} \in \Omega$, and let $\{\bar{y}\} = F_u(\bar{x})$. We claim first that if $x_n \longrightarrow \bar{x}$, and $\{y_n\} = F_u(x_n)$, then $y_n \longrightarrow \bar{y}$. If not, then there exists $\epsilon > 0$ and infinitely many points $y_{n_k} \notin B_\epsilon(\bar{y})$. Since the sequence $\{y_{n_k}\}$ is bounded, extracting a subsequence we may assume $y_{n_k} \longrightarrow \hat{y}$. But then it follows that $\hat{y} \in F_u(\bar{x})$ and hence by assumption that $\hat{y} = \bar{y}$, and this is a contradiction, proving the claim.

Using this claim we show that u has first-order partial derivatives. Without loss of generality, we can assume $u(0) = 0$ and $\{0\} = F_u(0)$, so $u(x) \geq -|x|^\alpha$ for all $x \in \Omega$. Suppose $\frac{\partial u}{\partial x_1}(0)$ does not exist. For $t > 0$ we have $\dfrac{u(te_1)}{t} \geq -t^{\alpha-1}$ and hence $\liminf_{t\to 0^+} \dfrac{u(te_1)}{t} \geq 0$. Suppose $\limsup_{t\to 0^+} \dfrac{u(te_1)}{t} = a > 0$. Let $t_n \longrightarrow 0^+$ such that $\dfrac{u(t_n e_1)}{t_n} \geq \dfrac{a}{2}$ and let $\{y_n\} = F_u(t_n e_1)$. It follows by the claim that $y_n \longrightarrow 0$, and we have $u(x) \geq u(t_n e_1) + |t_n e_1 - y_n|^\alpha - |x - y_n|^\alpha$ for all n and for all $x \in \Omega$. In particular, $0 = u(0) \geq u(t_n e_1) + |t_n e_1 - y_n|^\alpha - |y_n|^\alpha = u(t_n e_1) + \alpha|\xi - y_n|^{\alpha-2}\langle \xi - y_n, t_n e_1\rangle$ for some $\xi \in [0, t_n e_1]$. Dividing by t_n we get $0 \geq \dfrac{u(t_n e_1)}{t_n} + \alpha|\xi - y_n|^{\alpha-2}\langle \xi - y_n, e_1\rangle \geq \dfrac{a}{2} + \alpha|\xi - y_n|^{\alpha-2}\langle \xi - y_n, e_1\rangle \geq \dfrac{a}{3}$ for n large enough, a contradiction. Exactly the same argument works for $t < 0$, and hence we conclude that $\frac{\partial u}{\partial x_1}(0)$ exists. By the claim once again we can also conclude that the partial derivatives are continuous because if $y \in F_u(x)$, then $y = x + \alpha^{1/(\alpha-1)}|Du(x)|^{(2-\alpha)/(\alpha-1)}Du(x)$. □

Proof of Theorem 3.3. Let us recall that Ω is convex and $U(x) = \sup\{\mathrm{dist}^\alpha(y, \partial\Omega) - |x - y|^\alpha : y \in \Omega\}$. Fix $x_0 \in \Omega$. We show that $F_U(x_0)$ is a singleton.

Set $t = U(x_0) > 0$ and suppose by contradiction that $y_1, y_2 \in F_U(x_0)$ with $y_1 \neq y_2$. It follows that $\mathrm{dist}^\alpha(y_i, \partial\Omega) - |x_0 - y_i|^\alpha = t$ for $i = 1, 2$. We also have that $B_i := B_{(|x_0 - y_i|^\alpha + t)^{\frac{1}{\alpha}}}(y_i) \subseteq \Omega$ for $i = 1, 2$ and $\partial B_1 \cap \partial B_2 \neq \emptyset$.

Let Λ be the convex hull of $B_1 \cup B_2$ and let T be a supporting hyperplane to Λ that touches Λ at more than one point. Set $\Phi(y) = \mathrm{dist}^\alpha(y, T) - |x_0 - y|^\alpha$. We will prove in Lemma 4.2 that the set $S = \{y : \Phi(y) \geq t\}$ is strictly convex. Since $\Phi(y_i) = t$ for $i = 1, 2$ it follows that $[y_1, y_2] \subseteq S$. Then, for $y \in (y_1, y_2)$ we have $\mathrm{dist}^\alpha(y, \partial\Omega) - |x_0 - y|^\alpha \geq \mathrm{dist}^\alpha(y, \partial\Lambda) - |x_0 - y|^\alpha = \mathrm{dist}^\alpha(y, T) - |x_0 - y|^\alpha \geq \Phi(y) > t$, and this is a contradiction with the definition of U since $U(x_0) = t$. □

4. Nonhomogeneous Dirichlet problem

Let $\alpha > 1$ and let U be the solution of $|F_U(\Omega)| = 0$, $U = 0$ on $\partial\Omega$ from the previous section. We shall prove the following theorem; see [GN07, Lemma 6.19].

Theorem 4.1. *Let $x_0 \in \Omega$ and $t < U(x_0)$ and define*

$$u(x) = \sup\{v_{\lambda,y}(x) = \lambda - |x-y|^\alpha : v_{\lambda,y} \le 0 \text{ on } \partial\Omega \text{ and } v_{\lambda,y}(x_0) \le t\}. \tag{4.1}$$

Then $u \in C(\bar\Omega)$, $u = 0$ on $\partial\Omega$, $u(x_0) = t$, u is α-convex and satisfies the equation $F_u = \beta\delta_{x_0}$, for some $\beta \ge 0$. Moreover, when $t \ge 0$ we have the following estimates for β:

(a) *Suppose $1 < \alpha \le 2$. If $n(3-2\alpha)+1 \ge 0$, then*

$$\beta = |F_u(x_0)| \ge C\frac{(U(x_0)-t)^n}{\operatorname{dist}(x_0,\partial\Omega)^{n(\alpha-1)}}; \tag{4.2}$$

and if $n(3-2\alpha)+1 < 0$, then

$$\beta = |F_u(x_0)| \ge C\frac{(U(x_0)-t)^n}{\operatorname{dist}(x_0,\partial\Omega)^{\frac{n+1}{2}}\operatorname{diam}(\Omega)^{\frac{n(2\alpha-3)-1}{2}}}. \tag{4.3}$$

(b) *If on the other hand $\alpha \ge 2$, then we have*

$$\beta = |F_u(x_0)| \ge C\frac{(U(x_0)-t)^n}{\operatorname{dist}(x_0,\partial\Omega)^{\frac{n+1}{2}}\operatorname{diam}(\Omega)^{\frac{n(2\alpha-3)-1}{2}}}. \tag{4.4}$$

Here C is a positive constant depending only on α and n.

Finally, for $\alpha > 1$ and $t \ge 0$, the set $F_u(x_0)$ is convex.

Proof. The set of functions $v_{\lambda,y}$ is clearly nonempty, so u is well defined and also $u(x_0) \le t$. We will prove that $u(x_0) \ge t$. Let's assume first that $t \ge 0$. Since $t < U(x_0)$, there exists $y_0 \in \Omega$ such that $t < \operatorname{dist}^\alpha(y_0,\partial\Omega) - |x_0-y_0|^\alpha$. Since the function $\Psi(z) = \operatorname{dist}^\alpha(z,\partial\Omega) - |x_0-z|^\alpha$ is continuous and $\Psi(y_0) > t$ and $\Psi \le 0$ on $\partial\Omega$, this implies that there exists $z_0 \in \Omega$ such that $\Psi(z_0) = t$. Letting $v(x) = \operatorname{dist}^\alpha(z_0,\partial\Omega) - |x-z_0|^\alpha$, then $v \le 0$ on $\partial\Omega$ and $v(x_0) = t$; this implies $u(x_0) \ge t$. If $t < 0$, then we can take v_{t,x_0} in the definition of u. Therefore $u(x_0) = t$.

We prove that u is α-convex in Ω. Let $\bar x \in \Omega$. We first claim the supremum is attained; i.e., there exists $\bar\lambda \in \mathbb{R}$ and $\bar y \in \mathbb{R}^n$ such that $u(\bar x) = \bar\lambda - |\bar x - \bar y|^\alpha$, where $v_{\bar\lambda,\bar y}$ satisfies $v_{\bar\lambda,\bar y} \le 0$ on $\partial\Omega$ and $v_{\bar\lambda,\bar y}(x_0) \le t$. Assuming the claim we get that $u(x) \ge \bar\lambda - |x-\bar y|^\alpha = u(\bar x) + |\bar x - \bar y|^\alpha - |x-\bar y|^\alpha$, for all $x \in \Omega$, which implies that $\bar y \in F_u(\bar x)$; that is, u is α-convex.

To prove the claim, we have from the definition of u that $u(\bar x) = \lim_{n\to\infty}\lambda_n - |\bar x - y_n|^\alpha$. If $t \ge 0$, then we may assume that $y_n \in \Omega$; otherwise $v_{\lambda_n,y_n} \le 0$ in $\partial\Omega$ while $u \ge 0$ in Ω. We may also assume that $y_n \to \bar y \in \bar\Omega$ and hence also $\lambda_n \to \bar\lambda$ and the claim is then proved. If $t < 0$, suppose by contradiction that $|\bar x - y_n| \to +\infty$, in which case also $\lambda_n \to +\infty$. This implies that for n large, $y_n \notin \Omega$. Let $x_n \in \partial\Omega$ be such that $\operatorname{dist}(y_n,\partial\Omega) = |x_n - y_n|$. Set $v_n(x) = \lambda_n - |x-y_n|^\alpha$.

Then $v_n \le 0$ on $\partial\Omega$ and hence $\lambda_n \le \operatorname{dist}^\alpha(y_n, \partial\Omega) = |x_n - y_n|^\alpha$, which implies that $v_n(x) \le |x_n - y_n|^\alpha - |x - y_n|^\alpha$, so

$$\begin{aligned} v_n(\bar{x}) &\le |x_n - y_n|^\alpha - |\bar{x} - y_n|^\alpha \\ &= \alpha|\hat{x}_n - y_n|^{\alpha-2}\langle \hat{x}_n - y_n, x_n - \bar{x}\rangle \quad \text{for some } \hat{x}_n \in [\bar{x}, x_n] \\ &= \alpha|\hat{x}_n - y_n|^{\alpha-1}|x_n - \bar{x}|\cos\theta_n, \end{aligned}$$

where $\theta_n = \angle(y_n - \hat{x}_n, \bar{x} - x_n) = \angle(y_n - \hat{x}_n, \bar{x} - \hat{x}_n) = \angle(y_n - x_n, \bar{x} - x_n) \ge \frac{\pi}{2} + \delta(\epsilon)$ for some $\delta(\epsilon) > 0$, where $B_\epsilon(\bar{x}) \subseteq \Omega$ (here $\angle(x, y)$ denotes the angle between the vectors x and y). Hence $\cos\theta_n \le -C_\epsilon$. Then, $|x_n - y_n|^\alpha - |\bar{x} - y_n|^\alpha \le -\alpha|\hat{x}_n - y_n|^{\alpha-1}\epsilon C_\epsilon \le -\alpha|x_n - y_n|^{\alpha-1}\epsilon C_\epsilon \to -\infty$ as $n \to \infty$. This is a contradiction, since $v_n(\bar{x}) \to u(\bar{x})$.

Next we show that $u \in C(\bar{\Omega})$. First, notice that by definition, $u \le U$ in Ω. If $t \ge 0$, then $u \ge 0$ and we are done. Suppose $t < 0$. Fix $\bar{x}$ on $\partial\Omega$, and let η be the unit outer normal to $\partial\Omega$ at $\bar{x}$. For $s > 0$, let $v_s(x) = |\bar{x} - (\bar{x} + \eta s)|^\alpha - |x - (\bar{x} + \eta s)|^\alpha$. We have $v_s \le 0$ on $\partial\Omega$, and exactly as above we see that $v_s(x_0) \to -\infty$ as $s \to +\infty$. Hence, for s large enough, v_s is an admissible function and hence $u \ge v_s$ in Ω. Since $v(\bar{x}) = 0$, we get that $u \in C(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$.

We now show that $F_u = \beta\delta_{x_0}$ for some $\beta = \beta(x_0, t)$. Indeed, let $\bar{x} \in \Omega,\ \bar{x} \ne x_0$, and let $y \in F_u(\bar{x})$; we claim that $y \in F_u(\tilde{x})$ for some $\tilde{x} \ne \bar{x}$. Assuming the claim, we get from (2.2) that $|F_u(E)| = 0$ for each E with $x_0 \notin E$, and so F_u is concentrated at x_0.

To prove the claim, since $y \in F_u(\bar{x}),\ u(x) \ge u(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha$, for all $x \in \Omega$. Let $v(x) = u(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha = \lambda - |x - y|^\alpha$. Since $u = 0$ on $\partial\Omega$, we have $v \le 0$ on $\partial\Omega$ and $v(x_0) \le u(x_0) = t$. If $v(x_0) = t$, then $y \in F_u(x_0)$; and if $v(x_0) < t$, then, as before, there exists $\hat{x} \in \partial\Omega$ such that $v(\hat{x}) = 0$ and hence $y \in F_u(\hat{x})$.

Before estimating β, we need the following characterization of u (it will be easier to work with $\bar{u}$ rather than with u). If $\bar{u}(x) = \sup\{\operatorname{dist}^\alpha(y, \partial\Omega) - |x - y|^\alpha = v_y(x)$ such that $v_y(x_0) \le t,\ y \in \mathbb{R}^n\}$, then $u = \bar{u}$. Notice that $\bar{u} \le u$. To show that $\bar{u} \ge u$, let $\bar{x} \in \Omega$ and $\bar{y} \in F_u(\bar{x})$, so $u(x) \ge u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha$ for all $x \in \Omega$, and we claim that there exists $\hat{x} \in \partial\Omega$ such that $u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |\hat{x} - \bar{y}|^\alpha = 0$. Let's assume this claim holds. We have $0 \ge u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha$ for all $x \in \partial\Omega$ and $|\hat{x} - \bar{y}|^\alpha = u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha$, which implies that $|x - \bar{y}| \ge |\hat{x} - \bar{y}|$ for all $x \in \partial\Omega$, and hence $\operatorname{dist}(\bar{y}, \partial\Omega) = |\hat{x} - \bar{y}|$. This implies that $u(\bar{x}) = \operatorname{dist}^\alpha(\bar{y}, \partial\Omega) - |\bar{x} - \bar{y}|^\alpha \le \bar{u}(\bar{x})$ and we are done. We now prove the claim. If the claim is not true, there exists $\epsilon > 0$ such that $u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha + \epsilon < 0$ for all $x \in \partial\Omega$. By continuity, there exists $\delta > 0$ such that if $y \in B_\delta(\bar{y})$, then $u(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha + \frac{\epsilon}{2} \le u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha + \epsilon$ for all $x \in \partial\Omega$. Consider the set $\Gamma = \{y :\ u(\bar{x}) + |\bar{x} - y|^\alpha - |x_0 - y|^\alpha < t\}$. Then $\Gamma \cap B_\delta(\bar{y}) \ne \emptyset$. Fix $\tilde{y} \in \Gamma \cap B_\delta(\bar{y})$, and let $v(x) = u(\bar{x}) + |\bar{x} - \tilde{y}|^\alpha - |x - \tilde{y}|^\alpha + \dfrac{t - (u(\bar{x}) + |\bar{x} - \tilde{y}|^\alpha - |x_0 - \tilde{y}|^\alpha)}{M}$, where $M > 1$ is large enough such that $0 < \dfrac{t - (u(\bar{x}) + |\bar{x} - \tilde{y}|^\alpha - |x_0 - \tilde{y}|^\alpha)}{M} < \frac{\epsilon}{2}$. It follows that $v \le 0$ on $\partial\Omega$ and $v(x_0) \le t$, and this implies that $v \le u$ in Ω, but $v(\bar{x}) > u(\bar{x})$, and this contradiction proves the claim.

We next prove that

$$F_u(x_0) = \{y \in \mathbb{R}^n : \operatorname{dist}^\alpha(y, \partial\Omega) - |x_0 - y|^\alpha \ge t\} := F_t(x_0). \tag{4.5}$$

It will be easier to work with $F_u(x_0)$ written in this way. For, if $y \in F_u(x_0)$, then $u(x) \geq u(x_0)+|x_0-y|^\alpha-|x-y|^\alpha$ for all $x \in \Omega$, which implies that $|x-y|^\alpha \geq u(x_0)+|x_0-y|^\alpha$ for all $x \in \partial\Omega$, and hence, $\mathrm{dist}^\alpha(y,\partial\Omega) \geq u(x_0)+|x_0-y|^\alpha = t+|x_0-y|^\alpha$. On the other hand, if $\mathrm{dist}^\alpha(y,\partial\Omega)-|x_0-y|^\alpha \geq t$, then $|x-y|^\alpha \geq |x_0-y|^\alpha+t$ for all $x \in \partial\Omega$, and hence $0 \geq |x_0-y|^\alpha+t-|x-y|^\alpha = |x_0-y|^\alpha+u(x_0)-|x-y|^\alpha$ for all $x \in \partial\Omega$. Set $v(x) = |x_0-y|^\alpha+u(x_0)-|x-y|^\alpha$. Then $v \leq 0$ on $\partial\Omega$, $v(x_0) = t$, and hence $u \geq v$ in Ω, which implies $y \in F_u(x_0)$.

The remainder of the proof is devoted to establishing (4.2)-(4.4).

Recall that $u(x_0) = t < U(x_0)$, and say $U(x_0) = \mathrm{dist}^\alpha(y_0,\partial\Omega)-|x_0-y_0|^\alpha$ for some $y_0 \in \Omega$. Without loss of generality we will assume from now on that $y_0 = 0$, $\mathrm{dist}(y_0,\partial\Omega) = R$ and $x_0 = |x_0|e_1$ with $|x_0| < R$.

Since $B_R(0) \subseteq \Omega$ we have that $|F_t(x_0)| \geq |\{y \in B_R(0) : \mathrm{dist}^\alpha(y,\partial\Omega)-|y-x_0|^\alpha \geq t\}| \geq |\{y \in B_R(0) : \mathrm{dist}^\alpha(y,\partial B_R(0))-|y-x_0|^\alpha \geq t\}| = |\{y \in B_R(0) : (R-|y|)^\alpha - |y-x_0|^\alpha \geq t\}|$. We shall estimate the measure of the last set.

Estimation of β when $\alpha = 2$.

First notice that the set $\{y \in B_R(0) : (R-|y|)^2-|y-x_0|^2 \geq t\}$ is the ellipsoid

$$\left\{y : \frac{(y_1-\delta)^2}{a^2}+\frac{\Sigma_{i=2}^n (y_i)^2}{b^2} \leq 1\right\}$$

with

$$\delta = \frac{(R^2-|x_0|^2-t)|x_0|}{2(R^2-|x_0|^2)},\ a = \frac{(R^2-|x_0|^2-t)R}{2(R^2-|x_0|^2)},\ \text{and } b = \frac{(R^2-|x_0|^2-t)}{2(R^2-|x_0|^2)^{\frac{1}{2}}}.$$

The volume of this ellipsoid equals

$$C_n\frac{(R^2-|x_0|^2-t)^n R}{(R^2-|x_0|^2)^{\frac{n+1}{2}}}.$$

We also notice that the set $\{y \in B_R(0) : (R-|y|)^2-|y-x_0|^2 \geq t\}$ equals the set $\left\{\rho\xi : \xi \in S^{n-1}, 0 \leq \rho \leq \frac{R^2-|x_0|^2-t}{2(R-\langle x_0,\xi\rangle)}\right\}$, and using polar coordinates we get that its volume is equal to $C_n(R^2-|x_0|^2-t)^n \int_{S^{n-1}} \frac{1}{(R-\langle x_0,\xi\rangle)^n}d\xi$. This implies that $\int_{S^{n-1}} \frac{1}{(R-\langle x_0,\xi\rangle)^n}d\xi = C_n\frac{R}{(R^2-|x_0|^2)^{\frac{n+1}{2}}}$, which also implies for $\alpha > 1$ that

$$\int_{S^{n-1}} \frac{1}{(R^{\alpha-1}-\langle |x_0|^{\alpha-2}x_0,\xi\rangle)^n}d\xi = \frac{R^{\alpha-1}}{(R^{2(\alpha-1)}-|x_0|^{2(\alpha-1)})^{\frac{n+1}{2}}}, \tag{4.6}$$

for $|x_0| < R$.

Estimation of β for $\alpha > 1$.

Let $\xi \in S^{n-1}$ and consider $\phi(s) = (R-s)^\alpha-|s\xi-x_0|^\alpha$ for $0 \leq s \leq \frac{R^2-|x_0|^2}{2(R-\langle x_0,\xi\rangle)}$. We claim that ϕ is decreasing, concave for $1 < \alpha \leq 2$ and convex for $\alpha \geq 2$. First notice that for $0 \leq s \leq \frac{R^2-|x_0|^2}{2(R-\langle x_0,\xi\rangle)}$ we have $|s\xi-x_0| \leq (R-s)$. Next, we compute $\phi'(s) = -\alpha(R-s)^{\alpha-1}-\alpha|s\xi-x_0|^{\alpha-2}\langle s\xi-x_0,\xi\rangle \leq -\alpha(R-s)^{\alpha-1}+\alpha|s\xi-x_0|^{\alpha-1} < 0$ and $\phi''(s) = (\alpha-1)(R-s)^{\alpha-2}-|s\xi-x_0|^{\alpha-2}\left(1-(2-\alpha)(\frac{\langle s\xi-x_0,\xi\rangle}{|s\xi-x_0|})^2\right)$. Hence if $1 < \alpha < 2$, then $\phi''(s) < 0$; and if $\alpha > 2$, then $\phi''(s) > 0$. Therefore we have, for

$1 < \alpha \leq 2$, that $\phi(s) \geq R^\alpha - |x_0|^\alpha - s\dfrac{2(R - \langle x_0, \xi\rangle)(R^\alpha - |x_0|^\alpha)}{R^2 - |x_0|^2}$, and for $\alpha \geq 2$ that $\phi(s) \geq R^\alpha - |x_0|^\alpha - s(\alpha R^{\alpha-1} - \alpha|x_0|^{\alpha-2}\langle x_0, \xi\rangle)$. Given $0 \leq t < R^\alpha - |x_0|^\alpha$, we set in case $1 < \alpha \leq 2$, $R^\alpha - |x_0|^\alpha - s\dfrac{2(R - \langle x_0, \xi\rangle)(R^\alpha - |x_0|^\alpha)}{R^2 - |x_0|^2} = t$ and solve for s to get $s(\xi) = \dfrac{(R^\alpha - |x_0|^\alpha - t)(R^2 - |x_0|^2)}{2(R^\alpha - |x_0|^\alpha)(R - \langle x_0, \xi\rangle)}$. For $\alpha \geq 2$, we set $R^\alpha - |x_0|^\alpha - s(\alpha R^{\alpha-1} - \alpha|x_0|^{\alpha-2}\langle x_0, \xi\rangle) = t$ and we solve for s to get $s(\xi) = \dfrac{R^\alpha - |x_0|^\alpha - t}{\alpha R^{\alpha-1} - \alpha|x_0|^{\alpha-2}\langle x_0, \xi\rangle}$. Hence in any case we have that $\{y \in B_R(0) : (R - |y|)^\alpha - |y - x_0|^\alpha \geq t\}$ contains the set $\{\rho\xi : \xi \in S^{n-1}, 0 \leq \rho \leq s(\xi)\}$. Using polar coordinates we find that the volume of the last set is $\frac{1}{n}\int_{S^{n-1}} s(\xi)^n d\xi$. If $1 < \alpha \leq 2$, then

$$\begin{aligned}
&|\{\rho\xi : \xi \in S^{n-1}, 0 \leq \rho \leq s(\xi)\}| \\
&\quad = C_n \frac{(R^\alpha - |x_0|^\alpha - t)^n (R^2 - |x_0|^2)^n}{(R^\alpha - |x_0|^\alpha)^n} \int_{S^{n-1}} \frac{1}{(R - \langle x_0, \xi\rangle)^n} d\xi \\
&\quad = C_n \frac{(R^\alpha - |x_0|^\alpha - t)^n (R^2 - |x_0|^2)^n}{(R^\alpha - |x_0|^\alpha)^n} \frac{R}{(R^2 - |x_0|^2)^{\frac{n+1}{2}}}
\end{aligned}$$

from the calculation when $\alpha = 2$. If on the other hand, $\alpha \geq 2$, we get $|\{\rho\xi : \xi \in S^{n-1}, 0 \leq \rho \leq s(\xi)\} = C_n(R^\alpha - |x_0|^\alpha - t)^n \int_{S^{n-1}} \dfrac{1}{(R^{\alpha-1} - |x_0|^{\alpha-2}\langle x_0, \xi\rangle)^n} d\xi = \dfrac{(R^\alpha - |x_0|^\alpha - t)^n R^{\alpha-1}}{(R^{2(\alpha-1)} - |x_0|^{2(\alpha-1)})^{\frac{n+1}{2}}}$, from (4.6). From the mean value theorem, $R^\alpha - |x_0|^\alpha \leq \alpha R^{\alpha-1}(R - |x_0|)$, for $1 < \alpha \leq 2$, and for $\alpha \geq 2$ we have $R^{2(\alpha-1)} - |x_0|^{2(\alpha-1)} \leq 2(\alpha - 1)R^{2\alpha-3}(R - |x_0|)$, and using also that $R \leq R + |x_0| \leq 2R$, we get the following estimates: if $1 < \alpha \leq 2$, then $|F_t(x_0)| \geq C\dfrac{R^{\frac{n(3-2\alpha)+1}{2}}(R^\alpha - |x_0|^\alpha - t)^n}{(R - |x_0|)^{\frac{n+1}{2}}}$; and if $\alpha \geq 2$, then $|F_t(x_0)| \geq \dfrac{(R^\alpha - |x_0|^\alpha - t)^n}{R^{\frac{n(2\alpha-3)-1}{2}}(R - |x_0|)^{\frac{n+1}{2}}}$. Finally, noticing that $R - |x_0| \leq \operatorname{dist}(x_0, \partial\Omega)$, $R \leq \operatorname{diam}(\Omega)$ and $U(x_0) = R^\alpha - |x_0|^\alpha$, the estimates in the theorem follow.

To complete the proof it remains to show that for $t \geq 0$ and $\alpha > 1$ the set $F_t(x_0)$ is convex. We remark that this fact is not used in the estimation of $|F_t(x_0)|$. To show that $F_t(x_0)$ is convex we need the following lemma.

Lemma 4.2. *Let $C = \{(x', z) \in \mathbb{R}^n : z \in \mathbb{R};\ z^\alpha - (r^2 + (z - z_0)^2)^{\frac{\alpha}{2}} \geq t\}$, where $r = |x'|$, $z_0 > 0$. Then, the set C is convex for $t \geq 0$.*[*]

Proof. We will show that the function $r(z) = \left((z^\alpha - t)^{\frac{2}{\alpha}} - (z - z_0)^2\right)^{\frac{1}{2}}$ is a concave function on the set $z^\alpha - |z - z_0|^\alpha \geq t$. We have $rr' = (z^\alpha - t)^{\frac{2-\alpha}{\alpha}} z^{\alpha-1} - (z - z_0) > 0$ and

$$(r')^2 + rr'' = \frac{z^\alpha - (\alpha - 1)t - (z^\alpha - t)^{\frac{2(\alpha-1)}{\alpha}} z^{2-\alpha}}{(z^\alpha - t)^{\frac{2(\alpha-1)}{\alpha}} z^{2-\alpha}}.$$

[*]If $t > 0$, then C is strictly convex.

Hence, to show that $r'' < 0$ for $r > 0$, we must show that

$$\frac{z^\alpha - (\alpha-1)t - (z^\alpha - t)^{\frac{2(\alpha-1)}{\alpha}} z^{2-\alpha}}{(z^\alpha - t)^{\frac{2(\alpha-1)}{\alpha}} z^{2-\alpha}} \le (r')^2 = \frac{((z^\alpha - t)^{\frac{2-\alpha}{\alpha}} z^{\alpha-1} - (z - z_0))^2}{(z^\alpha - t)^{\frac{2}{\alpha}} - (z - z_0)^2},$$

which holds if and only if

$$0 \le (\alpha-1)t((z^\alpha - t)^{\frac{2}{\alpha}} - (z - z_0)^2) + z^\alpha((z - z_0) - (z^\alpha - t)z^{1-\alpha})^2.$$

This inequality is obviously true for $t \ge 0$. □

For $\bar{x} \in \partial\Omega$, let $T_{\bar{x}}$ be a supporting hyperplane to Ω at $\bar{x}$ and let $P_{\bar{x},t} = \{y \in \Omega : \operatorname{dist}^\alpha(y, T_{\bar{x}}) - |y - x_0|^\alpha \ge t\}$, which by the previous lemma is a convex set. Notice that $F_t(x_0) = \{y \in \Omega : \operatorname{dist}^\alpha(y, \partial\Omega) - |y - x_0|^\alpha \ge t\} = \bigcap_{\bar{x}\in\partial\Omega} P_{\bar{x},t}$, and hence it is a convex set. This completes the proof of Theorem 4.1. □

We now consider the case when $t < 0$ in Theorem 4.1.

Lemma 4.3. *For $1 < \alpha \le \dfrac{2n}{n-1}$ there exists $\delta > 1$ depending only on α such that if $-t \ge \delta \operatorname{dist}(x_0, \partial\Omega)^\alpha$ then*

$$|F_t(x_0) \cap \Omega^c| \ge C \frac{(-t)^{\frac{n+1}{2(\alpha-1)}}}{\operatorname{dist}(x_0, \partial\Omega)^{\frac{n(2-\alpha)+\alpha}{2(\alpha-1)}}}, \tag{4.7}$$

where C depends only on α and n.

Proof. Write $x = (x', x_n)$ and assume $0 \in \partial\Omega$, $\Omega \subseteq \{x : x_n \le 0\}$ and $x_0 = (0, -\epsilon) \in \Omega$ with $\operatorname{dist}(x_0, \partial\Omega) = \epsilon$. Assume that $-t \ge \delta\epsilon^\alpha$, where $\delta > 1$ will be chosen momentarily. For $y \in \mathbb{R}^n$ with $y_n \ge 0$, we have $\operatorname{dist}^\alpha(y, \partial\Omega) - |y - x_0|^\alpha \ge y_n^\alpha - (|y'|^2 + (y_n + \epsilon)^2)^{\frac{\alpha}{2}}$. Hence $H := \{y : y_n^\alpha - (|y'|^2 + (y_n + \epsilon)^2)^{\frac{\alpha}{2}} \ge t$ and $y_n \ge 0\ \} \subseteq F_t(x_0)$. Let $\hat{y} > 0$ satisfy the equation $y^\alpha - t = (y + \epsilon)^\alpha$. Then by slicing, the volume of H equals

$$V = C_n \int_0^{\hat{y}} ((y^\alpha - t)^{\frac{2}{\alpha}} - (y + \epsilon)^2)^{\frac{n-1}{2}} dy.$$

Let $\phi(y) = (y^\alpha - t)^{\frac{2}{\alpha}} - (y + \epsilon)^2 = \phi_1(y)\phi_2(y)$, where $\phi_1(y) = (y^\alpha - t)^{\frac{1}{\alpha}} - (y + \epsilon)$ and $\phi_2(y) = (y^\alpha - t)^{\frac{1}{\alpha}} + (y + \epsilon)$. Since ϕ_1 and ϕ_2 are convex, we have $\phi_1(y) \ge \phi_1'(\hat{y})(y - \hat{y})$ and $\phi_2(y) \ge \phi_2(0) + \phi_2'(0)y$. Hence $\phi_1(y)\phi_2(y) \ge p(y)$, with p a concave parabola. Set $p(\bar{y}) = \max\{p(y) : y \in [0, \hat{y}]\} := h$. Then, $p(y) \ge \dfrac{h}{\bar{y}} y$, for $y \in [0, \bar{y}]$ and $p(y) \ge \dfrac{h}{\hat{y} - \bar{y}}(\hat{y} - y)$, for $y \in [\bar{y}, \hat{y}]$. Therefore, we get

$$V \ge \int_0^{\hat{y}} (p(y))^{\frac{n-1}{2}} dy \ge h^{\frac{n-1}{2}} \hat{y}.$$

We estimate h and $\hat{y}$ from below. Notice that $-t = (\hat{y} + \epsilon)^\alpha - \hat{y}^\alpha = \alpha\xi^{\alpha-1}\epsilon$ for some $y < \xi < y + \epsilon$, and hence $-t \le \alpha(y + \epsilon)^{\alpha-1}\epsilon$, which implies that $\hat{y} \ge \left(\dfrac{-t}{\alpha\epsilon}\right)^{\frac{1}{\alpha-1}} - \epsilon$. Choosing $\delta = \alpha 2^{\alpha-1}$, we obtain $\hat{y} \ge \dfrac{1}{2}\left(\dfrac{-t}{\alpha\epsilon}\right)^{\frac{1}{\alpha-1}}$. It follows that $\dfrac{\hat{y}}{\hat{y} + \epsilon} \ge \frac{1}{2}$ and

hence that $1-\left(\frac{\hat{y}}{\hat{y}+\epsilon}\right)^{\alpha-1} \geq C_\alpha \frac{\epsilon}{\hat{y}+\epsilon}$ from the mean value theorem. A simple calculation shows that

$$h = \frac{1}{4}\left(1-\left(\frac{\hat{y}}{\hat{y}+\epsilon}\right)^{\alpha-1}\right)\left(\hat{y}+(-t)^{\frac{1}{\alpha}}+\epsilon\right)^2,$$

and hence we obtain $h \geq C_\alpha \epsilon(\hat{y}+\epsilon) \geq C_\alpha \epsilon \hat{y}$.

Therefore we get that $V \geq C_\alpha \epsilon^{\frac{n-1}{2}} \hat{y}^{\frac{n+1}{2}} \geq C_\alpha \frac{(-t)^{\frac{n+1}{2(\alpha-1)}}}{\epsilon^{\frac{n(2-\alpha)+\alpha}{2(\alpha-1)}}}$, which proves the lemma. □

Remark 4.4. In case $\alpha = 2$, an estimate similar to (4.3) can be proved for all $t \leq U(x_0)$. Recall that $U(x_0) = \operatorname{dist}^2(y_0, \partial\Omega) - |x_0 - y_0|^2$ for some $y_0 \in \Omega$. We first claim that $B_{\frac{U(x_0)-t}{2\operatorname{diam}(\Omega)}}(y_0) \subseteq F_t(x_0)$. Indeed, if $x \in \partial\Omega$ and $y \in \mathbb{R}^n$, then we have $|x-y|^2 - |x_0-y|^2 \geq U(x_0) + |x-y|^2 - |x-y_0|^2 - (|x_0-y|^2 - |x_0-y_0|^2) = U(x_0) + 2\langle x-x_0, y_0-y\rangle \geq U(x_0) - 2|x-x_0||y-y_0| \geq U(x_0) - 2\operatorname{diam}(\Omega)|y-y_0| \geq t$, provided that $|y-y_0| \leq \frac{U(x_0)-t}{2\operatorname{diam}(\Omega)}$, and the claim follows.

Next, let $\bar{x} \in A_{y_0}$; that is, $\bar{x} \in \partial\Omega$ and $\operatorname{dist}(y_0, \partial\Omega) = |\bar{x}-y_0|$. If $s \geq 0$ and $y_s = y_0 + s(\bar{x}-y_0)$, then $\operatorname{dist}(y_s, \partial\Omega) = |\bar{x}-y_s|$. Hence $\operatorname{dist}^2(y_s, \partial\Omega) - |x_0-y_s|^2 = |\bar{x}-y_s|^2 - |x_0-y_s|^2 = U(x_0) + |\bar{x}-y_s|^2 - |\bar{x}-y_0|^2 + |x_0-y_0|^2 - |x_0-y_s|^2 = U(x_0) + 2\langle \bar{x}-x_0, y_0-y_s\rangle \geq U(x_0) - 2|\bar{x}-x_0||y_s-y_0| \geq t$, provided $\frac{U(x_0)-t}{2|\bar{x}-x_0|} \geq |y_s-y_0|$. So, there exists $\bar{y} \in F_t(x_0)$, with $|\bar{y}-y_0| = \frac{U(x_0)-t}{2|\bar{x}-x_0|}$.

Notice that if $y_1, y_2 \in F_t(x_0)$, then the straight segment $[y_1, y_2] \subseteq F_t(x_0)$. Let Δ be the cone in $\mathbb{R}^n$ with base on $T_{y_0} \cap B_{\frac{U(x_0)-t}{2\operatorname{diam}(\Omega)}}(y_0)$ and vertex at $\bar{y}$, where T_{y_0} is the hyperplane passing through y_0 with normal $\bar{y}-y_0$. Then $\Delta \subseteq F_t(x_0)$ and Δ has measure $c_n \frac{(U(x_0)-t)^{n-1}}{(2\operatorname{diam}(\Omega))^{n-1}}|\bar{y}-y_0| = c_n \frac{(U(x_0)-t)^n}{(2\operatorname{diam}(\Omega))^{n-1}|\bar{x}-x_0|}$.

We now relate $|\bar{x}-x_0|$ to $\operatorname{dist}(x_0, \partial\Omega)$. Since $\bar{x} \in A_{y_0}$ is arbitrary, taking $\xi = x_0 - y_0$ in the definition of Λ_{y_0}, we obtain $\langle x_0-\bar{x}, x_0-y_0\rangle \leq 0$, which holds if and only if $|x_0-\bar{x}|^2 \leq |\bar{x}-y_0|^2 - |x_0-y_0|^2 = U(x_0)$. From the proof of Theorem 3.1, part (2), we have $U(x_0) \leq 2\operatorname{diam}(\Omega)\operatorname{dist}(x_0, \partial\Omega)$, and therefore we obtain the estimate $|F_t(x_0)| \geq |\Delta| \geq C_n \operatorname{diam}(\Omega)^{(1/2)-n} \operatorname{dist}(x_0, \partial\Omega)^{-1/2} (U(x_0)-t)^n$.

5. Main theorems

Theorem 5.1. *Let $\alpha > 1$ and let Ω be an open, bounded, convex domain in $\mathbb{R}^n$. Assume $u \in C(\bar{\Omega})$, $u = 0$ on $\partial\Omega$, and $0 \leq u(x_0) \leq U(x_0)$ for some $x_0 \in \Omega$. Then we have*

$$(U(x_0)-u(x_0))^n \leq C(\operatorname{dist}(x_0, \partial\Omega))^{\frac{n+1}{2}} \operatorname{diam}(\Omega)^{\frac{n(2\alpha-3)-1}{2}} |F_u(\Omega)| \tag{5.1}$$

whenever $n(2\alpha-3)-1 \geq 0$ and

$$(U(x_0)-u(x_0))^n \leq C(\operatorname{dist}(x_0, \partial\Omega))^{n(\alpha-1)} |F_u(\Omega)| \tag{5.2}$$

whenever $n(2\alpha-3)-1 \leq 0$. The constant C depends only on n and α.

Proof. Suppose $u(x_0) < U(x_0)$. Let $v(x) = \sup\{\lambda - |x-y|^\alpha := v_{\lambda,y}(x) : v_{\lambda,y}(x_0) \leq u(x_0)$ *and* $v_{\lambda,y} \leq 0$ *on* $\partial\Omega\}$. We claim $F_v(x_0) \subseteq F_u(\Omega)$. Let $y \in F_v(x_0)$, so

$v(x) \geq v(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha$ for all $x \in \Omega$. Consider

$$\sup_{\bar{\Omega}} \left(v(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha - u(x) \right),$$

and let $\hat{x} \in \bar{\Omega}$ be the point where the supremum is attained. From Theorem 4.1, $u(x_0) = v(x_0)$. We have $v(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha \leq 0$ for all $x \in \partial\Omega$. If $\hat{x} \in \partial\Omega$, then $u(x) \geq u(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha$ for all $x \in \bar{\Omega}$ and so $y \in F_u(x_0)$. If on the other hand $\hat{x} \in \Omega$, then $u(x) \geq u(\hat{x}) + |\hat{x} - y|^\alpha - |x - y|^\alpha$ for all $x \in \Omega$, and hence $y \in F_u(\hat{x})$. Consequently the claim is proved. From Theorem 4.1 applied to v we then obtain the result. □

Theorem 5.2. *Let $1 < \alpha \leq \dfrac{2n}{n-1}$. Let Ω be an open, bounded, convex domain in $\mathbb{R}^n$. Assume $u \in C(\bar{\Omega})$ with $u = 0$ on $\partial\Omega$. Let $x \in \Omega$ such that $u(x) < 0$. If*

$$|u(x)| \leq \alpha 2^{\alpha-1} \mathrm{dist}^\alpha(x, \partial\Omega), \tag{5.3}$$

then

$$|u(x)|^{\frac{n+\alpha}{\alpha}} \leq C_{\alpha,n} \mathrm{dist}^\alpha(x, \partial\Omega) |F_u(\Omega)|. \tag{5.4}$$

If on the other hand, (5.3) *does not hold, then*

$$|u(x)|^{\frac{n+1}{2(\alpha-1)}} \leq C_{\alpha,n} \, \mathrm{dist}(x, \partial\Omega)^{\frac{n(2-\alpha)+\alpha}{2(\alpha-1)}} |F_u(\Omega)|. \tag{5.5}$$

Proof. If (5.3) holds and since we always have $B_{|u(x)|^{\frac{1}{\alpha}}}(x) \subseteq F_u(\Omega)$, then (5.4) follows. If on the other hand, $|u(x)| > \alpha 2^{\alpha-1} \mathrm{dist}^\alpha(x, \partial\Omega)$, then we are under the hypothesis of Lemma 4.3, and the proof of (5.5) follows in the same way as in the previous theorem. □

Remark 5.3. If $\alpha > \dfrac{2n}{n-1}$ and $u(x) < 0$, then $u(x)$ cannot be estimated by any positive power of $\mathrm{dist}(x, \partial\Omega)$ and consequently neither can $U(x) - u(x)$.

Considering the cylinder $\Omega = \{(x', x_n) : |x'| < 2, |x_n| < 1\}$, we set $x = (x', x_n)$. Let $x_k = (0, 1 - \frac{1}{k})$. Notice that if $|x'| \geq 2$, then $\mathrm{dist}^\alpha(x, \partial\Omega) - |x - x_k|^\alpha \leq -2^\alpha$; this follows since for $\alpha > 2$ the function $h(y) = y^\alpha - (a^2 + (y+b)^2)^{\frac{\alpha}{2}}$ is decreasing for $y > 0$, for any fixed positive a and b. Hence, if $x \notin \Omega$ and $\mathrm{dist}^\alpha(x, \partial\Omega) - |x - x_k|^\alpha \geq -1$, then $|x'| < 2$, which implies that

$$\begin{aligned} &F_{-1}(x_k) \cap \Omega^c \\ &\quad \subseteq \left\{ x : |x'| < 2, x_n > 1, (x_n - 1)^\alpha - \left(|x'|^2 + \left(x_n - 1 + \frac{1}{k} \right)^2 \right)^{\alpha/2} > -1 \right\}, \end{aligned}$$

where $F_{-1}(x_k)$ is defined in (4.5), and the set on the right hand side is basically the set H defined in the proof of Lemma 4.3. We can now estimate from above the measure of this set in the same way as in Lemma 4.3. If $y > 0$, $t < 0$, then we have $(y^\alpha - t)^{\frac{2}{\alpha}} - y^2 = \frac{2}{\alpha} \xi^{\frac{2-\alpha}{\alpha}} (-t)$ for some ξ with $y^\alpha < \xi < y^\alpha - t$. We let $t = -1$ and

$\epsilon = 1/k$, and with the notation in the proof of Lemma 4.3, we then get that

$$\begin{aligned}|F_{-1}(x_k) \cap \Omega^c| &\leq \int_0^{\hat{y}_k} ((y^\alpha + 1)^{\frac{2}{\alpha}} - (y + 1/k)^2)^{\frac{n-1}{2}} dy \\ &\leq \int_0^\infty ((y^\alpha + 1)^{\frac{2}{\alpha}} - y^2)^{\frac{n-1}{2}} dy = \int_0^1 + \int_0^\infty \\ &\leq C_1 + \int_1^\infty y^{\frac{(2-\alpha)(n-1)}{2}} dy \leq C_{n,\alpha}\end{aligned}$$

for $\alpha > 2n/(n-1)$. Let u_k be α-convex such that $u_k = 0$ on $\partial\Omega$, $u_k(x_k) = -1$, whose existence follows from the first part of Theorem 4.1. Then $|F_{u_k}(\Omega)| \leq C$ for all k while $|u_k(x_k)| = 1$ and $\operatorname{dist}(x_k, \partial\Omega) = \frac{1}{k}$.

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Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122
E-mail address: gutierre@temple.edu

Instituto Argentino de Matemática, CONICET, Buenos Aires, Argentina
E-mail address: fedeleti@aim.com