AN ALEKSANDROV TYPE ESTIMATE
FOR $\alpha$-CONVEX FUNCTIONS

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Abstract. In the context of $\alpha$-convexity, using an operator similar to the Monge-Ampère operator based on the notion of normal mapping, we estimate the difference between a function $u$ and the solution of the homogeneous problem $U$ in terms of the measure of the normal mapping of $u$ and a power of the distance to the boundary.

1. Introduction

In the theory of the Monge-Ampère equation the following estimate due to Aleksandrov is of great importance: if $u$ is convex in $\Omega$, an open bounded convex subset of $\mathbb{R}^n$, and $u \in C(\bar{\Omega})$ with $u = 0$ on $\partial \Omega$, then
\begin{equation}
|u(x)|^n \leq C \text{dist}(x, \partial \Omega) \text{diam}(\Omega)^{n-1} |Du(\Omega)|,
\end{equation}
for all $x \in \Omega$, where
\begin{equation}
Du(\Omega) = \{ p \in \mathbb{R}^n : \exists y \in \Omega \text{ such that } u(x) \geq u(y) + p \cdot (x - y) \ \forall x \in \Omega \},
\end{equation}
with a constant $C$ depending only on $n$ and independent of $u$. The estimate plays a crucial role in the theory of sections of solutions to the Monge-Ampère equation and consequently in regularity theory; see [Caf90], [Gut01], [GH00]. More generally, if $u$ is not necessarily convex but satisfies $u(x_0) \leq 0$ at some $x_0 \in \Omega$, then (1.1) holds at $x = x_0$. Indeed, taking $v$ to be the convex function defining a cone with base in $\partial \Omega$ and vertex at the point $(x_0, u(x_0))$, and following the argument in [Gut01, Lemma 1.4.1], we obtain $Dv(x_0) \subset Du(\Omega)$. Then the proof of [Gut01, Theorem 1.4.2] applies in this case.

The purpose of this paper is to prove this estimate in the context of $\alpha$-convex functions, $\alpha > 1$; see Definition 2.3. In the language of optimal mass transportation these are functions that are convex with respect to the cost function $c(x, y) = |x - y|^\alpha$. In our estimate, the subdifferential $Du(\Omega)$ on the right hand side of (1.1) is replaced by the quantity
\begin{equation}
F_u(\Omega) = \{ y \in \mathbb{R}^n : \exists x \in \Omega \text{ such that } u(x) \geq u(x) + |x - y|^\alpha - |x - y|^\alpha \forall x \in \Omega \},
\end{equation}
and $|u(x)|$ on left hand side of (1.1) gets replaced by $U(x) - u(x)$, where $U$ is the solution of the homogeneous problem $|FU(\Omega)| = 0$ with $U = 0$ on $\partial \Omega$. The case $\alpha = 2$ is related to standard convexity since $F_2(x) = x + 2Du(x)$; see the
end of the proof of Lemma 3.4. The structure of the set $F_u$ is related to the condition (A3w) introduced in [TW09] for general cost functions, and it is proven there, in Example 4, that $|x - y|^{\alpha}$ satisfies this condition only when $\alpha = 2$ or when $-2 \leq \alpha < 1$. Consequently, from the results of Loeper [Loe09] Proposition 2.11 and Theorem 3.1], the set $F_u(\bar{x})$ defined in Definition 2.1 is in general not connected. We refer to the paper [GN07] for results on Monge-Ampère type equations arising in optimal mass transportation for general cost functions and properties of the subdifferential $F_u$. Optimal mass transportation has recently become a very active area of research; we mention, in particular, the fundamental work of Ma, Trudinger and Wang [MTW05], for smooth cost functions. For further references see [Vil07].

The main estimate proved in this paper is the following:

Let $\alpha > 1$ and let $\Omega$ be an open, bounded, convex domain in $\mathbb{R}^n$. If $u \in C(\Omega)$ with $u = 0$ on $\partial \Omega$ and such that $0 \leq u \leq U$ in $\Omega$, then for all $x \in \Omega$ we have

\begin{align}
(1.4) \quad & (U(x) - u(x))^n \leq C(\text{dist}(x, \partial \Omega))^{n+1} \text{diam}(\Omega) \frac{\alpha}{n(\alpha - 1)} |F_u(\Omega)| \\
\text{whenever } & n(2\alpha - 3) - 1 \geq 0, \\
(1.5) \quad & (U(x) - u(x))^n \leq C(\text{dist}(x, \partial \Omega))^{n(\alpha - 1)} |F_u(\Omega)| \\
\text{whenever } & n(2\alpha - 3) - 1 \leq 0. \quad \text{The constant } C \text{ depends only on } n \text{ and } \alpha, \text{ and } U \text{ is the solution of the homogeneous problem as stated above. Depending on the value of } \alpha, \text{ the hypothesis } u \geq 0 \text{ is essential for the validity of the estimates. Indeed, if } \alpha > 2n/(n - 1) \text{ and } u < 0, \text{ then it is not possible to give an estimate of } |u| \text{ by any positive power of the distance; see Remark 5.3. However, if } \alpha \leq 2n/(n - 1) \text{ and } u < 0, \text{ then such an estimate holds; see Theorem 5.2.}
\end{align}

The paper is organized as follows. Section 2 contains preliminary results. In Section 3, we solve the homogeneous Dirichlet problem giving an explicit characterization of the solution. In Section 4, we find the solution $u$ to the Dirichlet problem when the right hand side is a multiple $\beta$ of the Dirac delta function at a point $\bar{x} \in \Omega$, and we estimate the size of $\beta$ in terms of $U(\bar{x}) - u(\bar{x})$ and $\text{dist}(\bar{x}, \partial \Omega)$. The whole Section 4 is devoted to proving Theorem 4.1 and Lemma 4.3 and the main estimates are finally proved in Section 4.

2. Definitions and preliminary results

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, convex set, and $\alpha > 1$.

**Definition 2.1.** Let $u : \Omega \to \mathbb{R}$ and $\bar{x} \in \Omega$. We define

\begin{equation}
(2.1) \quad F_u(\bar{x}) := \{ y \in \mathbb{R}^n : u(x) \geq u(\bar{x}) + |x - y|^\alpha - |x - y|^\alpha \ \forall x \in \Omega \}.
\end{equation}

If $E \subseteq \Omega$, we define $F_u(E) = \bigcup_{x \in E} F_u(x)$.

**Remark 2.2.** If $u \in C(\overline{\Omega})$ and $\bar{x} \in \partial \Omega$, then we say that $y \in F_u(\bar{x})$ if there exists $\bar{x} \in \Omega$ such that $y \in F_u(\bar{x})$ and $u(x) \geq u(\bar{x}) + |x - y|^\alpha - |x - y|^\alpha$ for all $x \in \Omega$.

**Definition 2.3.** We say that $u : \Omega \to R$ is $\alpha$-convex in $\Omega$ if $F_u(\bar{x}) \neq \emptyset, \forall x \in \Omega$.

**Lemma 2.4.** If $u : \Omega \to R$ is $\alpha$-convex in $\Omega$, then $u$ is locally Lipschitz continuous in $\Omega$.

**Proof.** First we check the boundedness of $u$. That $u$ is bounded below is trivial. We show that $u$ is locally bounded above in $\Omega$. Let $K \subset \Omega$ be compact and suppose there exist $x_0 \in K$ and $\{x_n\} \subset K$ with $x_n \to x_0$ and $u(x_n) \to +\infty$. If
We have the following lemma, which will be used in the proof of Theorem 3.1.

Letting $\xi_n = \frac{y_n - x_n}{|y_n - x_n|}$, we may assume that $\xi_n \to e_1$. Consider the cone

$C = \{x : e_1 \cdot \left(\frac{x - x_n}{|x - x_n|}\right) \geq \delta\}$. We claim that $C \subset I$. If we let $x \in C$, there exists $N$ such that $\xi_n \cdot \left(\frac{x - x_n}{|x - x_n|}\right) \geq \delta/2$ for all $n \geq N$. Since $|x_n - y_n| \to \infty$, there exists $N'$ such that $\frac{2}{\delta} |x_n - y_n| \leq \frac{2}{\delta} \text{diam}(\Omega) \leq |x_n - y_n|$ for all $n \geq N'$. Thus,

$$\frac{\delta}{2} \leq \frac{|x_n - y_n|}{|x_n - y_n|} \cdot \frac{x - x_n}{|x - x_n|} \leq \frac{\delta}{2} \frac{|x_n - y_n|^2}{|x - x_n|}$$

for all $n$ sufficiently large and the claim is proved. Since $x_0 \in \Omega$ there exists $x' \in \Omega \cap C$, and therefore $x' \in \Omega \cap I$, a contradiction. Therefore, $u$ is bounded on compact subsets of $\Omega$.

We next show that $u$ is locally Lipschitz. Let $B$ be a ball with $2B \subset \Omega$. Then we show first that the set $F_u(B)$ is bounded. Otherwise, there exist $y_n \in F_u(x_n)$, with $x_n \in B$ and $|y_n| \to \infty$. Since $u$ is bounded above in $2B$ and bounded below in $\Omega$, we get that $|x_n - y_n|^\alpha - |x_n - y_n|^\alpha \leq M$ for all $x \in 2B \subset \Omega$. Let $x = x_n + \beta \xi_n$, with $\xi_n = \frac{y_n - x_n}{|y_n - x_n|}$. We have $x \in 2B$ for $\beta$ small, and therefore $|x_n - y_n|^\alpha - |x_n - y_n|^\alpha = |x_n - y_n|^\alpha - |x_n + \beta \xi_n - y_n|^\alpha = \alpha \beta |x_n + \beta \xi_n - y_n|^\alpha - 1$ for some $0 < \beta < \beta$ from the mean value theorem. But the last expression tends to $+\infty$ as $n \to \infty$, which yields a contradiction. Finally, let $B \subset \Omega$ and let $x_0, x_1 \in B$ and $y_0 \in F_u(x_0), y_1 \in F_u(x_1)$. Then $u(x) \geq u(x_0) + |x_0 - y_0|^\alpha - |x - y_0|^\alpha$ and $u(x) \geq u(x_1) + |x_1 - y_1|^\alpha - |x - y_1|^\alpha$ for all $x \in \Omega$. Thus,

$$|x_0 - y_0|^\alpha - |x_0 - y_0|^\alpha \leq u(x_1) - u(x_0) \leq |x_0 - y_1|^\alpha - |x_1 - y_1|^\alpha.$$

Again from the mean value theorem it follows that $u(x) - y_0)^\alpha - u(x_1) - u(x_0) \leq |x_0 - y_1|^\alpha - |x_1 - y_1|^\alpha$.

Remark 2.5. If $u$ is $\alpha$-convex in $\Omega$, then

$$|\{y \in \mathbb{R}^n : y \in F_u(x_1) \cap F_u(x_2), x_1 \neq x_2 \in \Omega\}| = 0.$$  

Indeed, the function $u^*(z) = \inf_{x \in \Omega} u(x) + |x - z|^\alpha$ is locally Lipschitz in $\mathbb{R}^n$. Suppose $y_1 \in F_u(x_1)$, where $x_1 \in \Omega$. Then $u^*(z) \leq u(x_1) + |x_1 - y_1|^\alpha = u(x_1) + |x_1 - y_1|^\alpha - |x_1 - y_1|^\alpha$, $\forall z \in \mathbb{R}^n$.

Therefore, we see that if $y \in F_u(x_1) \cap F_u(x_2)$ for some $x_1 \neq x_2 \in \Omega$, then $u^*$ cannot be differentiable at $y$. This proves the remark.

Remark 2.6. The conclusion in the previous remark also holds if $u \in C(\Omega)$ is $\alpha$-convex and the $y$'s are taken so that $y \in F_u(x_1) \cap F_u(x_2)$, where $x_1 \in \Omega$ and $x_2 \in \partial \Omega$.

For each $y_0 \in \Omega$ consider the set

$$A_{y_0} = \{x \in \partial \Omega : \text{dist}(y_0, \partial \Omega) = |y_0 - x|\}.$$  

We have the following lemma, which will be used in the proof of Theorem 3.1.
Lemma 2.7. For each \( y_0 \in \Omega \) and for each \( \xi \in \mathbb{R}^n \) with \( |\xi| = 1 \), there exists \( \bar{x} \in A_{y_0} \), a sequence \( y_k = y_0 + \delta_k \xi \) with \( \delta_k > 0 \) and \( \delta_k \to 0 \), and \( x_k \in A_{y_k} \) such that \( x_k \to \bar{x} \).

Proof. For each \( k \), let \( x_k \in A_{y_0} + \frac{k}{k} \xi \) and since \( \{x_k\} \) is a bounded sequence, passing through a subsequence, we can assume \( x_k \to \bar{x} \in \partial\Omega \). Then \( \text{dist}(y_0, \partial\Omega) = \lim_{k \to \infty} \text{dist}(y_0 + \frac{k}{k} \xi, \partial\Omega) = \lim_{k \to \infty} |y_0 + \frac{k}{k} \xi - x_k| = |y_0 - \bar{x}| \), i.e. \( \bar{x} \in A_{y_0} \).

3. Homogeneous Dirichlet problem

In this section, we solve the Dirichlet problem with zero boundary data and give a characterization of the solution. This problem was considered in [GN07] for general cost functions, but in our case we need to have a more precise characterization of the solution; see [GN07], Theorem 6.7.

Theorem 3.1. Let \( \Omega \) be a bounded, open, convex domain in \( \mathbb{R}^n \). Let \( \alpha > 1 \). Given \( y \in \mathbb{R}^n \), let \( v_{\lambda,y}(x) = \lambda - |x - y|^{\alpha} \) (\( v_{\lambda,y} \) is \( \alpha \)-convex in \( \mathbb{R}^n \)) and let

\[
U(x) = \sup\{v_{\lambda,y}(x) : v_{\lambda,y} \leq 0 \text{ on } \partial\Omega\}.
\]

We then have:

1. \( U \) is \( \alpha \)-convex in \( \Omega \);
2. \( U \in C(\overline{\Omega}) \) and \( U = 0 \) on \( \partial\Omega \);
3. \( |F_U(\Omega)| = 0 \);
4. \( U(x) = \sup\{\text{dist}^\alpha(y, \partial\Omega) - |x - y|^\alpha : y \in \Omega\} \);
5. if \( U(x_0) = \text{dist}^\alpha(y_0, \partial\Omega) - |x_0 - y_0|^\alpha \), then \( x_0 \in A_{y_0} \) with

\[
\Lambda_{y_0} = \{x \in \Omega : \text{for each } \xi \text{ with } |\xi| = 1 \text{ there exists } \bar{x} \in A_{y_0} \text{ such that } (|x - y_0|^\alpha - 2(y_0 - \bar{x}) - |x - y_0|^\alpha - 2(y_0 - x) \cdot \xi) \leq 0\}.
\]

where the set \( A_{y_0} \) is defined in (2.3).

Proof. 1. To define \( U \) it is enough to consider the set of functions \( v_{\lambda,y} \) with \( y \in \Omega \). Because if \( y \notin \Omega \), and \( v_{\lambda,y}(x) = \lambda - |x - y|^{\alpha} \) satisfies \( v_{\lambda,y} \leq 0 \) on \( \partial\Omega \), then \( v_{\lambda,y} \leq 0 \) in \( \Omega \). While if \( y \in \Omega \), then \( v(x) = \text{dist}^\alpha(y, \partial\Omega) - |x - y|^\alpha \) satisfies \( v \leq 0 \) on \( \partial\Omega \), and hence \( U(x) \geq v(x) \forall x \in \Omega \); in particular,

\[
(3.1) \quad U(y) \geq \text{dist}^\alpha(y, \partial\Omega) \forall y \in \Omega.
\]

Let \( x_0 \in \Omega \). From the definition of \( U \), \( U(x_0) = \lim_{n \to \infty} \lambda_n - |x_0 - y_n|^\alpha = \lim_{n \to \infty} v_n(x_0) \) with \( v_n(x) = \lambda_n - |x - y_n|^\alpha \) and \( v_n(x) \leq 0 \) on \( \partial\Omega \). Since \( y_n \in \Omega \), we may assume \( y_n \to y_0 \in \overline{\Omega} \), and so also \( \lambda_n \to \lambda_0 \). Thus, \( U(x_0) = \lambda_0 - |x_0 - y_0|^\alpha \) and \( v_n \to v_0 \) uniformly in \( \overline{\Omega} \) where \( v_0(x) = \lambda_0 - |x - y_0|^\alpha \). It follows that \( \lambda_0 \leq 0 \) on \( \partial\Omega \) and since \( v_0(x_0) = U(x_0) \geq \text{dist}^\alpha(x_0, \partial\Omega) > 0 \) we have \( y_0 \notin \Omega \). Since \( U \geq v_n \), it follows that \( U \geq v_0 \) in \( \Omega \) and therefore \( y_0 \in F_U(x_0) \), so \( U \) is \( \alpha \)-convex.

2. Since \( U \) is \( \alpha \)-convex in \( \Omega \), from Lemma 2.4 \( U \) is continuous in \( \Omega \). We show that \( U \) is continuous up to the boundary. Fix \( \bar{x} \in \partial\Omega \) and let \( \eta(\bar{x}) \) be the unit inner normal to some supporting plane to \( \partial\Omega \) at \( \bar{x} \), and let \( V(x) = 2\text{diam}(\Omega)^{\alpha-1}(x - \bar{x}, \eta(\bar{x})) \). Let \( v_{\lambda,y}(x) = \lambda - |x - y|^\alpha \) with \( v_{\lambda,y} \leq 0 \) on \( \partial\Omega \). Since \( |D(V - v_{\lambda,y})(x)| > 0 \) for \( x \in \Omega \) and \( V - v_{\lambda,y} \geq 0 \) on \( \partial\Omega \), it follows that \( V - v_{\lambda,y} \geq 0 \) in \( \Omega \). Therefore \( V(x) \geq U(x) \) for \( x \in \Omega \). This together with (3.1) yields that \( U \in C(\overline{\Omega}) \) and \( U = 0 \) on \( \partial\Omega \).

3. Let \( y \in F_U(\bar{x}) \) for some \( \bar{x} \in \Omega \), so \( U(x) \geq U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha \) for all \( x \in \Omega \) and we claim that there exists \( \hat{x} \in \partial\Omega \) such that \( U(\hat{x}) + |\hat{x} - y|^\alpha - |\hat{x} - \bar{x}|^\alpha = 0 \).
Otherwise, since $U = 0$ on $\partial \Omega$, we must have that $U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha < 0$ for all $x \in \partial \Omega$, and hence, there exists $\epsilon > 0$ such that $U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha + \epsilon < 0$ for all $x \in \partial \Omega$. Then by the definition of $U$, we must have $U(x) \geq U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha + \epsilon$ for all $x \in \Omega$, and in particular, $U(\bar{x}) \geq U(\bar{x}) + \epsilon$, a contradiction, and the claim is proved. Therefore, there exists $\tilde{x} \in \partial \Omega$ such that $U(\tilde{x}) + |\tilde{x} - y|^\alpha - |\bar{x} - y|^\alpha = 0$ and since $U(\bar{x}) = 0$ this clearly implies that $y \in F_U(\tilde{x})$. We have then proved that $F_U(\Omega) \subseteq \{ y : \exists \bar{x}, \Omega, x \in \partial \Omega \text{ such that } y \in F_U(\bar{x}) \cap F_U(x) \}$, which implies that $|F_U(\Omega)| = 0$.

We remark that if $\alpha = 2$, we have that $U(x) = |\tilde{x} - y|^2 - |x - y|^2$ for all $x \in [\bar{x}, \tilde{x}]$, and hence $y \in F_U(x)$ for all $x \in [\bar{x}, \tilde{x}]$, where $\tilde{x} \in \partial \Omega$ is as above.

(4) Let $\bar{x} \in \Omega$ be arbitrary and let $\tilde{y} \in F_U(\bar{x})$. From the claim in (3), we conclude that $U(x) \geq |\bar{x} - y|^\alpha - |x - \tilde{y}|^\alpha$ for all $x \in \Omega$ with equality at $\bar{x}$ and at $\tilde{x} \in \partial \Omega$. Hence, $|\bar{x} - \tilde{y}|^\alpha > |\bar{x} - y|^\alpha$, for all $x \in \partial \Omega$. This implies that $|\bar{x} - \tilde{y}|^\alpha = \text{dist}^\alpha(\bar{y}, \partial \Omega)$ and so $U(x) \geq \text{dist}^\alpha(\bar{y}, \partial \Omega) - |x - \tilde{y}|^\alpha$ for all $x \in \Omega$ with equality at $\tilde{x}$. Therefore, we get $U(\bar{x}) = \text{dist}^\alpha(\bar{y}, \partial \Omega) - |\bar{x} - \tilde{y}|^\alpha \leq \text{sup}\{ \text{dist}^\alpha(y, \partial \Omega) - |x - y|^\alpha : y \in \Omega \}$.

The reverse inequality follows by noting that for any $y$ fixed in $\Omega$, the function $v(x) = \text{dist}^\alpha(y, \partial \Omega) - |x - y|^\alpha$ satisfies $v \leq 0$ on $\partial \Omega$, $v$ $\alpha$-convex, and so $U \geq v$.

(5) Suppose $U(x_0) = \text{dist}^\alpha(y_0, \partial \Omega) - |x_0 - y_0|^\alpha$, and we will show $x_0 \in A_{y_0}$. Otherwise, there exists $\xi$ with $|\xi| = 1$ such that for all $x \in A_{y_0}$,

\[
(\text{3.2}) \quad (|x - y_0|^\alpha - (y_0 - x) - |x_0 - y_0|^\alpha (y_0 - x_0)) \cdot \xi > 0.
\]

From Lemma 2.1 applied to $\xi$, we know that there exists $\bar{x} \in A_{y_0}$ and $\delta_k > 0$, and $x_k \in A_{y_0} + \delta_k \xi$ such that $x_k \to \bar{x}$. Using $x = \bar{x}$ in equation (3.2), by definition of $U$ and the fact that $x_k \in A_{y_0} + \delta_k \xi$, we have

\[
(\text{3.3}) \quad 0 > |x_k - (y_0 + \delta_k \xi)|^\alpha - |x_0 - (y_0 + \delta_k \xi)|^\alpha - U(x_0)
\]

\[
= |\delta_k \xi - (x_k - y_0)|^\alpha - |\bar{x} - y_0|^\alpha - \text{dist}^\alpha(\bar{x} - y_0)^\alpha - |x_0 - y_0|^\alpha
\]

\[
= |x_k - y_0|^\alpha - |\bar{x} - y_0|^\alpha + \alpha \delta_k \left\{ \frac{|\delta_k \xi - (x_k - y_0)|^\alpha}{\text{dist}^\alpha(\bar{x} - y_0)} - |\bar{x} - y_0|^\alpha - (\text{dist}^\alpha(\bar{x} - y_0)^\alpha - |x_0 - y_0|^\alpha) \right\}
\]

by the mean value theorem for some $0 < \delta_k, \delta_k < \delta_k$. Notice that $\lim_{\delta_k \to 0} \{ \ldots \} = (\text{dist}^\alpha(y_0, \partial \Omega) - |x_0 - y_0|^\alpha (y_0 - x_0), \xi) > 0$ and since $x_k \in \partial \Omega$ and $\bar{x} \in A_{y_0}$ we also have $|x_k - y_0|^\alpha - |\bar{x} - y_0|^\alpha \geq 0$ and hence, we conclude that for $\delta_k$ small enough, (3.3) is positive, which is a contradiction, thus proving the claim. This completes the proof.

Remark 3.2. We analyze in passing the case $\alpha = 2$. In this case, $A_{y_0} = \{ x : \forall \xi, |\xi| = 1, \exists \bar{x} \in A_{y_0}, \text{ such that } (x - \bar{x}, \xi) \leq 0 \}$ is convex hull$(A_{y_0})$ and we have the following conclusions:

If $x_0 \in \Omega$ and $U(x_0) = \text{dist}^2(y_0, \partial \Omega) - |x_0 - y_0|^2$, then $A_{y_0} = \{ x \in \partial \Omega : \text{dist}^2(y_0, \partial \Omega) - |x_0 - y_0|^2 \}$ is not a singleton. Moreover, if $U(x_0) = \text{dist}^2(y_0, \partial \Omega) - |x_0 - y_0|^2 = |x - y_0|^2 - |x_0 - y_0|^2$, then $U(x) = |x - y_0|^2 - |x - y_0|^2$ for all $x \in \text{convex hull}(A_{y_0})$. Also, $x_0 \in B_{|y_0 - \bar{x}|/2}(\bar{y_0})$ and $|x_0 - \bar{x}|^2 \leq U(x_0)$. This can be realized by taking $\xi = x_0 - y_0$ in the definition of $A_{y_0}$ and $x_0$ to be the corresponding point in $A_{y_0}$.

Now, let us look at the case $\alpha > 1$ and consider the set $A_{y_0}$. Set $p_{y_0} = (x - y_0)|x - y_0|^{\alpha - 2}$. One can check that $A_{y_0} = p_{y_0}^{-1}(\text{convex hull}(p_{y_0}(A_{y_0})))$. 

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Let \( \xi = x_0 - y_0 \). Then there exists \( \bar{x} \in A_{y_0} \) such that \( (|\bar{x} - y_0|^{\alpha - 2}(y_0 - \bar{x}) - |x_0 - y_0|^{\alpha - 2}(y_0 - x_0), x_0 - y_0) \leq 0 \), which gives \( |x_0 - y_0|^\alpha \leq |\bar{x} - y_0|^{\alpha - 2}(\bar{x} - y_0, x_0 - y_0) \).

Taking for instance \( y_0 = 0 \) and \( \bar{x} \) along \( e_1 \), we see that if \( x_0 \in A_{y_0} \), then \( x_0 \) is on the set obtained by rotating the polar curve \( r = R(\cos \theta)^{\frac{1}{\alpha - 2}} \) around the \( e_1 \)-axis, where \( R = |\bar{x}| \).

### 3.1. Regularity of \( U \)

We prove the following theorem.

**Theorem 3.3.** The function \( U \) in Theorem 3.1 is \( C^1(\Omega) \).

We first prove a lemma.

**Lemma 3.4.** Let \( u \) be \( \alpha \)-convex in \( \Omega \). Then \( u \in C^1(\Omega) \) if and only if \( F_u(x) \) is a singleton for each \( x \in \Omega \).

**Proof.** It is clear that if for some \( x \in \Omega \), \( F_u(x) \) has more than one point, then \( u \) is not differentiable at \( x \).

To prove the other implication, fix \( \bar{x} \in \Omega \), and let \( \{y\} = F_u(\bar{x}) \). We claim first that if \( x_n \to \bar{x} \), and \( \{y_n\} = F_u(x_n) \), then \( y_n \to y \). If not, there exists \( \epsilon > 0 \) and infinitely many points \( y_{n_k} \notin B_\epsilon (\bar{y}) \). Since the sequence \( \{y_{n_k}\} \) is bounded, extracting a subsequence we may assume \( y_{n_k} \to \hat{y} \). But then it follows that \( \hat{y} \in F_u(\bar{x}) \) and hence by assumption that \( \hat{y} = y \), and this is a contradiction, proving the claim.

Using this claim we show that \( u \) has first-order partial derivatives. Without loss of generality, we can assume \( u(0) = 0 \) and \( \{0\} = F_u(0) \), so \( u(x) \geq -|x|^\alpha \) for all \( x \in \Omega \). Suppose \( \frac{\partial u}{\partial x_t}(0) \) does not exist. For \( t > 0 \) we have \( \frac{u(te_1)}{t} \geq -t^{\alpha - 1} \) and hence \( \liminf_{t \to 0^+} \frac{u(te_1)}{t} \geq 0 \). Suppose \( \limsup_{t \to 0^+} \frac{u(te_1)}{t} = a > 0 \). Let \( t_n \to 0^+ \) such that \( \frac{u(te_1)}{t_n} \geq \frac{a}{2} \) and let \( \{y_n\} = F_u(t_ne_1) \). It follows by the claim that \( y_n \to 0 \), and we have \( u(x) \geq u(te_1) + |t_ne_1 - y_n|^\alpha - |x - y_n|^\alpha \) for all \( n \) and for all \( x \in \Omega \). In particular, \( 0 = u(0) \geq u(te_1) + |t_ne_1 - y_n|^\alpha - |x - y_n|^\alpha = u(te_1) + \alpha \xi - y_n|^\alpha - 2(\xi - y_n, t_ne_1) \) for some \( \xi \in [0, t_ne_1] \). Dividing by \( t_n \) we get \( 0 \geq \frac{u(te_1)}{t} + \alpha \xi - y_n|^\alpha - 2(\xi - y_n, e_1) \geq -a \alpha \xi - y_n|^\alpha - 2(\xi - y_n, e_1) \geq a \) for \( n \) large enough, a contradiction. Exactly the same argument works for \( t < 0 \), and hence we conclude that \( \frac{\partial u}{\partial x_t}(0) \) exists. By the claim once again we can also conclude that the partial derivatives are continuous because if \( y \in F_u(x) \), then \( y = x + a^{1/(\alpha - 1)}|Du(x)|^{(2-\alpha)/(\alpha - 1)}Du(x) \).

**Proof of Theorem 3.3.** Let us recall that \( \Omega \) is convex and \( U(x) = \sup \{\text{dist}^\alpha(y, \partial \Omega) - |x - y|^\alpha : y \in \Omega \} \). Fix \( x_0 \in \Omega \). We show that \( F_U(x_0) \) is a singleton.

Set \( t = U(x_0) > 0 \) and suppose by contradiction that \( y_1, y_2 \in F_U(x_0) \) with \( y_1 \neq y_2 \). It follows that \( \text{dist}^\alpha(y_i, \partial \Omega) - |x_0 - y_i|^\alpha = t \) for \( i = 1, 2 \). We also have that \( B_i := B_{(|x_0 - y_i|^\alpha + t)^{1/\alpha}}(y_i) \subseteq \Omega \) for \( i = 1, 2 \) and \( \partial B_1 \cap \partial B_2 \neq \emptyset \).

Let \( \Lambda \) be the convex hull of \( B_1 \cup B_2 \) and let \( T \) be a supporting hyperplane to \( \Lambda \) that touches \( \Lambda \) at more than one point. Set \( \Phi(y) = \text{dist}^\alpha(y, \Omega) - |x_0 - y|^\alpha \). We will prove in Lemma 4.2 that the set \( S = \{y : \Phi(y) \geq t\} \) is strictly convex. Since \( \Phi(y_i) = t \) for \( i = 1, 2 \) it follows that \( |y_1, y_2| \subseteq S \). Then, for \( y \in (y_1, y_2) \) we have \( \text{dist}^\alpha(y, \partial \Omega) - |x_0 - y|^\alpha \geq \text{dist}^\alpha(y, \partial \Lambda) - |x_0 - y|^\alpha = \text{dist}^\alpha(y, \Omega) - |x_0 - y|^\alpha \geq \Phi(y) > t \), and this is a contradiction with the definition of \( U \) since \( U(x_0) = t \).
4. Nonhomogeneous Dirichlet problem

Let $\alpha > 1$ and let $U$ be the solution of $|F_U(\Omega)| = 0$, $U = 0$ on $\partial \Omega$ from the previous section. We shall prove the following theorem; see [GN07, Lemma 6.19].

**Theorem 4.1.** Let $x_0 \in \Omega$ and $t < U(x_0)$ and define

\[ u(x) = \sup \{ v_{\lambda,y}(x) = \lambda - |x - y|^\alpha : v_{\lambda,y} \leq 0 \text{ on } \partial \Omega \text{ and } v_{\lambda,y}(x_0) \leq t \}. \]

Then $u \in C(\Omega)$, $u = 0$ on $\partial \Omega$, $u(x_0) = t$, $u$ is $\alpha$-convex and satisfies the equation $F_u = \beta \delta_{x_0}$, for some $\beta \geq 0$. Moreover, when $t \geq 0$ we have the following estimates for $\beta$:

(a) Suppose $1 < \alpha \leq 2$. If $n(3 - 2\alpha) + 1 \geq 0$, then

\[ \beta = |F_u(x_0)| \geq C \frac{(U(x_0) - t)^n}{\text{dist}(x_0, \partial \Omega)^n(\alpha - 1)}; \]

and if $n(3 - 2\alpha) + 1 < 0$, then

\[ \beta = |F_u(x_0)| \geq C \frac{(U(x_0) - t)^n}{\text{dist}(x_0, \partial \Omega)^n\frac{n+1}{2}\text{diam}(\Omega)^{\frac{n(2\alpha-3)}{2}}}. \]

(b) If on the other hand $\alpha \geq 2$, then we have

\[ \beta = |F_u(x_0)| \geq C \frac{(U(x_0) - t)^n}{\text{dist}(x_0, \partial \Omega)^n\frac{n+1}{2}\text{diam}(\Omega)^{\frac{n(2\alpha-3)}{2}}}. \]

Here $C$ is a positive constant depending only on $\alpha$ and $n$.

Finally, for $\alpha > 1$ and $t \geq 0$, the set $F_u(x_0)$ is convex.

**Proof.** The set of functions $v_{\lambda,y}$ is clearly nonempty, so $u$ is well defined and also $u(x_0) \leq t$. We will prove that $u(x_0) \geq t$. Let’s assume first that $t \geq 0$. Since $t < U(x_0)$, there exists $y_0 \in \Omega$ such that $t < \text{dist}^\alpha(y_0, \partial \Omega) - |x_0 - y_0|^\alpha$. Since the function $\Psi(z) = \text{dist}^\alpha(z, \partial \Omega) - z$ is continuous and $\Psi(y_0) > t$ and $\Psi \leq 0$ on $\partial \Omega$, this implies that there exists $z_0 \in \Omega$ such that $\Psi(z_0) = t$. Letting $v(x) = \text{dist}^\alpha(z_0, \partial \Omega) - |x - z_0|^\alpha$, then $v \leq 0$ on $\partial \Omega$ and $v(x_0) = t$; this implies $u(x_0) \geq t$. If $t < 0$, then we can take $u_{t,x_0}$ in the definition of $u$. Therefore $u(x_0) = t$.

We prove that $u$ is $\alpha$-convex in $\Omega$. Let $\bar{x} \in \Omega$. We first claim the supremum is attained; i.e., there exists $\bar{\lambda} \in \mathbb{R}$ and $\bar{y} \in \mathbb{R}^n$ such that $u(\bar{x}) = \bar{\lambda} - |\bar{x} - \bar{y}|^\alpha$, where $v_{\bar{\lambda},\bar{y}}$ satisfies $v_{\bar{\lambda},\bar{y}} \leq 0$ on $\partial \Omega$ and $v_{\bar{\lambda},\bar{y}}(x_0) \leq t$. Assuming the claim we get that $u(x) \geq \bar{\lambda} - |x - \bar{y}|^\alpha = u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha$, for all $x \in \Omega$, which implies that $\bar{y} \in F_u(\bar{x})$; that is, $u$ is $\alpha$-convex.

To prove the claim, we have from the definition of $u$ that $u(\bar{x}) = \lim_{n \to \infty} \lambda_n - |\bar{x} - y_n|^\alpha$. If $t \geq 0$, then we may assume that $y_n \in \Omega$; otherwise $v_{\lambda_n,y_n} \leq 0$ in $\partial \Omega$ while $u \geq 0$ in $\Omega$. We may also assume that $y_n \to \bar{y} \in \Omega$ and hence also $\lambda_n \to \bar{\lambda}$ and the claim is then proved. If $t < 0$, suppose by contradiction that $|\bar{x} - y_n| \to +\infty$, in which case also $\lambda_n \to +\infty$. This implies that for $n$ large, $y_n \notin \Omega$. Let $x_n \in \partial \Omega$ be such that $\text{dist}(y_n, \partial \Omega) = |x_n - y_n|$. Set $v_n(x) = \lambda_n - |x - y_n|^\alpha$. 
Then $v_n \leq 0$ on $\partial \Omega$ and hence $\lambda_n \leq \text{dist}^\alpha(y_n, \partial \Omega) = |x_n - y_n|^\alpha$, which implies that 
\[ v_n(x) \leq |x_n - y_n|^\alpha - |x - y|^\alpha, \]
so
\[
v_n(x) = \alpha |x_n - y_n|^\alpha - |x - y|^\alpha = \alpha |x_n - y_n|^{\alpha-2} (x_n - y_n, x_n - x) \]
for some $x_n \in \bar{x}, x_n$.

\[ = \alpha |x_n - y_n|^{\alpha-1} |x_n - x| \cos \theta_n, \]
where $\theta_n = \angle(y_n - \bar{x}, x_n, \bar{x}) = \angle(\bar{x}, x_n, \bar{x}, x_n) = \angle(y_n - x_n, \bar{x} - x_n) \geq \frac{\pi}{2} + \delta(\epsilon)$ for some $\delta(\epsilon) > 0$, where $B_\delta(\bar{x}) \subseteq \Omega$ (here $\angle(x, y)$ denotes the angle between the vectors $x$ and $y$). Hence $\cos \theta_n \leq -C_\epsilon$. Then $|x_n - y_n|^\alpha - |\bar{x} - y_n|^\alpha \leq -\alpha |x_n - y_n|^{\alpha-1}C_\epsilon \leq -\alpha |x_n - y_n|^{\alpha-1} \rightarrow -\infty$ as $n \rightarrow \infty$. This is a contradiction, since $v_n(x) \rightarrow u(x)$.

Next we show that $u \in C(\bar{\Omega})$. First, notice that by definition, $u \leq U$ in $\Omega$. If $t \geq 0$, then $u \geq 0$ and we are done. Suppose $t < 0$. Fix $\bar{x} \in \partial \Omega$, and let $\eta$ be the unit outer normal to $\partial \Omega$ at $\bar{x}$. For $s > 0$, let $v_s(x) = |\bar{x} - (\bar{x} + \eta s)|^{\alpha} - |x - (\bar{x} + \eta s)|^{\alpha}$. We have $v_s \leq 0$ on $\partial \Omega$, and exactly as above we see that $v_s(x_0) \rightarrow -\infty$ as $s \rightarrow +\infty$.

Hence, for $s$ large enough, $v_s$ is an admissible function and hence $u \geq v_s$ in $\Omega$. Since $v(\bar{x}) = 0$, we get that $u \in C(\bar{\Omega})$ and $u = 0$ on $\partial \Omega$.

We now show that $F_u = \beta \delta x_0$ for some $\beta = \beta(x_0, t)$. Indeed, let $\bar{x} \in \Omega$, $\bar{x} \neq x_0$, and let $y \in F_u(x)$; we claim that $y \in F_u(x)$ for some $x \neq \bar{x}$. Assuming the claim, we get from (2.2) that $|F_u(E)| = 0$ for each $E$ with $x_0 \notin E$, and so $F_u$ is concentrated at $x_0$.

To prove the claim, since $y \in F_u(x)$, $u(x) \geq u(\bar{x}) + |y - \bar{x}|^\alpha - |x - y|^\alpha$, for all $x \in \Omega$. Let $v(x) = u(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha = \lambda - |x - y|^\alpha$. Since $u = 0$ on $\partial \Omega$, we have $v \leq 0$ on $\partial \Omega$ and $v(x_0) \leq u(x_0) = t$. If $v(x_0) = t$, then $y \in F_u(x_0)$; and if $v(x_0) < t$, then, as before, there exists $x \in \partial \Omega$ such that $v(x) = 0$ and hence $y \in F_u(x)$.

Before estimating $\beta$, we need the following characterization of $u$ (it will be easier to work with $\bar{u}$ rather than with $u$). If $\bar{u}(x) = \sup \{ \text{dist}^\alpha(y, \partial \Omega) - |x - y|^\alpha : y \in \mathbb{R}^n \}$, then $u = \bar{u}$. Notice that $\bar{u} \leq u$. To show that $\bar{u} \geq u$, let $\bar{x} \in \Omega$ and $\bar{y} \in F_u(x)$, so $u(\bar{x}) \geq u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha$ for all $x \in \Omega$, and we claim that there exists $x \in \partial \Omega$ such that $u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha = 0$. Let's assume this claim holds. We have $0 \geq u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha$ for all $x \in \partial \Omega$ and $|\bar{x} - \bar{y}|^\alpha = u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha$, which implies that $|x - \bar{y}| \geq |\bar{x} - \bar{y}|$ for all $x \in \partial \Omega$, and hence $\text{dist}(\bar{y}, \partial \Omega) = |\bar{x} - \bar{y}|$. This implies that $u(\bar{x}) = \text{dist}^\alpha(\bar{y}, \partial \Omega) - |\bar{x} - \bar{y}|^\alpha \leq \bar{u}(\bar{x})$ and we are done. We now prove the claim. If the claim is not true, there exists $\epsilon > 0$ such that $u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha + \epsilon < 0$ for all $x \in \partial \Omega$. By continuity, there exists $\delta > 0$ such that if $y \in B_\delta(\bar{y})$, then $u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha + \frac{\epsilon}{2} \leq u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha + \epsilon$ for all $x \in \partial \Omega$. Consider the set $\Gamma = \{ y : u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x_0 - \bar{y}|^\alpha < t \}$. Then $\Gamma \cap B_\delta(\bar{y}) \neq \emptyset$. Fix $\tilde{y} \in \Gamma \cap B_\delta(\bar{y})$, and let $v(x) = u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha + t - (u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x_0 - \bar{y}|^\alpha)$, where $M > 1$ is large enough such that $0 < \frac{t - (u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x_0 - \bar{y}|^\alpha)}{M} < \frac{\epsilon}{2}$. It follows that $v \leq 0$ on $\partial \Omega$ and $v(x_0) \leq t$, and this implies that $u \leq u$ in $\Omega$, but $v(\bar{x}) > u(\bar{x})$, and this contradiction proves the claim.

We next prove that
\[
F_u(x_0) = \{ y \in \mathbb{R}^n : \text{dist}^\alpha(y, \partial \Omega) - |x_0 - y|^\alpha \geq t \} := F_t(x_0).
\]
It will be easier to work with $F_u(x_0)$ written in this way. For, if $y \in F_u(x_0)$, then $u(x) \geq u(x_0) + |x_0 - y|\alpha - |x - y|\alpha$ for all $x \in \Omega$, which implies that $|x - y|\alpha \geq u(x_0) + |x_0 - y|\alpha$ for all $x \in \partial \Omega$, and hence, $\text{dist}(y, \partial \Omega) \geq u(x_0) + |x_0 - y|\alpha = t + |x_0 - y|\alpha$.

On the other hand, if $\text{dist}(y, \partial \Omega) - |x_0 - y|\alpha \geq t$, then $|x - y|\alpha \geq |x_0 - y|\alpha + t$ for all $x \in \partial \Omega$, and hence $0 \geq |x_0 - y|\alpha + t - |x - y|\alpha = |x_0 - y|\alpha + u(x_0) - |x - y|\alpha$ for all $x \in \partial \Omega$. Set $v(x) = |x_0 - y|\alpha + u(x_0) - |x - y|\alpha$. Then $v \leq 0$ on $\partial \Omega$, $v(x_0) = t$, and hence $u \geq v$ in $\Omega$, which implies $y \in F_u(x_0)$.

The remainder of the proof is devoted to establishing (4.2)-(4.4).

Recall that $u(x_0) = t < U(x_0)$, and say $U(x_0) = \text{dist}(y_0, \partial \Omega) - |x_0 - y_0|\alpha$ for some $y_0 \in \Omega$. Without loss of generality we will assume from now on that $y_0 = 0$, $\text{dist}(y_0, \partial \Omega) = R$ and $x_0 = |x_0|e_1$ with $|x_0| < R$.

Since $B_R(0) \subseteq \Omega$ we have that $|F_t(x_0)| \geq \{|y \in B_R(0) : \text{dist}(y, \partial \Omega) - |y - x_0|\alpha \geq t\}| \geq \{|y \in B_R(0) : \text{dist}(y, \partial B_R(0)) - |y - x_0|\alpha \geq t\}| = \{|y \in B_R(0) : (R - |y|)\alpha - |y - x_0|\alpha \geq t\}$. We shall estimate the measure of the last set.

**Estimation of $\beta$ when $\alpha = 2$.**

First notice that the set $\{y \in B_R(0) : (R - |y|)^2 - |y - x_0|^2 \geq t\}$ is the ellipsoid

$$\left\{y : \frac{(y_1 - \delta)^2}{a^2} + \frac{\sum_{n=2}^n(y_i)^2}{b^2} \leq 1 \right\}$$

with

$$\delta = \frac{(R^2 - |x_0|^2 - t)|x_0|}{2(R^2 - |x_0|^2)}, \quad a = \frac{R^2 - |x_0|^2 - t}{2(R^2 - |x_0|^2)}, \quad b = \frac{R^2 - |x_0|^2 - t}{2(R^2 - |x_0|^2)^{\frac{1}{2}}}.$$  

The volume of this ellipsoid equals

$$C_n\left(\frac{R^2 - |x_0|^2 - t}{R^2 - |x_0|^2}\right)^n R.$$

We also notice that the set $\{y \in B_R(0) : (R - |y|)^2 - |y - x_0|^2 \geq t\}$ equals the set

$$\{\rho \xi \in S^{n-1}, 0 \leq \rho \leq \frac{R^2 - |x_0|^2 - t}{2(R - \langle x_0, \xi \rangle)}\}$$

and using polar coordinates we get that its volume is equal to $C_n(R^2 - |x_0|^2)^n \int_{S^{n-1}} \frac{1}{(R - \langle x_0, \xi \rangle)^n} d\xi$. This implies that

$$\int_{S^{n-1}} \frac{1}{(R - \langle x_0, \xi \rangle)^n} d\xi = C_n\frac{R}{(R^2 - |x_0|^2)^{\frac{n}{2}}}$$

which also implies for $\alpha > 1$ that

$$\int_{S^{n-1}} \frac{1}{(R^\alpha - |x_0|^{\alpha - 2}\xi - x_0, \xi)^n} d\xi = \frac{R^{\alpha - 1}}{(R^{2(\alpha - 1)} - |x_0|^{2(\alpha - 1)})^{\frac{n}{2}}}$$

for $|x_0| < R$.

**Estimation of $\beta$ for $\alpha > 1$.**

Let $\xi \in S^{n-1}$ and consider $\phi(s) = (R-s)^\alpha - |s\xi - x_0|^\alpha$ for $0 \leq s \leq \frac{R^2 - |x_0|^2}{2(R - \langle x_0, \xi \rangle)}$.

We claim that $\phi$ is decreasing, concave for $1 < \alpha \leq 2$ and convex for $\alpha > 2$. First notice that for $0 \leq s \leq \frac{R^2 - |x_0|^2}{2(R - \langle x_0, \xi \rangle)}$ we have $|s\xi - x_0| \leq (R-s)$. Next, we compute $\phi'(s) = -\alpha(R-s)^{\alpha - 1} - |s\xi - x_0|^{\alpha - 2}(s\xi - x_0, \xi) \leq -\alpha(R-s)^{\alpha - 1} + \alpha|s\xi - x_0|^{\alpha - 1} < 0$ and $\phi''(s) = (\alpha - 1)(R-s)^{\alpha - 2} - |s\xi - x_0|^{\alpha - 2} \left(1 - 2(\alpha - 1)\frac{(s\xi - x_0, \xi)^2}{|s\xi - x_0|^2}\right)$. Hence if $1 < \alpha < 2$, then $\phi''(s) < 0$; and if $\alpha > 2$, then $\phi''(s) > 0$. Therefore we have, for
Proof. We will show that the function \( r(z) = \left( (z^\alpha - t)^{\frac{1}{\alpha}} - (z - z_0)^2 \right)^{\frac{1}{2}} \) is a concave function on the set \( z^\alpha - |z - z_0|^\alpha \geq t \). We have \( rr' = (z^\alpha - t)^{\frac{2(\alpha - 1)}{\alpha}} z^{\alpha - 1} - (z - z_0) > 0 \) and

\[
(r')^2 + rr'' = \frac{z^\alpha - (\alpha - 1)t - (z^\alpha - t)^{\frac{2(\alpha - 1)}{\alpha}} z^{2 - \alpha}}{(z^\alpha - t)^{\frac{2(\alpha - 1)}{\alpha}} z^{2 - \alpha}}.
\]
Hence, to show that \( r'' < 0 \) for \( r > 0 \), we must show that
\[
\frac{z^\alpha - (\alpha - 1)t - (z^\alpha - t)\frac{2(\alpha-1)}{\alpha} z^{2-\alpha}}{(z^\alpha - t)\frac{2(\alpha-1)}{\alpha} z^{2-\alpha}} \leq (r')^2 = (\frac{(z^\alpha - t)^{\frac{\alpha-1}{\alpha}} - (z - z_0)^2}{z^\alpha - t})^2 - \frac{2(\alpha-1)}{\alpha} (z^\alpha - t)^{\frac{\alpha-1}{\alpha}} (z - z_0)^2 - \frac{2(\alpha-1)}{\alpha} (z - z_0)^2,
\]
which holds if and only if
\[
0 \leq (\alpha - 1)t(z^\alpha - t)^{\frac{\alpha-1}{\alpha}} - (z - z_0)^2 + z^\alpha((z - z_0) - (z^\alpha - t)z^{1-\alpha})^2.
\]
This inequality is obviously true for \( t \geq 0 \). \( \Box \)

For \( \bar{x} \in \partial \Omega \), let \( T_{\bar{x}} \) be a supporting hyperplane to \( \Omega \) at \( \bar{x} \) and let \( P_{\bar{x},t} = \{ y \in \Omega : \text{dist}^\alpha(y, T_{\bar{x}}) - |y - x_0|^\alpha \geq t \} \), which by the previous lemma is a convex set. Notice that \( F_t(x_0) = \{ y \in \Omega : \text{dist}^\alpha(y, \partial \Omega) - |y - x_0|^\alpha \geq t \} = \bigcap_{x \in \partial \Omega} P_{\bar{x},t} \), and hence it is a convex set. This completes the proof of Theorem 4.1. \( \Box \)

We now consider the case when \( t < 0 \) in Theorem 4.1.

**Lemma 4.3.** For \( 1 < \alpha \leq 2n \) there exists \( \delta > 1 \) depending only on \( \alpha \) such that if \( -t \geq \delta \text{dist}(x_0, \partial \Omega)^\alpha \) then
\[
|F_t(x_0) \cap \Omega^c| \geq C \frac{(\alpha - 1)|\Omega|}{\text{dist}(x_0, \partial \Omega)^{\frac{2(\alpha-1)}{\alpha}}},
\]
where \( C \) depends only on \( \alpha \) and \( n \).

**Proof.** Write \( x = (x', x_n) \) and assume \( 0 \in \partial \Omega \), \( \Omega \subseteq \{ x : x_n \leq 0 \} \) and \( x_0 = (0, -\epsilon) \in \Omega \) with \( \text{dist}(x_0, \partial \Omega) = \epsilon \). Assume that \( -t \geq \delta \epsilon \), where \( \delta > 1 \) will be chosen momentarily. For \( y \in \mathbb{R}^n \) with \( y_n \geq 0 \), we have \( |\text{dist}^\alpha(y, \partial \Omega) - |y - x_0|^\alpha \geq y_n^\alpha - (|y'|^2 + (y_n + \epsilon)^2)^{\frac{1}{2}} \). Hence \( H := \{ y : y_n^\alpha - (|y'|^2 + (y_n + \epsilon)^2)^{\frac{1}{2}} \geq t \text{ and } y_n \geq 0 \} \subseteq F_t(x_0) \). Let \( y > 0 \) satisfy the equation \( y_n^\alpha - t = (y + \epsilon)^\alpha \). Then by slicing, the volume of \( H \) equals
\[
V = C_n \int_0^{\hat{y}} (y_n^\alpha - t)^{\frac{1}{2}} - (y + \epsilon)^\alpha \frac{\alpha-1}{\alpha} dy.
\]
Let \( \phi(y) = (y_n^\alpha - t)^{\frac{1}{2}} - (y + \epsilon)^\alpha \) and \( \phi_1(y) = (y_n^\alpha - t)^{\frac{1}{2}} - (y + \epsilon) \) and \( \phi_2(y) = (y_n^\alpha - t)^{\frac{1}{2}} + (y + \epsilon) \). Since \( \phi_1 \) and \( \phi_2 \) are convex, we have \( \phi_1(y) \geq \phi_2(y) \), which implies that \( \phi_1(y) \phi_2(y) \geq p(y) \), with \( p \) a concave parabola. Set \( p(y) = \max \{ p(y) : y \in [0, \hat{y}] \} := h \). Then, \( p(y) \geq h \frac{y}{\hat{y}} \), for \( y \in [0, \hat{y}] \) and \( p(y) \geq h \frac{\hat{y}}{y} \), for \( y \in [\hat{y}, \bar{y}] \). Therefore, we get
\[
V \geq \int_0^{\hat{y}} (p(y))^{\frac{\alpha-1}{\alpha}} dy \geq h^{\frac{\alpha-1}{\alpha}} \frac{\hat{y}}{y}.
\]
We estimate \( h \) and \( \hat{y} \) from below. Notice that \( -t = (y + \epsilon)^\alpha - \hat{y}^\alpha = \alpha \xi^{\alpha-1} \epsilon \) for some \( y < \xi < y + \epsilon \), and hence \( -t \leq \alpha (y + \epsilon)^{\alpha-1} \epsilon \), which implies that \( \hat{y} \geq \left( \frac{t}{\alpha \epsilon} \right)^{\frac{1}{\alpha-1}} - \epsilon \).

Choosing \( \delta = \alpha^{2\alpha-1} \), we obtain \( \hat{y} \geq \frac{1}{2} \left( \frac{t}{\alpha \epsilon} \right)^{\frac{1}{\alpha-1}} \). It follows that \( \frac{\hat{y}}{y + \epsilon} \geq \frac{1}{2} \) and
hence that $1 - \left(\frac{\dot{y}}{y + \epsilon}\right)^{\alpha - 1} \geq C_n \frac{\epsilon}{y + \epsilon}$ from the mean value theorem. A simple calculation shows that

$$h = \frac{1}{4} \left[1 - \left(\frac{\dot{y}}{y + \epsilon}\right)^{\alpha - 1}\right] \left(\dot{y} + (-t)^{\frac{1}{\alpha}} + \epsilon\right)^2,$$

and hence we obtain $h \geq C_\alpha \epsilon (\dot{y} + \epsilon) \geq C_\alpha \epsilon \dot{y}$.

Therefore we get that $V \geq C_\alpha \epsilon \frac{\dot{y}^{\frac{1}{\alpha}}}{y + \epsilon} \geq C_\alpha \frac{\epsilon}{y + \epsilon}^{\frac{\alpha - 1}{\alpha}}$, which proves the lemma.  

\[ \square \]

**Remark 4.4.** In case $\alpha = 2$, an estimate similar to (4.3) can be proved for all $t \leq U(x_0)$. Recall that $U(x_0) = \text{dist}^2(y_0, \partial \Omega) - |x_0 - y_0|^2$ for some $y_0 \in \Omega$. We first claim that $B_{U(x_0)}(y_0) \subseteq F_t(x_0)$. Indeed, if $x \in \partial \Omega$ and $y \in \mathbb{R}^n$, then we have $|x - y|^2 - |x_0 - y|^2 \geq U(x_0) + |x - y|^2 - |x_0 - y|^2 = U(x_0) + |x_0 - y|^2 - |x_0 - y|^2 = U(x_0) + 2\langle x_0 - y \rangle \geq U(x_0) - 2|x_0 - y|^2 \geq U(x_0) - 2\text{diam}(\Omega) |y - y_0| \geq t$, provided that $|y - y_0| \leq \frac{U(x_0) - t}{2\text{diam}(\Omega)}$, and the claim follows.

Next, let $\bar{x} = A_{y_0}$, then, $x \in \partial \Omega$ and $\text{dist}(y_0, \partial \Omega) = |\bar{x} - y_0|$. If $s \geq 0$ and $y_0 = y_0 + s(\bar{x} - y_0)$, then $\text{dist}(y_0, \partial \Omega) = |\bar{x} - y_0|$. Hence $\text{dist}^2(y_0, \partial \Omega) - |x_0 - y_0|^2 = |\bar{x} - y_0|^2 - |x_0 - y_0|^2 = U(x_0) + |\bar{x} - y_0|^2 - |x_0 - y_0|^2 = U(x_0) + 2\langle \bar{x} - x_0, y_0 - y_0 \rangle \geq U(x_0) - 2|x_0 - y_0| |y_0 - y_0| \geq t$, provided $\frac{U(x_0) - t}{2|x_0 - y_0|} \geq |y_0 - y_0|$. So, there exists $\bar{y} \in F_t(x_0)$, with $|\bar{y} - y_0| = \frac{U(x_0) - t}{2|x_0 - y_0|}$.

Notice that if $y_1, y_2 \in F_t(x_0)$, then the straight segment $[y_1, y_2] \subseteq F_t(x_0)$. Let $\Delta$ be the cone in $\mathbb{R}^n$ with base on $T_{y_0} \cap B_{U(x_0)}(y_0)$ and vertex at $\bar{y}$, where $T_{y_0}$ is the hyperplane passing through $y_0$ with normal $\bar{y} - y_0$. Then $\Delta \subseteq F_t(x_0)$ and $\Delta$ has measure $c_n \frac{U(x_0) - t}{2\text{diam}(\Omega)^{n-1}} |\bar{y} - y_0| = c_n \frac{U(x_0) - t}{2\text{diam}(\Omega)^{n-1}} |x_0 - x_0|$. We now relate $|\bar{x} - x_0|$ to $\text{dist}(x_0, \partial \Omega)$. Since $\bar{x} \in A_{y_0}$ is arbitrary, taking $\xi = x_0 - y_0$ in the definition of $A_{y_0}$, we obtain $|x_0 - \bar{x}, x_0 - y_0| \leq 0$, which holds if and only if $|x_0 - \bar{x}|^2 \leq |\bar{x} - y_0|^2 - |x_0 - y_0|^2 = U(x_0)$. From the proof of Theorem 3.1 part (2), we have $U(x_0) \leq 2\text{diam}(\Omega) \text{dist}(x_0, \partial \Omega)$, and therefore we obtain the estimate $|F_t(x_0)| \geq |\Delta| \geq C_n \frac{\text{diam}(\Omega)^{1/2-n}}{\text{dist}(x_0, \partial \Omega)^{-1/2}} (U(x_0) - t)^n$.

5. MAI THEOREMS

**Theorem 5.1.** Let $\alpha > 1$ and let $\Omega$ be an open, bounded, convex domain in $\mathbb{R}^n$. Assume $u \in C(\Omega)$, $u = 0$ on $\partial \Omega$, and $0 \leq u(x_0) \leq U(x_0)$ for some $x_0 \in \Omega$. Then we have

$$ (U(x_0) - u(x_0))^n \leq C(\text{dist}(x_0, \partial \Omega))^{\frac{n+1}{2}} \text{diam}(\Omega) \frac{n(2\alpha - 3) - 1}{2\alpha - 3} |F_u(\Omega)| $$

whenever $n(2\alpha - 3) - 1 \geq 0$ and

$$ (U(x_0) - u(x_0))^n \leq C(\text{dist}(x_0, \partial \Omega))^{\frac{n(\alpha - 1)}{2}} |F_u(\Omega)| $$

whenever $n(2\alpha - 3) - 1 \leq 0$. The constant $C$ depends only on $n$ and $\alpha$.

**Proof.** Suppose $u(x_0) < U(x_0)$. Let $v(x) = \text{sup} \{\lambda - |x - y|^2 : v_{\lambda,y}(x) \leq u(x_0)\}$ and $v_{\lambda,y} \leq 0$ on $\partial \Omega$. We claim $F_v(x_0) \subseteq F_u(\Omega)$. Let $y \in F_v(x_0)$, so
\begin{align*}
v(x) &\geq v(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha \text{ for all } x \in \Omega. \text{ Consider} \\
&\quad \sup_{\Omega} (v(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha - u(x)),
\end{align*}
and let \( \hat{x} \in \bar{\Omega} \) be the point where the supremum is attained. From Theorem 4.1 we have
\( u(x_0) = v(x_0) \). We have \( v(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha \leq 0 \) for all \( x \in \partial \Omega \). If \( \hat{x} \in \partial \Omega \),
then \( u(x) \geq u(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha \) for all \( x \in \Omega \) and so \( y \in F_u(x_0) \). If on the other hand \( \hat{x} \in \Omega \),
then \( u(x) \geq u(\hat{x}) + |\hat{x} - y|^\alpha - |x - y|^\alpha \) for all \( x \in \Omega \), and hence \( y \in F_u(\hat{x}) \). Consequently the claim is proved. From Theorem 4.1 applied to \( v \) we then obtain the result.

\begin{theorem}
Let \( 1 < \alpha \leq \frac{2n}{n-1} \). Let \( \Omega \) be an open, bounded, convex domain in \( \mathbb{R}^n \). Assume \( u \in C(\Omega) \) with \( u = 0 \) on \( \partial \Omega \). Let \( x \in \Omega \) such that \( u(x) < 0 \). If
\begin{equation}
|u(x)| \leq \alpha 2^{\alpha-1}\text{dist}^\alpha(x, \partial \Omega),
\end{equation}
then
\begin{equation}
|u(x)|^{\frac{n+\alpha}{n-1}} \leq C_{\alpha, n}\text{dist}^\alpha(x, \partial \Omega)|F_u(\Omega)|.
\end{equation}
If on the other hand, \( \text{(5.3)} \) does not hold, then
\begin{equation}
|u(x)|^{\frac{n+\alpha}{2(n-1)}} \leq C_{\alpha, n}\text{dist}^\alpha(x, \partial \Omega)^{\frac{n(2-\alpha)+\alpha}{2(n-1)}}|F_u(\Omega)|.
\end{equation}
\end{theorem}

\textbf{Proof.} If \( \text{(5.3)} \) holds and since we always have \( B_{|u(x)|^{\frac{1}{\alpha}}}(x) \subseteq F_u(\Omega) \), then \( \text{(5.4)} \) follows. If on the other hand, \( |u(x)| > \alpha 2^{\alpha-1}\text{dist}^\alpha(x, \partial \Omega) \), then we are under the hypothesis of Lemma 4.3 and the proof of \( \text{(5.5)} \) follows in the same way as in the previous theorem.

\begin{remark}
If \( \alpha > \frac{2n}{n-1} \) and \( u(x) < 0 \), then \( u(x) \) cannot be estimated by any positive power of \( \text{dist}(x, \partial \Omega) \) and consequently neither can \( U(x) = u(x) \).

Considering the cylinder \( \Omega = \{ x' \in \mathbb{R}^{n-1} : |x'| < 2, |x_n| < 1 \} \), we set \( x = (x', x_n) \). Let \( x_k = (0, 1 - \frac{k}{2}) \). Notice that if \( |x'| \geq 2 \), then \( \text{dist}^\alpha(x, \partial \Omega) - |x - x_k|^\alpha \leq -2^{\alpha} \); this follows since for \( \alpha > 2 \) the function \( h(y) = y^\alpha - (a^2 + (y + b)^2)^\frac{\alpha}{2} \) is decreasing for \( y > 0 \), for any fixed positive \( a \) and \( b \). Hence, if \( x \notin \Omega \) and \( \text{dist}^\alpha(x, \partial \Omega) - |x - x_k|^\alpha \geq -1 \), then \( |x'| < 2 \), which implies that
\begin{align*}
F_{-1}(x_k) \cap \Omega^c \\
\subseteq \left\{ x : |x'| < 2, x_n > 1, (x_n - 1)^\alpha - \left(|x'|^2 + \frac{1}{k} \right)^{\alpha/2} > -1 \right\},
\end{align*}
where \( F_{-1}(x_k) \) is defined in \( \text{[15]} \), and the set on the right hand side is basically the set \( H \) defined in the proof of Lemma 4.3. We can now estimate from above the measure of this set in the same way as in Lemma 4.3. If \( y > 0, t < 0 \), then we have
\( (y^\alpha - t)^{\frac{\alpha}{2}} - y^2 = \frac{\alpha}{2} \xi^{\frac{\alpha}{2} - 1}(-t) \) for some \( \xi \) with \( \alpha < \xi < y^\alpha - t \). We let \( t = -1 \) and
\( \epsilon = 1/k \), and with the notation in the proof of Lemma 4.3 we then get that

\[
|F_{-1}(x_k) \cap \Omega^c| \leq \int_0^{\tilde{y}} \left( (y^{n+1})^{\frac{2}{n}} - (y + 1/k)^2 \right)^{\frac{1}{n-1}} dy
\]

\[
\leq \int_0^{\infty} \left( (y^{n+1})^{\frac{2}{n}} - y^2 \right)^{\frac{1}{n-1}} dy = \int_1^{\infty} \int_0^{\infty}
\]

\[
\leq C_1 + \int_1^{\infty} y^{\left(\alpha - 1\right)} dy \leq C_{n, \alpha}
\]

for \( \alpha > 2n/(n-1) \). Let \( u_k \) be \( \alpha \)-convex such that \( u_k = 0 \) on \( \partial \Omega \), \( u_k(x_k) = -1 \), whose existence follows from the first part of Theorem 4.1. Then \( |F_{u_k}(\Omega)| \leq C \) for all \( k \) while \( |u_k(x_k)| = 1 \) and \( \text{dist}(x_k, \partial \Omega) = \frac{1}{k} \).

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